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Stress distribution in an elastic layer resting on a Winkler foundation with an emptiness

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Abstract. The paper deals with the plane problem of an elastic layer resting on a Winkler foundation with an emptiness. The stresses in the layer are caused by a given normal loading on its upper boundary plane. The mathematical formulation of the problem leads to a mixed boundary value problem and it is solved using Fourier transform methods and Fredholm integral equation of the second kind. The detailed analysis is derived analytically and numerically for an elliptic distribution of boundary loadings. The results for the normal displacement and the stresses on the lower boundary of the layer are presented in figures.

Key words: displacements, stresses, elastic layer, Winkler foundation, emptiness.

1. Introduction

The problems of stress distributions in an elastic layer resting on a rigid or deformable foundation are of significant interest in geology, in geophysics and in engineering constructions. The knowledge of mechanical interactions of elastic bodies with substrates is of practical importance in the design of building foundations, analysis of railroad tracks, highway engineering. Few mathematical models of substrates are used there. One of them is the Winkler model being a one-parameter model which based on the modulus of subgrade reaction. The Winkler foundations are adopted in many papers and monographs. In paper [1] the plane contact problem of two infinite elastic layers lying on a Winkler foundation is considered. Paper [2] deals with the contact problems of an elastic wedge supported by the Winkler foundation. The wedge is in the plane frictionless contact with a rigid flat plate. In paper [3] the axisymmetric contact problem of a rigid conical, paraboloidal or ellipsoidal indenter on an elastic layer supported by the Winkler foundation is investigated. The solution is based on the fundamental solutions and an integral equation, which is solved numerically. Paper [4] presents the plane contact problem of an elastic homogeneous and isotropic layer supported by a Winkler foundation. The upper surface of the layer is in plane contact with a rigid punch. The axisymmetric contact problem for an elastic layer resting on the rigid half-space with a near-boundary cylindrical excavitation filled with a deformable material is considered in [5]. The material is modelled by a Winkler medium and the layer is pressed by a rigid sphere or by a rigid flat cylinder. The stress analysis near a crack tip in an elastic layer resting on a Winkler foundation is investigated in [6]. The edge crack normal to the lower boundary plane is taken into account. The stress intensity factor is obtained on the basis of the solution of the Fredholm integral equation and some numerical results are presents.

The problems of elastic plates resting on a Winkler foundations are investigated in [7–10]. The estimation of values of parameter k being the Winkler constant is presented in [11]. Some comments on the modeling of foundations are given in [12].

Various problems connected with the analysis of soil-foundation integration were discussed in monograph [13]. The author presents elastic models of soil behaviour, solutions of problems of beams and plates resting on foundations. The monograph [14] is devoted to simple methods for analysis of various structures supported by an elastic foundation (beams, plates, frames, walls) using the Winkler foundation, elastic half-space, elastic layer. The contact problem for a functionally graded layer resting on the Winkler foundation is considered in [15]. The layer is loaded by a rigid cylindrical punch. The Poisson ratio is taken as constant and the modulus of elasticity is assumed to vary exponentially through the thickness of the layer.

The paper deals with the analysis of stresses in an elastic layer resting on a Winkler foundation with an emptiness. The surface of substrate is assumed to be a plane with an infinitely long strip emptiness. The emptiness is sufficiently deep what precludes a contact of the layer with the foundation. The stresses in the layer are caused by a given normal loading of its upper surface.

2. Formulation of problem

Consider the plane static problem of an elastic layer resting on a Winkler foundation with an emptiness. Let the problem be related to a Cartesian coordinate system (x, y, z) such that the

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lower boundary of layer is located along the plane $y = 0$, Fig. 1. Let h denote the thickness of layer. The upper boundary $y = h$ is assumed to be loaded by normal forces symmetrically with respect to the axis Oy and it is independent of z. The elastic layer rests on the Winkler foundation characterized by the stiffness denoted by k. Moreover, the Winkler substrate is assumed to be a half-space $y < 0$ with an infinitely long in the axis Oz direction strip emptiness in the region $-a < x < a$. The emptiness is sufficiently deep such that a contact of the elastic layer with the foundation in the region is impossible. Let λ, μ be Lamé constants and ν be Poisson ratio of the layer. Let $\vec{u}(x, y) = [u_x(x, y), u_{y1}(x, y), 0]$ denote the displacement vector and σ_{xx}, σ_{xy}, σ_{yy} be the non-zero components of the stress tensor.

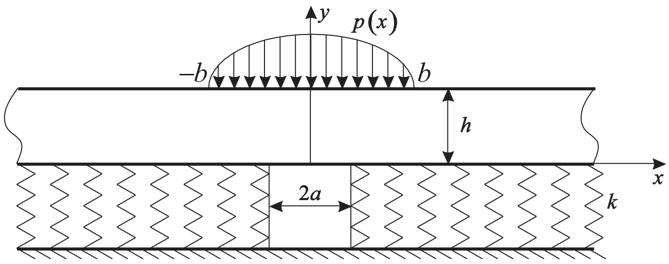

Fig. 1. Scheme of considered problem

The considered problem is described by equations of the plane theory of elasticity

$$\mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + (\lambda + \mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0,$$

$$\mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + (\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0,$$

and the following boundary conditions:

a) on the upper boundary $y = h$

$$\sigma_{yy}(x, h) = -p(x), \quad x \in R, \tag{1a}$$

$$\sigma_{xy}(x, h) = 0, \quad x \in R; \tag{1b}$$

b) on the interface $y = 0$

$$\sigma_{xy}(x, 0) = 0, \quad x \in R; \tag{2a}$$

$$\sigma_{yy}(x, 0) = 0, \quad |x| < a, \tag{2b}$$

$$\sigma_{yy}(x, 0) = ku_y(x, 0), \quad |x| \geq a. \tag{2c}$$

where $p(\cdot)$ is a given even function defining the intensity of loadings.

The solution of equations for the static state of plane strain within the framework of linear elasticity can be written in the following form [6, 18]:

$$u_x(x, y) = \frac{1}{2\mu}\sqrt{\frac{2}{\pi}}\int_0^\infty \{[A + B\alpha y + 2(1-\nu)D]\sinh(\alpha y) + [C + D\alpha y + 2(1-\nu)B]\cosh(\alpha y)\}\sin(\alpha x)\,d\alpha, \tag{3a}$$

$$u_y(x, y) = \frac{-1}{2\mu}\sqrt{\frac{2}{\pi}}\int_0^\infty \{[A + B\alpha y - (1-2\nu)D]\cosh(\alpha y) + [C + D\alpha y - (1-2\nu)B]\sinh(\alpha y)\}\cos(\alpha x)\,d\alpha, \tag{3b}$$

and

$$\sigma_{xx}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \alpha\{[A + B\alpha y + 2D]\sinh(\alpha y) + [C + D\alpha y + 2B]\cosh(\alpha y)\}\cos(\alpha x)\,d\alpha, \tag{4a}$$

$$\sigma_{xy}(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \alpha\{[A + B\alpha y + D]\cosh(\alpha y) + [C + D\alpha y + B]\sinh(\alpha y)\}\sin(\alpha x)\,d\alpha, \tag{4b}$$

$$\sigma_{yy}(x, y) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \alpha\{(A + B\alpha y)\sinh(\alpha y) + (C + D\alpha y)\cosh(\alpha y)\}\cos(\alpha x)\,d\alpha, \tag{4c}$$

where A, B, C, D are unknown functions of the variable α which should be determined by using boundary conditions (1, 2).

3. Reduction of the problem to Fredholm integral equation

The considered mixed boundary value problem can be reduced to a integral equation. For this aim the following auxiliary problem determined by the boundary conditions

$$\sigma_{yy}(x, h) = -p(x), \qquad x \in R, \tag{5a}$$

$$\sigma_{xy}(x, h) = 0, \qquad x \in R; \tag{5b}$$

$$\sigma_{xy}(x, 0) = 0, \qquad x \in R; \tag{5c}$$

$$\sigma_{yy}(x, 0) - ku_y(x, 0) = f(x), \qquad x \in R, \tag{5d}$$

where $f(\cdot)$ is unknown integrable function, will be solved.

From boundary conditions (5) and equations (3) and (4) the following algebraic equations for unknowns A, B, C, D are obtained:

$$(A + B\alpha h)\sinh(\alpha h) + (C + D\alpha h)\cosh(\alpha h) = \alpha^{-1}\tilde{p}(\alpha), \tag{6a}$$

$$(A + B\alpha h + D)\cosh(\alpha h) + \\ + (C + D\alpha h + B)\sinh(\alpha h) = 0, \tag{6b}$$

$$A + D = 0, \tag{6c}$$

$$-\alpha C + \frac{k}{2\mu}(A - (1-2\nu)D) = \tilde{f}(\alpha), \tag{6d}$$

where

$$\tilde{p}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty p(x)\cos(\alpha x)dx, \tag{7a}$$

$$\tilde{f}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(\alpha x)dx. \tag{7b}$$

The solution of equations (6) takes the form:

$$A = -D, \tag{8a}$$

$$C = -\frac{\tilde{f}(\alpha)}{\alpha} - \frac{k^*}{\alpha h}D, \tag{8b}$$

$$B = -\frac{(C + D\alpha h)\sinh(\alpha h)}{\alpha h \cosh(\alpha h) + \sinh(\alpha h)}, \tag{8c}$$

$$D = \frac{\tilde{p}(\alpha)}{\alpha W(\alpha h)} + \frac{\tilde{f}(\alpha)}{\alpha} W_f(\alpha h), \tag{8d}$$

where

$$k^* = \frac{k(1-\nu)h}{\mu}, \tag{9a}$$

$$W(\alpha h) = \\ = \frac{(\alpha h)^2 - \sinh^2(\alpha h) - k^*(\alpha h)^{-1}[\sinh(\alpha h)\cosh(\alpha h) + \alpha h]}{\alpha h \cosh(\alpha h) + \sinh(\alpha h)}, \tag{9b}$$

$$W_f(\alpha h) = \\ = \frac{\sinh(\alpha h)\cosh(\alpha h) + \alpha h}{(\alpha h)^2 - \sinh^2(\alpha h) - k^*(\alpha h)^{-1}[\sinh(\alpha h)\cosh(\alpha h) + \alpha h]}. \tag{9c}$$

The parameter k^* given by (9a) is dimensionless. The vertical displacement $u_y(x, 0)$ can be calculated using equations (3b) and (8a):

$$u_y(x, 0) = \frac{1-\nu}{\mu} \sqrt{\frac{2}{\pi}} \int_0^\infty D(\alpha)\cos(\alpha x)d\alpha, \tag{10}$$

and from (8d) it follows that

$$u_y(x, 0) = \\ = \frac{1-\nu}{\mu} \sqrt{\frac{2}{\pi}} \int_0^\infty \left[\frac{\tilde{p}(\alpha)}{\alpha W(\alpha h)} + \frac{\tilde{f}(\alpha)}{\alpha} W_f(\alpha h)\right] \cos(\alpha x)d\alpha. \tag{11}$$

Taking into equations (5d) and (7b) as well as boundary conditions (2b) and (2c) the following relation is obtained:

$$\tilde{f}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty [\sigma_{yy}(t, 0) - k u_y(t, 0)]\cos(\alpha t)dt = \\ = -k\sqrt{\frac{2}{\pi}} \int_0^a u_y(t, 0)\cos(\alpha t)dt. \tag{12}$$

Substituting equation (12) into (11) the following integral equation is obtained

$$u_y(x, 0) + \\ + \frac{k^*}{h}\frac{2}{\pi} \int_0^\infty \alpha^{-1} W_f(\alpha h)\cos(\alpha x)d\alpha \int_0^a u_y(t, 0)\cos(\alpha t)dt = \\ = \frac{1-\nu}{\mu}\frac{2}{\pi} \int_0^\infty \frac{\cos(\alpha x)}{\alpha W(\alpha h)} d\alpha \int_0^\infty p(t)\cos(\alpha t)dt, \quad x \in R. \tag{13}$$

The relations (13) is the Fredholm integral equation of the second kind for $|x| < a$, for unknown vertical displacement $u_y(x, 0)$. On the other hand, knowing $u_y(x, 0)$ for $|x| < a$, the relation (13) permits to determine the vertical displacement $u_y(x, 0)$ for $|x| \geq a$. The Fredholm integral equation (13) for unknown $u_y(x, 0)$, $|x| < a$ will be solved numerically.

4. Numerical solution and analysis of the considered problem

For the further analysis the following special case of loadings on the upper boundary plane is assumed:

$$p(x) = p_0\sqrt{1 - (x/b)^2}\, H(b - |x|), \tag{14}$$

where b is given constant, $b > 0$ and $H(\cdot)$ is Heaviside'a step function. The considered problem is symmetric with respect of axis $0y$. Fourier transform of function $p(x)$ can be written in the form [19]

$$\frac{\tilde{p}(\alpha)}{p_0} = \sqrt{\frac{2}{\pi}} \int_0^b \sqrt{1 - (x/b)^2}\cos(\alpha x)dx = \sqrt{\frac{\pi}{2}}\frac{J_1(b\alpha)}{\alpha}, \tag{15}$$

where $J_1(b\alpha)$ is the Bessel function of first kind. Introducing the dimensionless parameters and functions

$$\xi = \frac{x}{a},\ \tau = \frac{t}{a},\ \alpha = a^{-1}s,\ h = aH,\ b = ab^*, \\ k^* = Hk_1^*,\ u_y(x, 0) = \frac{(1-\nu)p_0 a}{\mu} u_y^*(a\xi, 0), \tag{16}$$

and using (15) from Fredholm integral equation (13) it follows that

$$u_y^*(\xi,0)+k_1^*\frac{2}{\pi}\int_0^\infty s^{-1}W_f(sH)\cos(s\xi)ds\times$$
$$\times\int_0^1 u_y^*(\tau,0)\cos(s\tau)d\tau=\int_0^\infty\frac{\cos(s\xi)J_1(b^*s)}{s^2W(sH)}ds,\ \ \xi\in R. \tag{17}$$

Denoting by

$$K(\xi,\tau)=\frac{2}{\pi}\int_0^\infty s^{-1}W_f(sH)\cos(s\xi)\cos(s\tau)ds, \tag{18a}$$

$$F(\xi)=\int_0^\infty\frac{\cos(s\xi)J_1(b^*s)}{s^2W(sH)}ds, \tag{18b}$$

the Fredholm integral equation (17) for unknown $u_y^*(\xi,0)$ can be rewritten in the form:

$$u_y^*(\xi,0)+k_1^*\int_0^1 u_y^*(\tau,0)K(\xi,\tau)d\tau=F(\xi),\ \ \xi\in[0,1]. \tag{19}$$

Taking into account the asymptotic properties of function $W_f(sH)$ given by equation (9c):

$$\lim_{s\to\infty}W_f(sH)=-1, \tag{20}$$

it follows that kernel $K(\xi,\tau)$ of integral equation (19) is singular and has the singularity analogical to function $\ln|\xi-\tau|$. Because

$$\lim_{s\to 0}\frac{W_f(sH)}{s}=-\frac{1}{k_1^*}, \tag{21}$$

function $s^{-1}W_f(sH)$ can be written in the form:

$$\frac{W_f(sH)}{s}=-\frac{1}{s+k_1^*}+W_f^*(sH). \tag{22}$$

The above expression causes that kernel $K(\xi,\tau)$ can be presented in the form:

$$K(\xi,\tau)=-K_0(\xi,\tau)+K_1(\xi,\tau), \tag{23}$$

where

$$K_0(\xi,\tau)=\frac{2}{\pi}\int_0^\infty\frac{\cos(s\xi)\cos(s\tau)}{s+k_1^*}ds, \tag{24a}$$

$$K_1(\xi,\tau)=\frac{2}{\pi}\int_0^\infty W_f^*(sH)\cos(s\xi)\cos(s\tau)ds. \tag{24b}$$

The function $K_0(\xi,\tau)$ is singular part and $K_1(\xi,\tau)$ is regular part of the kernel $K(\xi,\tau)$.

The integral equation (19) will be solved numerically using the method of collocation. The collocation points ξ_i, $i=1,\dots,n$ are positive roots of equation

$$P_{2n}(\xi)=0, \tag{25}$$

where $P_{2n}(x)$ is Legendre polynomial of degree $2n$ [20].

Denoting by

$$u_y^*(\xi,0)=u_{yi},\ \ \xi\in(a_{i-1},a_i),\ \ i=1,\dots,n, \tag{26}$$

where

$$a_0=0,\ \ a_n=1,\ \ a_i=\frac{\xi_i+\xi_{i+1}}{2},\ \ i=1,\dots,n-1. \tag{27}$$

From equation (19) it follows that u_{yi} satisfy the system of linear algebraic equations:

$$\sum_{j=1}^n\left(\delta_{ij}-k_1^*M_{ij}^{(0)}+k_1^*M_{ij}^{(1)}\right)u_{yj}=F(\xi_i),\ \ i=1,\dots,n, \tag{28}$$

where δ_{ij} is Kronecker symbol, $M_{ij}^{(0)}$ and $M_{ij}^{(1)}$ are the matrixes obtained by application of numerical procedures for integrals which included singular and regular parts of the kernel:

$$\begin{aligned}
&\int_0^1 u_y^*(\tau,0)K_0(\xi_i,\tau)d\tau=\\
&=\frac{2}{\pi}\int_0^1 u_y^*(\tau,0)d\tau\int_0^\infty\frac{\cos(s\xi_i)\cos(s\tau)}{s+k_1^*}ds=\\
&=\frac{2}{\pi}\sum_{j=1}^n u_{yj}\int_{a_{j-1}}^{a_j}d\tau\int_0^\infty\frac{\cos(s\xi_i)\cos(s\tau)}{s+k_1^*}ds=\\
&=\frac{2}{\pi}\sum_{j=1}^n u_{yj}\int_0^\infty\frac{\cos(s\xi_i)}{s+k_1^*}ds\int_{a_{j-1}}^{a_j}\cos(s\tau)d\tau=\\
&=\frac{2}{\pi}\sum_{j=1}^n u_{yj}\int_0^\infty\frac{\cos(s\xi_i)\left(\sin(sa_j)-\sin(sa_{j-1})\right)}{s(s+k_1^*)}ds=\\
&=\sum_{j=1}^n M_{ij}^{(0)}u_{yj};
\end{aligned} \tag{29a}$$

$$\begin{aligned}
&\int_0^1 u_y^*(\tau,0)K_1(\xi_i,\tau)d\tau=\\
&=\frac{2}{\pi}\int_0^1 u_y^*(\tau,0)d\tau\int_0^\infty W_f^*(s,H)\cos(s\xi_i)\cos(s\tau)ds=\\
&=\frac{2}{\pi}\sum_{j=1}^n u_{yj}w_j\int_0^\infty W_f^*(s,H)\cos(s\xi_i)\cos(s\tau_j)ds=\\
&=\sum_{j=1}^n M_{ij}^{(1)}u_{yj},
\end{aligned} \tag{29b}$$

where w_j, $j=1,\dots,n$ are weights of Gauss quadrature [20]. The improper integrals in equations (18b) and (29) have been calculated numerically by using Gauss quadrature.

Knowing a solution of equations (28) and parameters u_{yi} the vertical displacement $u_y(\xi, 0)$, $\xi > 1$ is calculated in the analogical procedure.

The stress components $\sigma_{xy}(\xi, 0)$ and $\sigma_{yy}(\xi, 0)$ are well-known from boundary conditions (2a–2c). The stress component $\sigma_{xx}(\xi, 0)$ can be calculated on the basis (4b) with (4c):

$$\sigma_{xx}(x,0) = 2\sqrt{\frac{2}{\pi}}\int_0^\infty \alpha\big(C(\alpha)+B(\alpha)\big)\cos(\alpha x)\,d\alpha + \sigma_{yy}(x,0), \tag{30}$$

$$\sigma_{yy}(x,0) = -\sqrt{\frac{2}{\pi}}\int_0^\infty \alpha C\cos(\alpha x)\,d\alpha,$$

Substituting equations (8) into (30) it leads

$$\sigma_{xx}(x,0) = -2\sqrt{\frac{2}{\pi}}\int_0^\infty \big(\tilde{f}(\alpha)W_{\sigma f}(\alpha h) + \tilde{p}(\alpha)W_{\sigma p}(\alpha h)\big)\cos(\alpha x)\,d\alpha + \sigma_{yy}(x,0). \tag{31}$$

where

$$W_{\sigma f}(\alpha h) = \frac{\alpha h\cosh(\alpha h)}{\alpha h\cosh(\alpha h)+\sinh(\alpha h)} + \frac{k^*\cosh(\alpha h)+\alpha h\sinh(\alpha h)}{\alpha h\cosh(\alpha h)+\sinh(\alpha h)}\cdot W_f(\alpha h), \tag{32a}$$

$$W_{\sigma p}(\alpha h) = \frac{k^*\cosh(\alpha h)+\alpha h\sinh(\alpha h)}{(\alpha h)^2-\sinh^2(\alpha h)-k^*(\alpha h)^{-1}[\sinh(\alpha h)\cosh(\alpha h)+\alpha h]}. \tag{32b}$$

Substituting equation (12) into (31) and applying dimensionless parameters and functions (16) it follows that:

$$\sigma^*_{xx}(\xi,0) = \frac{4k_1^*}{\pi}\int_0^1 u_y^*(\tau,0)\,d\tau\int_0^\infty W_{\sigma f}(sH)\cos(s\xi)\cos(s\tau)\,ds - 2\int_0^\infty s^{-1}W_{\sigma p}(sH)J_1(b^*s)\cos(s\xi)\,ds + \sigma^*_{yy}(\xi,0). \tag{33}$$

where

$$\sigma_{xx}(x,0) = p_0\sigma^*_{xx}(a\xi,0),\quad \sigma_{yy}(x,0) = p_0\sigma^*_{yy}(a\xi,0). \tag{34}$$

An analysis of functions $W_{\sigma f}$ and $W_{\sigma p}$ shown their relatively fast convergence to zero for $s\to\infty$, the integrals in equation (33) are calculated by using Gauss quadrature.

The numerical analysis of dimensionless displacement $u_y^*(\xi, 0)$ and dimensionless stress $\sigma^*_{xx}(\xi, 0)$ $\sigma^*_{yy}(\xi, 0)$ are dependent on three dimensionless parameters: k_1^* (or k^*), H and b^*. The following values of the parameters are taken into account: $k_1^* = 1, 2, 4$; $H = 0.5, 1$; $b^* = 2$. Moreover, the parameter of numerical method n is assumed as $n = 10, 20$.

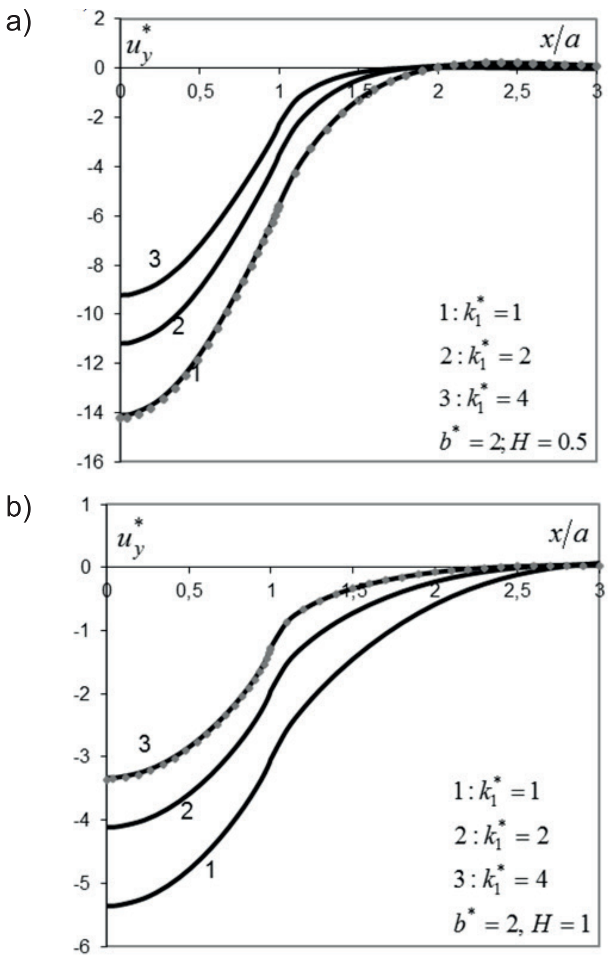

Fig. 2. The distribution of displacement on the boundary surface $y = 0$ where countinous line $n = 10$; rhomboids: $n = 20$

Figures 2a and 2b present the dimensionless normal displacement $u_y^*(\xi, 0)$ as functions of $\xi = x/a$, (where a is the half of emptiness dimension).

Figure 2a shows the deflection of lower boundary plane of elastic stratum represented by displacement $u_y^*(\xi, 0)$ for the case of elastic layer with thickness $h = 0.5a$ and parameter k_1^* (connected with Winkler parameter k, Poisson ratio ν, shear modulus μ and half of emptiness width a, $k_1^* = k^*/H = k(1-\nu)a/\mu$) $k_1^* = 1; 2; 4$, as well as parameter b^*=2, ($b^* = b/a$ is dimensionless half of the width of loading region on the upper boundary plane). Figure 2b presents dimensionless displacement $u_y^*(\xi, 0)$ for $k_1^* = 1; 2; 4$, $b^* = 2$ and twice time thick elastic stratum then in Fig. 2a, namely for $H = 1$. It can be observed according with an intuition that the deflections of lower boundary are greater for thinner stratum. However, the segment in which the deflection of the lower boundary from the axis $\xi = x/a$ for $\xi > 1$, is longer for thicker stratum. In Figures 2a and 2b the curves represented by continuous lines were calculated for $n = 10$, the

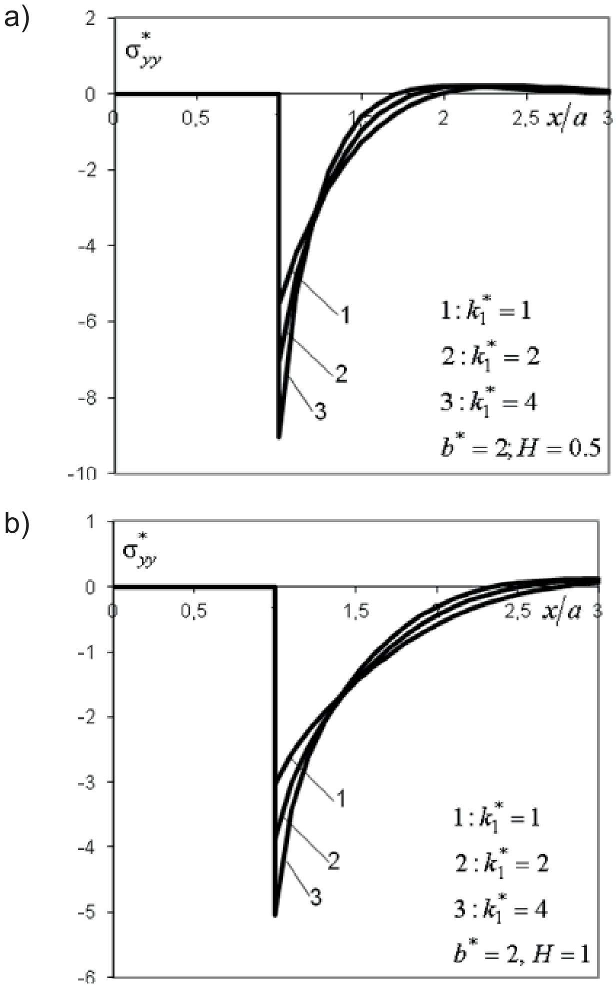

Fig. 3. The distribution of stress tensor component σ_{yy} on the boundary surface $y = 0$

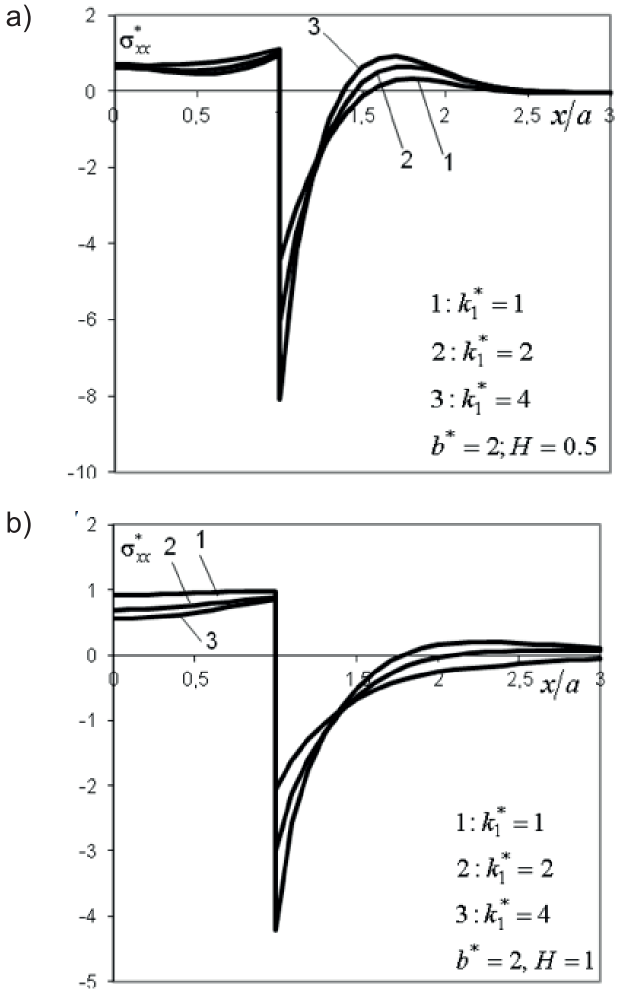

Fig. 4. The distribution of stress tensor component σ_{xx} on the boundary surface $y = 0$

rhomboids are for $n = 20$, so for the calculations the number of collocations points $n = 10$ is enough.

Figures 3a and 3b show the dimensionless normal stress component $\sigma_{yy}^*(\xi, 0)$ on the lower boundary of stratum. According with boundary condition (2b) $\sigma_{yy}^*(\xi, 0) = 0$ for $0 \leq \xi = x/a < 1$, and $\xi = 1$ is the point of discontinuity. The smallest values of $\sigma_{yy}^*(\xi, 0)$ are achieved for $\xi \to 1^+$ and for $k_1^* = 4$ in the both figures. The curves represented values of $\sigma_{yy}^*(\xi, 0)$ intersect in the point which is dependent on the thickness of the elastic layer.

In Figures 4a and 4b the dimensionless stress component $\sigma_{xx}^*(\xi, 0)$ as functions of ξ are presented. For $0 \leq \xi < 1$ the component takes positive values, $\xi = 1$ is the point of discontinuity. The smallest values of $\sigma_{xx}^*(\xi, 0)$ are achieved for $\xi \to 1^+$. Analogically to $\sigma_{yy}^*(\xi, 0)$ the curves represented of $\sigma_{xx}^*(\xi, 0)$ intersect for $\xi \approx 1.2$ and tend to zero for $\xi \to \infty$.

5. Final remarks

The problem of stress distributions in the elastic layer resting on the Winkler foundation with emptiness was considered. The mixed boundary conditions on the lower boundary of the layer caused necessity of solution of Fredholm integral equation of the second kind (13), which was derived for any cases of normal symmetric with respect of axis $0y$, loadings on the upper boundary plane of the layer. The equation (13) was solved numerically under assumption that the intensity of the boundary loading is elliptic. It seems that the distributions of normal displacements and stresses on the lower boundary are important, the numerical analysis has been concentrated on these aims. The obtained results are presented in the form of figures, which showed among other things the deflection of the elastic layer caused by the emptiness. The knowledge of normal displacement above the emptiness or normal stress component on the lower boundary without the emptiness allows for reducing the problem of finding the displacements and stress in the inside of layer.

The obtained stress components σ_{xx} and σ_{yy} are non-singular (but discontinuous) at the edge of the emptiness. It is caused by the form of the assumed Winkler type of boundary conditions (the component σ_{yy} is proportional to the u_y for $|x| \geq a, y = 0$).

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References

[1] A. Birinci and R. Erdol, "A frictionless contact problem for two elastic layers supported by a Winkler foundation", *Structural Engineering and Mechanics*, 15, 331–344 (2003).
[2] J. Marzęda, V. Pauk, and M. Woźniak, "Contact of a rigid flat punch with a wedge supported by the Winkler foundation", *Journal of Theoretical and Applied Mechanics* 39, 563–575 (2001).
[3] J.P. Dempsey, Z.G. Zhao, and H. Li, "Axisymmetric indentation of an elastic layer supported by a Winkler foundation", *International Journal of Solids and Structures*, 27, 73–87 (1991).
[4] J.P. Dempsey, Z.G. Zhao, L. Minnetyan, and H.Li, "Plane contact of an elastic layer supported by a Winkler foundation", *Journal of Applied Mechanics*, 57, 974–980 (1991).
[5] M. Woźniak, A. Hummel, and V.J. Pauk, "Axisymmetric contact problems for an elastic layer resting on a rigid base with a Winkler type excavitation", *International Journal of Solids and Structures*, 39, 4117–4131 (2002).
[6] S.J. Matysiak and V.J. Pauk, "Edge crack in an elastic layer resting on Winkler foundation", *Engineering Fracture Mechanics* 70, 2353–2361 (2003).
[7] F.L. Liu, "Rectangular thick plates on Winkler foundation: differential quadrature element solution", *International Journal of Solids and Structures*, 37, 1743–1763 (2000).
[8] A. Karasin and G. Alitas, "An approximate solution for plates resting on Winkler foundation", *International Journal of Civil Engineering and Technology* 5, 114–124 (2014).
[9] H. Kobayashi and K. Sonoda, "Rectangular Midlin plates on elastic foundations", *International Journal of Mechanical Sciences*, 31, 679–692 (1989).
[10] Y. Kumar and R. Lai, "Vibrations of nonhomogenous orthotropic rectangular plates with bilinear thickness variation resting on Winkler foundation", *Meccanica*, 47, 893–915 (2012).
[11] A. Daloglu and C. Vallabhan, "Values of k for slab on Winkler foundation", *Journal of Geotechnical and Geoenvironmental Engineering*, 126, 463–471 (2000).
[12] S.Ch. Datta and R. Roy, "A critical review on idealization and modelling for interaction among soil-foundation structure system", *Computers and Structures*, 80, 1579–1594 (2002).
[13] A.P.S. Selvadurai, "Elastic Analysis of Soil-Foundation Interaction", *Developments in Geotechnical Engineering*, Vol. 17, Elsevier, Amsterdam (1979).
[14] E. Tsudik, *Analysis of Structures on Elastic Foundation*, J. Ross Publ., Plantation, FL. (2013).
[15] I. Cömez, "Contact problem of a functionally graded layer resting on a Winkler foundation", *Acta Mechanica*, 224, 2833–2843 (2013).
[16] S.J. Matysiak and V.J. Pauk, "Some problems of stress distribution in a periodic stratified elastic stratum", *Journal of Theoretical and Applied Mechanics*, 21, 795–811 (1993).
[17] S.J. Matysiak and C. Woźniak, "Micromorphic effects in a modelling of periodic multilayered elastic composites", *International Journal of Engineering Science*, 25, 549–559 (1987).
[18] S.J. Matysiak and V.J. Pauk, "On crack problem in an elastic ponderable layer", *International Journal of Fracture*, 96, 371–380 (1999).
[19] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, 1980.
[20] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.