BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 65, No. 3, 2017
DOI: 10.1515/bpasts-2017-0034

Stability of fractional positive continuous-time linear systems with state matrices in integer and rational powers

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Abstract. The stability of fractional standard and positive continuous-time linear systems with state matrices in integer and rational powers is addressed. It is shown that the fractional systems are asymptotically stable if and only if the eigenvalues of the state matrices satisfy some conditions imposed on the phases of the eigenvalues. The fractional standard systems are unstable if the state matrices have at least one positive eigenvalue.

Key words: stability, fractional, positive, linear, continuous-time, system, integer, rational, order.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [1, 2] A variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [3–5]. The positive fractional linear systems have been investigated in [6–8]. The stability is a basic notion of the analysis of dynamical linear and nonlinear systems [2, 9–11]. Stability of fractional linear continuous-time systems has been investigated in the papers [3, 12, 10, 13–14]. The notion of practical stability of positive fractional linear systems has been introduced in [15]. Some recent interesting results in fractional systems theory and its applications can be found in [4, 10, 13, 16]. The positive linear systems consisting of n subsystems with different fractional orders have been addressed in [17]. The controllability and minimum energy control of fractional systems have been analyzed in [18–20] and the reachability of fractional positive linear systems in [18].

In this paper the stability of fractional positive continuous-time linear systems with state matrices in integer and rational powers will be addressed.

The paper is organized as follows. In Section 2 preliminaries concerning the fractional positive continuous-time linear systems are recalled. The fractional standard linear systems with state matrices in integer and rational powers are investigated in Section 3. Similar problems for positive fractional linear systems are analyzed in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries

Consider the fractional autonomous continuous-time linear system

$$ {}_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)\,,\ A \in \Re^{n\times n}, \tag{1}$$

where

$$\frac{d^\alpha x(t)}{dt^\alpha} = \frac{1}{\Gamma(q-\alpha)}\int_0^t \frac{x^{(q)}(\tau)}{(t-\tau)^{\alpha+1-q}}d\tau,$$
$$q-1<\alpha<q,\ \ q=1,2,\ \ x^{(q)}(\tau)=\frac{d^q x(\tau)}{d\tau^q} \tag{2}$$

is the Caputo derivative of α order of $x(t) \in \Re^n$ and

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt\,,\ \mathrm{Re}(x)>0 \tag{3}$$

is the Euler gamma function.
The solution of (1) has the form [8]

$$x(t) = \Phi_0(t)x_0\,,\ \ x_0 = x(0), \tag{4}$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{5}$$

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Definition 1. [8] The system (1) (or equivalently the matrix A) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x_0 \in \Re^n. \tag{6}$$

Theorem 1. [8] The system (1) (the matrix A) is asymptotically stable if and only if

$$\alpha\frac{\pi}{2} < \varphi_i < \alpha\frac{3\pi}{2} \quad \text{for } i = 1,...,n,\ q-1 < \alpha < q,\ q = 1,2 \tag{7}$$

where $s_i = |s_i|e^{j\varphi_i}$ are the eigenvalues of the matrix A, i.e. the zeros of the characteristic polynomial of A

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0. \tag{8}$$

Definition 2. [8] The fractional system (1–3) is called (internally) positive if $x(t) \in \Re_+^n$, $t \in [0, +\infty]$ for all $x_0 \in \Re_+^n$.

Definition 3. [2, 18] A matrix $A = [a_{ij}] \in \Re^{n\times n}$ is called Metzler matrix if $a_{ij} \geq 0$ for $i \neq j$, $i, j = 1, …, n$.

Theorem 2. [8] The fractional system (1–3) is positive if and only if

$$A \in M_n \text{ and } 0 < \alpha < 1. \tag{9}$$

Definition 4. [8] The positive fractional system (1) – (3) (the matrix A) is called asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x_0 \in \Re_+^n. \tag{10}$$

Theorem 3. [8] The positive fractional system (1–3) (the matrix A) is asymptotically stable if and only if the eigenvalues $s_i = |s_i|e^{j\varphi_i}$, $i = 1, …, n$ of the matrix $A \in M_n$ satisfy the condition

$$\frac{\pi}{2} < \varphi_i < \frac{3\pi}{2} \text{ for } i = 1,...,n \text{ and } 0 < \alpha < 1. \tag{11}$$

Theorem 4. [21–22] Let s_l, $l = 1, …, n$ be the eigenvalues (not necessarily distinct) of the matrix $A \in \Re^{n\times n}$ and $f(s_l)$ be well defined on the spectrum $\sigma = \{s_1, s_2, …, s_n\}$ of A. Then $f(s_l)$, $l = 1, …, n$ are the eigenvalues of the matrix A.

For example if s_l, $l = 1, …, n$ are the nonzero eigenvalues of the matrix $A \in \Re^{n\times n}$ then s_l^{-1}, $l = 1, …, n$ are the eigenvalues of the inverse matrix A^{-1}.

Theorem 5. The fractional linear system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t),\ A \in \Re^{n\times n},\ 0 < \alpha < 2 \tag{12}$$

is asymptotically stable if and only if the system

$$\frac{d^\alpha x(t)}{dt^\alpha} = -Ax(t),\ A \in \Re^{n\times n},\ 0 < \alpha < 2 \tag{13}$$

is unstable.

Proof. By Theorem 3 if s_k, $k = 1, …, n$ are the eigenvalues of A then $-s_l$, $l = 1, …, n$ are the eigenvalues of $-A$. Therefore, the system (12) is asymptotically stable if and only if the system (13) is unstable. □

3. Fractional linear systems

Case 1. $k = 2, 3, …$ First we shall consider the asymptotic stability of the fractional continuous-time linear systems of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = A^k x(t),\ A \in \Re^{n\times n},\ 0 < \alpha < 2 \tag{14}$$

for $k = 2, 3, ….$

Theorem 6. The fractional continuous-time linear system (14) is asymptotically stable for $k = 2, 3, …$ if and only if the eigenvalues $s_l = |s_l|e^{j\varphi_l}$, $l = 1, …, n$ of the matrix A satisfy the condition

$$\frac{\pi}{2}\alpha < k\varphi_l < 2\pi - \frac{\pi}{2}\alpha \tag{15}$$

Proof. By Theorem 3 if s_l, $l = 1, …, n$ are the eigenvalues of the matrix A then $s_l^k = |s_l^k|e^{jk\varphi_l}$, $l = 1, …, n$ are the eigenvalues of the matrix A^k, $k = 2, 3, …$ Applying to the system (14) Theorem 1 we obtain the condition (15). □

Example 1. Consider the fractional system (14) for $0 < \alpha < 2$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{16}$$

for $k = 2, 3, …$

The characteristic polynomial of (16) has the form

$$\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 1 & s+1 \end{vmatrix} = s^2 + s + 1 \tag{17}$$

and its zeros are

$$s_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\frac{2\pi}{3}},\quad s_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{2\pi}{3}} = e^{j\frac{4\pi}{3}}. \tag{18}$$

From (18) and (15) it follows that the system with (16) is asymptotically stable for $k = 1$ and $0 < \alpha < 4/3$. The system is also asymptotically stable for $k = 2$ and $0 < \alpha < 4/3$ since

$$s_1^2 = \left(e^{j\frac{2\pi}{3}}\right)^2 = e^{j\frac{4\pi}{3}} \text{ and } s_2^2 = \left(e^{j\frac{4\pi}{3}}\right)^2 = e^{j\frac{2\pi}{3}}. \quad (19)$$

The system is unstable for $k = 3$ and $0 < \alpha < 2$ since

$$s_1^3 = \left(e^{j\frac{2\pi}{3}}\right)^3 = e^{j2\pi} = e^{j0^\circ} \text{ and}$$
$$s_2^3 = \left(e^{j\frac{4\pi}{3}}\right)^3 = e^{j4\pi} = e^{j0^\circ} \quad (20)$$

and the condition (15) is not satisfied.

It is easy to show that the fractional system (14) with (16) and $0 < \alpha < 2$ is asymptotically stable for $k = 2l$, $l = 1, 2, \ldots$ and unstable for $k = 2l + 1$, $l = 1, 2, \ldots$

Example 2. Consider the system (14) with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ and } 0 < \alpha < 2. \quad (21)$$

The characteristic polynomial of (21) has the form

$$\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 3s + 2 \quad (22)$$

and its zeros are: $s_1 = -1$, $s_2 = -2$.

The system for $k = 1$ is asymptotically stable for $0 < \alpha < 2$ but for $k = 2$ is unstable since

$$s_1^2 = (-1)^2 = e^{j0^\circ} \text{ and } s_2^2 = (-2)^2 = 4e^{j0^\circ} \quad (23)$$

and the condition (15) is not satisfied.

For $k = 3$ and $0 < \alpha < 2$ the system is asymptotically stable since

$$s_1^3 = (-1)^3 = -1 = e^{j180^\circ} \text{ and } s_2^3 = (-2)^2 = 8e^{j180^\circ} \quad (24)$$

and the condition (15) is satisfied.

In general case it is easy to show that the system (14) with (21) is asymptotically stable for $k = 2l + 1$, $l = 0, 1, \ldots$ and unstable for $k = 2l$, $l = 1, 2, \ldots$ and $0 < \alpha < 2$.

Case 2. $k = -1, -2, \ldots$ Consider the asymptotic stability of the system (14) for $k = -1, -2, \ldots$

Theorem 7. The fractional continuous-time linear system (14) is asymptotically stable for $k = -1, -2, \ldots$ if and only if the eigenvalues $s_l = |s_l|e^{j\varphi_l}$, $l = 1, \ldots, n$ of the matrix A satisfy the condition

$$\frac{\pi}{2}\alpha < -k\varphi_l < 2\pi - \frac{\pi}{2}\alpha \quad (25)$$

Proof. The proof is similar to the proof of Theorem 6.

Example 3. (Continuation of Example 1) The inverse matrix of (16) has the form

$$A^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \quad (26)$$

and its eigenvalues are: $s_1^{-1} = e^{-j\frac{2\pi}{3}} = e^{j\frac{4\pi}{3}}$, $s_2^{-1} = e^{j\frac{2\pi}{3}}$.

The fractional system with (26) is asymptotically stable since the condition (25) is satisfied for $k = -1$ and $0 < \alpha < 4/3$.

Note that for (16) and $k = -2$ we have

$$A^{-2} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad (27)$$

and its eigenvalues are: $s_1^{-2} = s_1 = e^{j\frac{2\pi}{3}}$, $s_2^{-2} = s_2 = e^{j\frac{4\pi}{3}}$.

Therefore, the fractional system with (27) is also asymptotically stable.

For (16) and $k = -3$ we have

$$A^{-3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (28)$$

and the fractional system with (28) is unstable.

Note that the fractional system with (16) is unstable for $k = -3, -6, \ldots$ since

$$(s_1^{-1})^k = e^{j\frac{4k\pi}{3}} = 1 \text{ and } (s_2^{-1})^k = e^{j\frac{2k\pi}{3}} = 1 \quad (29)$$

for $k = -3, -6, \ldots$

Example 4. (Continuation of Example 2) The inverse matrix of (21) has the form

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \quad (30)$$

and its eigenvalues are: $s_1^{-1} = -1 = e^{-j\pi} = e^{j\frac{4\pi}{3}}$, $s_2^{-1} = -\frac{1}{2} = \frac{1}{2}e^{-j\pi}$.

The fractional system with (30) is asymptotically stable since the condition (25) is satisfied for $k = -1$ and $0 < \alpha < 2$.

For $k = -2$ the eigenvalues of the matrix

$$A^{-2} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \quad (31)$$

are $s_1^{-2} = 1 = e^{j0^\circ}$, $s_2^{-2} = \frac{1}{4}e^{j0^\circ}$ and the fractional system with (31) is unstable since the condition (25) is not satisfied for $k = -2$.

It is easy to show that the fractional system with (21) is asymptotically stable for $k = -(1 + 2l)$, $l = 0, 1, \ldots$ and it is unstable for $k = -2l$, $l = 1, 2, \ldots$ and $0 < \alpha < 2$.

Case 3. $k = \frac{p}{q}$ or $k = -\frac{p}{q}$, $p, q = \{1, 2, \ldots\}$

Theorem 8. The fractional continuous-time linear system (14) is asymptotically stable for $k = \pm p/q$, $p, q = \{1, 2, \ldots\}$ if and only if the eigenvalues $s_l = |s_l|e^{j\varphi_l}$, $l = 1, 2, \ldots, n$ of the matrix A satisfy the condition

$$\frac{\pi}{2}\alpha < \pm\frac{p}{q}\varphi_l < 2\pi - \frac{\pi}{2}\alpha \text{ for } l = 1,\ldots,n. \quad (32)$$

Proof. If s_l, $l = 1, \ldots, n$ are the eigenvalues of A then by Theorem 3 $s_l^{\pm p/q}$, $l = 1, \ldots, n$ are the eigenvalues of the matrix $A^{\pm p/q}$, and next applying Theorem 1 to the system (14) we obtain the condition (31). □

Example 5. (Continuation of Example 2) For $\frac{p}{q} = \frac{2}{3}$ the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^{\frac{2}{3}} \quad (33)$$

are $s_1^{\frac{2}{3}} = (-1)^{\frac{2}{3}} = (e^{j\pi})^{\frac{2}{3}} = e^{j\frac{2\pi}{3}}$, $s_2^{\frac{2}{3}} = (-2)^{\frac{2}{3}} = (2e^{j\pi})^{\frac{2}{3}} = 2^{\frac{2}{3}}e^{j\frac{2\pi}{3}}$ and they satisfy the condition (32) for $0 < \alpha < 4/3$. Therefore, the fractional system (14) with the matrix (21) is asymptotically stable for $p = 2$, $q = 3$ and $0 < \alpha < 4/3$.

For $\frac{p}{q} = -\frac{2}{3}$ the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}^{-\frac{2}{3}} \quad (34)$$

are, $s_1^{-\frac{2}{3}} = (-1)^{-\frac{2}{3}} = (e^{j\pi})^{-\frac{2}{3}} = (e^{j\pi})^{\frac{2}{3}} = e^{j\frac{2\pi}{3}}$ and they also satisfy the condition (32) for $0 < \alpha < 4/3$. The fractional system (14) with (34) is also asymptotically stable.

Theorem 9. The fractional continuous-time linear system (14) is unstable for all values of k (integer and rational) and $0 < \alpha < 2$ if its matrix A has at least one real positive eigenvalue.

Proof. If at least one eigenvalue s_l, $l \in \{1, \ldots, n\}$ is positive then by Theorem 3 at least one eigenvalue of the matrix A^k is also positive for all values (integer or rational) of k and the system is unstable for $0 < \alpha < 2$. □

Example 6. Consider the fractional system (14) with the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 2 \end{bmatrix} \text{ and } 0 < \alpha < 2. \quad (35)$$

The characteristic polynomial of (35) has the form

$$\det[I_3 s - A] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 2 & s-2 \end{vmatrix} = s^3 - 2s^2 + 2s - 1 \quad (36)$$

and its zeros are $s_1 = 1$, $s_2 = 1/2 + j\sqrt{3}/2$, $s_3 = 1/2 - j\sqrt{3}/2$.

The fractional system is unstable for all values of k and $0 < \alpha < 2$ since $s_1^k = 1$ for k integer and rational.

4. Positive fractional linear systems

Case 1. $k = 2, 3, \ldots$

First we shall consider the asymptotic stability of the positive fractional continuous-time linear system (14) for $k = 2, 3, \ldots$, $0 < \alpha < 1$ and $A \in M_n$.

Theorem 10. The positive fractional linear system (14) with $A \in M_n$, $0 < \alpha < 1$ is asymptotically stable for $k = 2, 3, \ldots$ if and only if the eigenvalues $s_l = |s_l|e^{j\varphi_l}$, $l = 1, 2, \ldots, n$ of the matrix $A \in M_n$ satisfy the condition

$$\frac{\pi}{2} < k\varphi_l < \frac{3\pi}{2} \text{ for } k = 2,3,\ldots \text{ and } l = 1,\ldots,n. \quad (37)$$

Proof. If s_l, $l = 1, \ldots, n$ are the eigenvalues of $A \in M_n$ then by Theorem 4 $s_l^k = |s_l^k|e^{jk\varphi_l}$, $k = 2, 3, \ldots$ and $l = 1, \ldots, n$ are the eigenvalues of A^k, $k = 2, 3, \ldots$. Applying to the positive fractional system (14) Theorem 3 we obtain the condition (37). □

Example 7. Consider the fractional system (14) for $0 < \alpha < 1$ and

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \quad (38)$$

for $k = 2, 3, \ldots$.

The fractional system (14) with (38) for $0 < \alpha < 1$ is positive since $A \in M_2$. The eigenvalues of (38) are $s_1 = -1 = e^{j\pi}$, $s_2 = -2 = 2e^{j\pi}$ and the condition (37) is satisfied for $k = 2l + 1$, $l = 0, 1, 2, \ldots$ and it is not satisfied for $k = 2l$, $l = 1, 2, \ldots$ Therefore, the fractional positive system with (38) is asymptotically stable for $0 < \alpha < 1$ and $k = 2l + 1$, $l = 0, 1, 2, \ldots$ and it is unstable for $k = 2l$, $l = 1, 2, \ldots$

Case 2. $k = -1, -2, \dots$

Theorem 11. The fractional positive linear system (14) for $0 < \alpha < 1$ is asymptotically stable for $k = -(2l + 1)$, $l = 0, 1, 2, \dots$ if and only if the eigenvalues $s_l = |s_l|e^{j\varphi_l}$, $l = 1, 2, \dots, n$ of the matrix $A \in M_n$ satisfy the condition

$$\frac{\pi}{2} < -k\varphi_l < \frac{3\pi}{2} \text{ for } k = 1,2,\dots \text{ and } l = 1,\dots,n. \quad (39)$$

Proof. If s_l, $l = 1, \dots, n$ are the eigenvalues of $A \in M_n$ then by Theorem 4 $s_l^{-k} = |s_l^{-k}|e^{-jk\varphi_l}$, $k = 1, 2, \dots$ and $l = 1, \dots, n$ are the eigenvalues of A^{-k}, $k = 1, 2, \dots$ and by Theorem 3 the fractional system is asymptotically stable if and only if the condition (39) is satisfied. □

Example 8. (Continuation of Example 7) The inverse matrix of (38) has the form

$$A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} \quad (40)$$

and its eigenvalues are $s_1^{-1} = -1 = e^{j\pi}$, $s_2 = -1/2 = 1/2 e^{j\pi}$. For $k = -1$ the condition (39) is satisfied and the fractional positive system (14) for $0 < \alpha < 1$ with the matrix (38) is asymptotically stable for $k = -1$.

In a similar way it is easy to check that the fractional system (14) for $0 < \alpha < 1$ with (38) is asymptotically stable for $k = -(2l + 1)$, $l = 0, 1, 2, \dots$ and unstable for $k = -2l$, $l = 1, 2, \dots$.

Note that the system with (40) is not positive since $A^{-1} \notin M_2$.

Example 9. Consider the fractional positive linear system for $0 < \alpha < 1$ with

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 4 \\ 1 & 0 & -4 \end{bmatrix} \in M_3. \quad (41)$$

The characteristic polynomial of (41) has the form

$$\det[I_3 s - A] = \begin{vmatrix} s+2 & -1 & -1 \\ 0 & s+3 & -4 \\ -1 & 0 & s+4 \end{vmatrix} = s^3 + 9s^2 + 25s + 17 \quad (42)$$

and its zeros are $s_1 = -1$, $s_2 = -4 + j$, $s_3 = -4 - j$.

Remark 1. In [23] it was shown that the matrix $A \in M_3$ has real negative zero $s_1 = -\alpha_1$ and two complex conjugate zeros $s_2 = -\alpha + j\beta$, $s_3 = -\alpha - j\beta$ ($\alpha > 0, \beta > 0$) if the coefficients of the characteristic polynomial

$$\det[I_3 s - A] = s^3 + a_2 s^2 + a_1 s + a_0 \quad (43)$$

satisfy the conditions

$$a_2^2 > 3a_1 \text{ and } \alpha > \alpha_1. \quad (44)$$

It is easy to check that the zeros of (42) satisfy the conditions (44).

For the matrix (41) we have

$$s_1^{-1} = -1 = e^{j\pi},\ s_2^{-1} = \frac{1}{-4+j} = \frac{1}{17}e^{-j0.58\pi}, \quad s_3^{-1} = \frac{1}{-4-j} = \frac{1}{17}e^{j0.58\pi} \quad (45)$$

and the condition (39) is satisfied for $k = -1$. Therefore, the fractional system with (41) for $k = -1$ is asymptotically stable.

For $k = -2$ the fractional system with (41) is unstable since $s_1^{-2} = 1$ and the condition (41) is not satisfied.

In a similar way it can be shown that the fractional system with (41) and $0 < \alpha < 1$ is asymptotically stable for $k = -(2l + 1)$, $l = 0, 1, 2, \dots$ and unstable for $k = -2l$, $l = 1, 2, \dots$

Note that the fractional system with the inverse matrix

$$A^{-1} = \begin{bmatrix} -2 & 1 & 1 \\ 0 & -3 & 4 \\ 1 & 0 & -4 \end{bmatrix}^{-1} = -\frac{1}{17}\begin{bmatrix} 12 & 4 & 7 \\ 4 & 7 & 8 \\ 3 & 1 & 6 \end{bmatrix} \quad (46)$$

is not positive since $A^{-1} \notin M_3$.

In general case we have the following theorem.

Theorem 12. If nonsingular $A \in M_n$ and it is asymptotically stable then its inverse $A^{-1} \notin \Re_+^{n\times n}$ and it is unstable.

Proof. The proof will be accomplished by induction. The hypothesis is valid for $n = 2$ since by assumption

$$a_{ij} \geq 0,\ i, j = 1,2,\ a_{11}a_{22} - a_{12}a_{21} > 0,$$
$$A^{-1} = \begin{bmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} -a_{22} & -a_{12} \\ -a_{21} & -a_{11} \end{bmatrix} \text{ and } -A^{-1} \in \Re_+^{2\times 2} \quad (47)$$

Assuming that the hypothesis is valid for $n - 1$ it will be shown that it is also true for $n - 1$.

It is easy to verify that if

$$A_n = \begin{bmatrix} -a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & -a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & -a_{nn} \end{bmatrix} = \begin{bmatrix} A_{n-1} & u_n \\ v_n & -a_{nn} \end{bmatrix} \in M_n \quad (48)$$

then

$$A_n^{-1} = \begin{bmatrix} A_{n-1}^{-1} + \frac{A_{n-1}^{-1} u_n v_n A_{n-1}^{-1}}{a_n} & -\frac{A_{n-1}^{-1} u_n}{a_n} \\ -\frac{v_n A_{n-1}^{-1}}{a_n} & \frac{1}{a_n} \end{bmatrix}, \quad (49)$$

$$a_n = a_{nn} - v_n A_{n-1}^{-1} u_n$$

since $A_n A_n^{-1} = A_n^{-1} A_n = I_n$. For (49) we have $-A_n^{-1} \in \Re_+^{n\times n}$ since $-A_{n-1}^{-1} \in \Re_+^{(n-1)\times(n-1)}$.

By assumption $A_n \in M_n$ is asymptotically stable and its eigenvalues $s_k = |s_k| e^{j\varphi_k}$, $k = 1, \ldots, n$ satisfy the condition

$$\frac{\pi}{2} < \varphi_k < \frac{3\pi}{2} \text{ for } k = 1,\ldots,n \quad (50)$$

and $a_{nn} > 0$. By Theorem 4 the eigenvalues $-s_k$, $k = 1, \ldots, n$ of the matrix $-A_n$ do not satisfy the condition (50) and the matrix $-A_n^{-1}$ is unstable. □

Case 3. $k = \pm\frac{p}{q}$, $p, q = \{1, 2, \ldots\}$

Theorem 13. The fractional positive linear system (14) for $0 < \alpha < 1$ is asymptotically stable for $k = \pm p/q$, $p, q \in \{1, 2, \ldots\}$ if and only if the eigenvalues $s_l = |s_l| e^{j\varphi_l}$, $l = 1, 2, \ldots, n$ of the matrix $A \in M_n$ satisfy the condition

$$\frac{\pi}{2} < \pm\frac{p}{q}\varphi_l < \frac{3\pi}{2} \text{ for } p,q \in \{1,2,\ldots\} \text{ and } l = 1,\ldots,n. \quad (51)$$

Proof. The proof is similar to the proof of Theorem 11.

Example 10. (Continuation of Example 7) The eigenvalues of the matrix (38) are $s_1 = -1 = e^{j\pi}$, $s_2 = -2 = 2e^{j\pi}$ and of the matrix (38) in power $\pm p/q = \pm 2/3$ are

$$s_1^{\pm\frac{2}{3}} = e^{\pm j\frac{2\pi}{3}}, \; s_2^{\pm\frac{2}{3}} = 2e^{\pm j\frac{2\pi}{3}}. \quad (52)$$

The eigenvalues (52) satisfy the condition (51) and the fractional positive system with (38) for $0 < \alpha < 1$ and $k = \pm 2/3$ is asymptotically stable.

5. Concluding remarks

The stability of fractional standard and positive continuous-time linear systems with state matrices in integer and rational powers has been addressed. It has been shown that the fractional systems are asymptotically stable if and only if the eigenvalues of the state matrices satisfy some conditions imposed only on the phases of the eigenvalues (Theorems 6 – 8 and 10 – 13). The fractional standard linear systems are unstable for all integer and rational powers of the state matrices if the state matrix has at least one positive eigenvalue (Theorem 9). It is also shown that if nonsingular Metzler matrix is asymptotically stable then its inverse matrix has nonpositive entries (Theorem 12). The considerations have been illustrated by numerical examples of the matrices with real and complex conjugate eigenvalues.

The considerations can be extended to fractional positive discrete-time linear systems.

Acknowledgment. This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

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