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PROBABILISTIC ANALYSIS OF TWO RELIABILITY MODELS OF A SINGLE-UNIT SYSTEM WITH PREVENTIVE MAINTENANCE BEYOND WARRANTY AND DEGRADATION

ANALIZA PROBABILISTYCZNA DWÓCH MODELI NIEZAWODNOŚCI SYSTEMU JEDNOELEMENTOWEGO WYKORZYSTUJĄCYCH POJĘCIA POGWARANCYJNEJ OBSŁUGI PROFILAKTYCZNEJ ORAZ DEGRADACJI

This paper presents two reliability models of a single-unit system with the concept of preventive maintenance (PM) beyond warranty and degradation. In both the models, repair of any failure during warranty is cost-free to the users, provided failures are not due to the negligence of users. There is a single repairman who always remains with the system. Beyond warranty, the unit goes under PM and works as new after PM (in both models). In model-1, the unit works as new after its repair beyond warranty whereas; in model-2, the unit becomes degraded. After failure, the degraded unit is replaced by a new one. The failure time of the system follows negative exponential distribution while PM, replacement and repair time distributions are taken as arbitrary with different probability density functions. Supplementary variable technique is adopted to derive the expressions for some economic measures such as reliability, mean time to system failure (MTSF), availability and profit function. Using Abel's lemma, the behaviour of the system in steady-state has been examined. To highlight the behaviour of reliability and profit function, numerical results are considered for particular values of various parameters and repair cost. Profit comparison of both the models is also made to see the usefulness of the concept of degradation.

Keywords: *probabilistic analysis, reliability, preventive maintenance, warranty, degradation.*

W artykule przedstawiono dwa modele niezawodności systemu jednoelementowego wykorzystujące pojęcia pogwarancyjnej obsługi profilaktycznej oraz degradacji. Oba modele zakładają, że w okresie gwarancyjnym użytkownik nie ponosi żadnych kosztów związanych z naprawą uszkodzeń, chyba że uszkodzenie powstało wskutek zaniedbania ze strony użytkownika. Obsługi są wykonywane przez jedną ekipę remontową, która zawsze pozostaje na stanowisku. Po upływie okresu gwarancyjnego, urządzenie podlega obsłudze profilaktycznej i po jej przeprowadzeniu działa jak nowe (w obu modelach). Model 1 zakłada, że element po naprawie pogwarancyjnej działa jak nowy, natomiast w Modelu 2, element ulega degradacji. Zdegradowany element, który uległ uszkodzeniu, zostaje wymieniony na nowy. Rozkład czasu uszkodzenia jest rozkładem wykładniczym ujemnym, a rozkłady czasu obsługi profilaktycznej, wymiany i naprawy są traktowane jako arbitralne, o różnych funkcjach gęstości prawdopodobieństwa. Zastosowana technika dodatkowej zmiennej pozwoliła na wyprowadzenie wyrażeń dla niektórych miar ekonomicznych, takich jak niezawodność, średni czas do uszkodzenia systemu (MTSF), gotowość i funkcja zysków. Zachowanie systemu w stanie ustalonym badano z wykorzystaniem lematu Abela. Aby przedstawić zachowanie funkcji niezawodności i zysków, analizowano wyniki numeryczne dla poszczególnych wartości różnych parametrów oraz kosztów naprawy. Porównanie zyskowności badanych modeli umożliwiło weryfikację przydatności pojęcia degradacji.

Słowa kluczowe: *analiza probabilistyczna, niezawodność, obsługa profilaktyczna, gwarancja, degradacja.*

1. Introduction

In modern marketplace, warranty has its own priority in business for manufacturers to protect their benefits and to compete with other manufacturers. By using regenerating point and semi-Markov technique, various researchers including Kadyan et al. [3], Yang and Dhillon [6], Perez Ocon and Ruiz Castro [8], Philip and Cristiano [9] and Yuan and Meng [11] have studied reliability models of one or more unit systems under different sets of assumptions on failure and repair policies. When the failure rate or repair rate or both are time-dependent, the system loses its Markov character and becomes non-Markovian. By introducing one or more supplementary variables, the non-Markovian nature of the process is changed to Markovian. Firstly, Cox [1] used supplementary variable technique in analyzing non-Markovian stochastic process. Singh et al. [10] studied a system having two units in series configuration with controller and Nailwal and Singh [7] analyzed an operating system with inspection in different weather condition by using supplementary variable technique without considering the concept of cost-free warranty. But, warranty assured the customers that the products they are buying perform satisfactorily for a particular period of time and markets the product.

Also, performing PM has become prevalent to improve the condition of the deteriorated product (or system) and reduce the cost of repairing deteriorated product. Kadyan [2] discussed reliability and profit analysis of a single-unit system with preventive maintenance without considering degradation of the unit after its repair.

However, the failed unit does not always work as new after its repair. Due to continuous usage and ageing effect, failure rate of a unit may increase after its repair. In such a situation, unit works with reduced capacity after its repair and so is called a degraded unit. Kumar et al. [4,5] analysed redundant systems with degradation of the unit after repair without any warranty.

In view of the above observations, here we developed two reliability models of a single-unit system with the concept of PM beyond warranty and degradation. In both the models, repair of any failure during warranty is cost-free to the users, provided failures are not due to the negligence of users such as cracked screen, accident, misuse, physical damage, damage due to liquid and unauthorized modifications etc. There is a single repairman, who always remains with the system. Beyond warranty, the unit goes under PM and works as new after PM (in both the models). In model-1, the unit works as new after its repair beyond warranty whereas; in model-2, the unit becomes degraded. In model-2, the degraded unit is replaced by a new one after its failure. The failure time of the system follows negative exponential distribution while PM, replacement and repair time distributions are taken as arbitrary with different probability density functions. Supplementary variable technique is adopted to derive the expressions for some economic measures such as reliability, MTSF, availability and profit function. Using Abel's lemma ([6] & [9]), the behaviour of the system in steady-state has been examined. To highlight the behaviour of reliability and profit function, numerical results are also considered for particular values of various parameters and repair cost. Profit comparison of both models is made to see the usefulness of the concept of degradation.

2. Notations

λ / λ_1 Constant failure rate of the new unit within/beyond warranty.

λ_2 Constant failure rate of the degraded unit beyond warranty.

λ_m Transition rate with which a unit goes under PM for improvement.

α Transition rate with which warranty of the system is completed.

$\mu(x), S(x) / \mu_1(x), S_1(x)$ Repair rate of the unit and probability density function, for the elapsed repair time x within/ beyond warranty.

$\mu_2(y), S_2(y)$ PM rate of the unit and probability density function, for the elapsed PM time y.

$\mu_3(z), S_3(z)$ Replacement rate of the failed degraded unit and probability density function, for the elapsed replacement time z.

$p_0(t) / p_1(t)$ Probability density that at time t, the system is within/ beyond warranty and in good state.

$p_i(x,t)$ Probability density that at time t, the system is in state S_i, $i=2,4$ and the system is under repair with elapsed repair time x.

$p_3(y,t)$ Probability density that at time t, the system is in state S_3 and the unit is under PM with elapsed PM time y.

$p_5(t)$ Probability density that at time t, the system is operable and in degraded state.

$p_6(z,t)$ Probability density that at time t, the system is in state S6 and the failed degraded unit is under replacement with elapsed replacement time z.

$p(s)$ Laplace transform of function $p(t)$

$$S(x) = \mu(x)\, e^{\left[-\int_0^x \mu(x)dx\right]}$$

$$S_1(x) = \mu_1(x)\, e^{\left[-\int_0^x \mu_1(x)dx\right]}$$

$$S_2(y) = \mu_2(y)\, e^{\left[-\int_0^y \mu_2(y)dy\right]}$$

$$S_3(z) = \mu_3(z)\, e^{\left[-\int_0^z \mu_3(z)dz\right]}$$

3. State-Specification

The following states of the system are common for both models:

S_0 / S_1 The new unit is operative within/ beyond warranty.

S_2 / S_4 The new unit is in failed state within/ beyond warranty.

S_3 The new unit is under PM beyond warranty.

The remaining states for model-2 are:

S_5 The degraded unit is operative beyond warranty.

S_6 The failed degraded unit is under replacement beyond warranty.

4. Formulation of mathematical model-1

Using the probabilistic arguments and limiting transitions, we have the following difference-differential equations:

$$\left[\frac{d}{dt}+\lambda+\alpha\right]p_0(t)=\int_0^\infty \mu(x)p_2(x,t)dx \quad (1)$$

$$\left[\frac{d}{dt}+\lambda_1+\lambda_m\right]p_1(t)=\alpha\, p_0(t)+\int_0^\infty \mu_1(x)p_4(x,t)dx+\int_0^\infty \mu_2(y)p_3(y,t)dy \quad (2)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu(x)\right]p_2(x,t)=0 \quad (3)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_2(y)\right]p_3(y,t)=0 \quad (4)$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)\right]p_4(x,t)=0 \quad (5)$$

Boundary Conditions

$$p_2(0,t)=\lambda\, p_0(t) \quad (6)$$

$$p_3(0,t)=\lambda_m p_1(t) \quad (7)$$

$$p_4(0,t) = \lambda_1 p_1(t) \quad (8)$$

Initial conditions

$p_i(0) = 1;$ when $i = 0$

$$p_i(0)=0; \text{ when } i \neq 0 \quad (9)$$

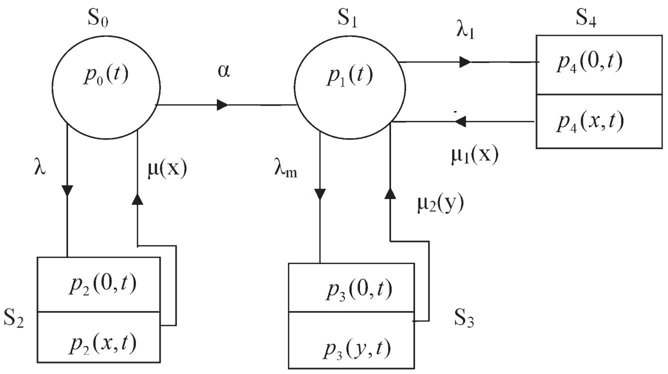

Fig. 1. State transition diagram of the model-1

5 Analysis for model-1

5.1. Solution of the equations

Taking Laplace transforms of equations (1)-(8) and using (9), we obtain:

$$\left[s+\lambda+\alpha\right]p_0(s) = 1+\int_0^\infty \mu(x) p_2(x,s)dx \quad (10)$$

$$\left[s+\lambda_1+\lambda_m\right]p_1(s) = \alpha\, p_0(s)+\int_0^\infty \mu_1(x)p_4(x,s)dx+\int_0^\infty \mu_2(y)p_3(y,s)dy \quad (11)$$

$$\left[\frac{\partial}{\partial x}+s+\mu(x)\right]p_2(x,s)=0 \quad (12)$$

$$\left[\frac{\partial}{\partial y}+s+\mu_2(y)\right]p_3(y,s) \quad (13)$$

$$\left[\frac{\partial}{\partial x}+s+\mu_1(x)\right]p_4(x,s)=0 \quad (14)$$

$$p_2(0,s) = \lambda p_0(s) \quad (15)$$

$$p_3(0,s) = \lambda_m p_1(s) \quad (16)$$

$$p_4(0,s) = \lambda_1 p_1(s) \quad (17)$$

Taking integration of equations (12), (13) and (14), we get the following equations:

$$p_2(x,t) = p_2(0,t)\,\mathrm{e}^{\left[-sx-\int_0^x \mu(x)dx\right]} \quad (18)$$

$$p_3(y,t) = p_3(0,t)\,\mathrm{e}^{\left[-sy-\int_0^y \mu_2(y)dy\right]} \quad (19)$$

and

$$p_4(x,t) = p_4(0,t)\,\mathrm{e}^{\left[-sx-\int_0^x \mu_1(x)dx\right]} \quad (20)$$

Using equations (15) and (18), equation (10) yields:

$$\left[s+\lambda+\alpha\right]p_0(s) = 1+p_2(0,t)\int_0^\infty \mu(x)\,\mathrm{e}^{\left[-sx-\int_0^x \mu(x)dx\right]}dx = 1 = \lambda p_0(s)S(s)$$

$$p_0(s) = \frac{1}{T(s)} \quad (21)$$

where $T(s) = s+\alpha+\lambda\left(1-S(s)\right)$ (22)

Using equations (16), (17), (19) and (20), equation (11) yields:

$$\left[s+\lambda_1+\lambda_m\right]p_1(s) = \alpha\, p_0(s)+p_4(0,t)\int_0^\infty \mu_1(x)\,\mathrm{e}^{\left[-sx-\int_0^x \mu_1(x)dx\right]}dx+p_3(0,t)\int_0^\infty \mu_2(y)\,\mathrm{e}^{\left[-sy-\int_0^y \mu_2(y)dy\right]}dy$$

$$= \alpha\, p_0(s)+\lambda_1 p_1(s)S_1(s)+\lambda_m p_1(s)S_2(s)$$

$$p_1(s) = \frac{A(s)}{T(s)} \quad (23)$$

Where $A(s) = \dfrac{\alpha}{\left(s+\lambda_1-\lambda_1 S_1(s)-\lambda_m S_2(s)\right)}$ (24)

Now, the Laplace transform of the probability that the system is in the failed state is given by:

$$p_2(s) = \int_0^\infty p_2(s,x)dx = \lambda p_0(s)\frac{\left(1-S(s)\right)}{s}$$

$$p_2(s) = \frac{\lambda B(s)}{T(s)} \quad (25)$$

Where $B(s) = \dfrac{\left(1-S(s)\right)}{s}$ (26)

Similarly, $p_3(s) = \displaystyle\int_0^\infty p_3(s,y)dy = \lambda_m p_1(s)\frac{\left(1-S_2(s)\right)}{s}$

$$p_3(s) = \frac{\left(\lambda_m A(s)C(s)\right)}{T(s)} \quad (27)$$

Where $C(s) = \dfrac{\left(1-S_2(s)\right)}{s}$ (28)

Similarly, $p_4(s) = \displaystyle\int_0^\infty p_4(s,x)dx = \lambda_1 p_1(s)\frac{\left(1-S_1(s)\right)}{s}$

$$p_4(s)=\frac{\left(\lambda_1 A(s)D(s)\right)}{T(s)} \tag{29}$$

where $D(s)=\frac{\left(1-S_1(s)\right)}{s}$ (30)

It is worth noticing that:

$$p_0(s)+p_1(s)+p_2(s)+p_3(s)+p_4(s)=\frac{1}{s} \tag{31}$$

5.2. Evaluation of Laplace Transforms of Up and Down state probabilities

The Laplace transforms of the probabilities that the system is in Up State ($p_{up}(t)$) (i.e. Good State) and Down State ($p_{down}(t)$) (i.e. Failed State) at time t are as follows:

$$p_{up}(s)=p_0(s)+p_1(s)$$

$$p_{up}(s)=\frac{\left(1+A(s)\right)}{T(s)} \tag{32}$$

$$p_{down}(s)=p_2(s)+p_3(s)+p_4(s)$$

$$p_{down}(s)=\frac{\left(\lambda B(s)+\lambda_m A(s)C(s)+\lambda_1 A(s)D(s)\right)}{T(s)} \tag{33}$$

5.3. Steady-State behavior of the system

Using Abel's Lemma ([6] & [9]) i.e. $\lim_{t\to\infty} F(t)=\lim_{s\to 0} sF(s)=F$ in equations (32) and (33), Provided the limit on the right hand side exists, the following time independent probabilities have been obtained:

$$P_{up}=\frac{1}{\left(1-\lambda_1 S_1'(0)-\lambda_m S_2'(0)\right)} \tag{34}$$

$$P_{down}=\frac{-\lambda_1 S_1'(0)-\lambda_m S_2'(0)}{\left(1-\lambda_1 S_1'(0)-\lambda_m S_2'(0)\right)} \tag{35}$$

5.4. Reliability of the system (R(t))

The differential–difference equations for reliability of the system are:

$$\left[\frac{d}{dt}+\lambda+\alpha\right]p_0(t)=0 \tag{36}$$

$$\left[\frac{d}{dt}+\lambda_1+\lambda_m\right]p_1(t)=\alpha\, p_0(t) \tag{37}$$

Taking Laplace transform of equations (36) and (37), using (9), we get:

$$\left[s+\lambda+\alpha\right]p_0(s)=1 \tag{38}$$

$$\left[s+\lambda_1+\lambda_m\right]p_1(s)=\alpha\, p_0(s) \tag{39}$$

The solution can be written as:

$$p_0(s)=\frac{1}{\left(s+\alpha+\lambda\right)} \tag{40}$$

$$p_1(s)=\frac{\alpha}{\left(s+\alpha+\lambda\right)\left(s+\lambda_1+\lambda_m\right)}$$

$$R(s)=p_0(s)+p_1(s) \tag{41}$$

$$=\frac{1}{\left(s+\alpha+\lambda\right)}+\frac{\alpha}{\left(s+\alpha+\lambda\right)\left(s+\lambda_1+\lambda_m\right)}$$

Taking inverse Laplace transform, we get:

$$R(t)=e^{-(\lambda+\alpha)t}\left[\frac{\left(\lambda-\lambda_1-\lambda_m\right)}{\left(\lambda-\lambda_1-\lambda_m+\alpha\right)}\right]+e^{-(\lambda_1+\lambda_m)t}\left[\frac{\alpha}{\left(\lambda-\lambda_1-\lambda_m+\alpha\right)}\right] \tag{42}$$

5.5. Mean time to system failure (MTSF)

$$MTSF=\int_0^\infty R(t)dt$$

$$MTSF=\int_0^\infty\left\{e^{-(\lambda+\alpha)t}\left[\frac{\left(\lambda-\lambda_1-\lambda_m\right)}{\left(\lambda-\lambda_1-\lambda_m+\alpha\right)}\right]+e^{-(\lambda_1+\lambda_m)t}\left[\frac{\alpha}{\left(\lambda-\lambda_1-\lambda_m+\alpha\right)}\right]\right\}dt$$

$$MTSF=\left[\frac{\left(\lambda-\lambda_1-\lambda_m\right)}{\left(\lambda-\lambda_1-\lambda_m+\alpha\right)\left(\lambda+\alpha\right)}\right]+\left[\frac{\alpha}{\left(\lambda-\lambda_1-\lambda_m+\alpha\right)\left(\lambda_1+\lambda_m\right)}\right] \tag{43}$$

6. Formulation of mathematical model-2

Equations (1), (3), (4) and (5) defined in model-1 are same for model-2 and remaining equations for model-2 are:

$$\left[\frac{d}{dt}+\lambda_1+\lambda_m\right]p_1(t)=\alpha\, p_0(t)+\int_0^\infty \mu_3(z)p_6(z,t)dx+\int_0^\infty \mu_2(y)p_3(y,t)dy \tag{44}$$

$$\left[\frac{d}{dt}+\lambda_2\right]p_5(t)=\int_0^\infty \mu_1(x)p_4(x,t)dx \tag{45}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+\mu_3(z)\right]p_6(z,t)=0 \tag{46}$$

Boundary Conditions

Boundary $p_2(0,t), p_3(0,t)$ and $p_4(0,t)$ are same as defined in model-1 and remaining is:

$$p_6(0,t)=\lambda_2 p_5(t) \tag{47}$$

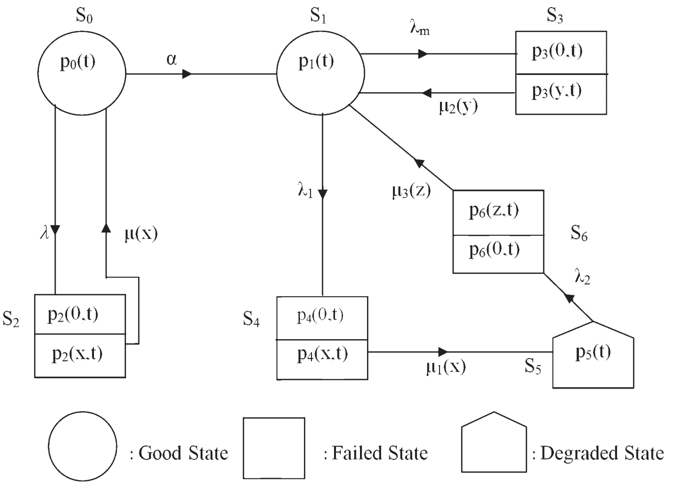

Fig. 2. Transition diagram of the model-2

7. Analysis for model-2

7.1. Solution of the equations

Proceeding in similar way as in model-1, the expressions for $p_0(s)$, $p_2(s), p_3(s)$ and $p_4(s)$ are same as defined in model-1 and remaining expressions are:

$$p_1(s) = \frac{A(s)}{T(s)} \tag{48}$$

Where, $A(s) = \dfrac{\alpha(s+\lambda_2)}{(s+\lambda_1+\lambda_m)(s+\lambda_2) - \lambda_m S_2(s)(s+\lambda_2) - \lambda_1 S_1(s) S_3(s)}$ (49)

And $T(s)$ is same as defined in model-1.

$$p_5(s) = \frac{\lambda_1 A(s) S_1(s)}{T(s)(s+\lambda_2)} \tag{50}$$

Now,

$$p_6(s) = \int_0^\infty p_6(s,z)dz = \lambda_2 p_5(s)\frac{(1-S_3(s))}{s}$$

$$p_6(s) = \frac{(\lambda_2\lambda_1 A(s) S_1(s) E(s))}{T(s)(s+\lambda_2)} \tag{51}$$

where $E(s) = \dfrac{(1-S_3(s))}{s}$ (52)

It is worth noticing that:

$$p_0(s) + p_1(s) + p_2(s) + p_3(s) + p_4(s) + p_5(s) + p_6(s) = \frac{1}{s} \tag{53}$$

7.2. Evaluation of Laplace Transforms of Up and Down state probabilities

$$p_{up}(s) = p_0(s) + p_1(s) + p_5(s)$$

$$p_{up}(s) = \frac{\left(1 + A(s) + \dfrac{\lambda_1 S_1(s) A(s)}{(s+\lambda_2)}\right)}{T(s)} \tag{54}$$

$$p_{down}(s) = p_2(s) + p_3(s) + p_4(s) + p_6(s)$$

$$p_{down}(s) = \frac{\left(\lambda B(s) + \lambda_m C(s) A(s) + \lambda_1 A(s) D(s) + \left(\dfrac{\lambda_1\lambda_2 S_1(s) A(s) E(s)}{(s+\lambda_2)}\right)\right)}{T(s)} \tag{55}$$

7.3 Steady-State behavior of the system

$$p_{up} = \frac{(\lambda_1+\lambda_2)}{\left(\lambda_1+\lambda_2-\lambda_2\lambda_m S_2^{'}(0) - \lambda_1\lambda_2 S_1^{'}(0) - \lambda_1\lambda_2 S_3^{'}(0)\right)} \tag{56}$$

$$p_{down} = \frac{-\lambda_2\lambda_m S_2^{'}(0) - \lambda_1\lambda_2 S_1^{'}(0) - \lambda_1\lambda_2 S_3^{'}(0)}{\left(\lambda_1+\lambda_2-\lambda_2\lambda_m S_2^{'}(0) - \lambda_1\lambda_2 S_1^{'}(0) - \lambda_1\lambda_2 S_3^{'}(0)\right)} \tag{57}$$

7.4. Reliability and mean time to system failure (MTSF)

Reliability and MTSF of this model is same as that of the model-1.

8. Special cases

8.1. Availability of the system for model-1

When repair and PM times follow exponential distribution i.e. $S(s) = \dfrac{\mu}{(s+\mu)}$, $S_1(s) = \dfrac{\mu_1}{(s+\mu_1)}$ and $S_2(s) = \dfrac{\mu_2}{(s+\mu_2)}$

where μ and μ_1 are constant repair rates and μ_2 is constant PM rate. Putting these values in equations (21)-(24), we get:

$$p_0(s) = \frac{1}{I(s)} \tag{58}$$

Where $I(s) = \dfrac{\left(s^2 + s(\mu+\lambda+\alpha) + \alpha\mu\right)}{(s+\mu)}$ (59)

$$p_1(s) = \frac{F(s)}{I(s)} \tag{60}$$

Where

$$F(s) = \frac{\alpha(s+\mu_1)(s+\mu_2)}{(s+\lambda_1+\lambda_m)(s+\mu_1)(s+\mu_2) - \lambda_1\mu_1(s+\mu_2) - \lambda_m\mu_2(s+\mu_1)} \tag{61}$$

$$p_{up}(s) = p_0(s) + p_1(s)$$

$$\frac{\left(s^3 + s^2\left(\lambda_1 + \lambda_m + \alpha + \mu_1 + \mu_2\right) + s\left(\lambda_1\mu_2 + \mu_1\mu_2 + \alpha\mu_2 + \lambda_m\mu_1 + \alpha\mu_1\right) + \alpha\mu_1\mu_2\right)\left(s + \mu_1\right)}{s\left(s^2 + s\left(\lambda + \alpha + \mu\right) + \alpha\mu\right)\left(s^2 + s\left(\lambda_1 + \lambda_m + \mu_1 + \mu_2\right) + \left(\lambda_1\mu_2 + \mu_1\mu_2 + \lambda_m\mu_1\right)\right)} \tag{62}$$

Taking inverse Laplace transform of equation (62), we get:

$$\begin{aligned} p_{up}(t) = {} & \frac{c_0\mu}{z_1z_2z_3z_4} + \left\{\frac{\left(z_1^3 + c_2z_1^2 + c_1z_1 + c_0\right)\left(z_1 + \mu\right)}{z_1\left(z_1 - z_2\right)\left(z_1 - z_3\right)\left(z_1 - z_4\right)}\right\}\exp\left(z_1t\right) \\ & - \left\{\frac{\left(z_2^3 + c_2z_2^2 + c_1z_2 + c_0\right)\left(z_2 + \mu\right)}{z_2\left(z_1 - z_2\right)\left(z_2 - z_3\right)\left(z_2 - z_4\right)}\right\}\exp\left(z_2t\right) \\ & + \left\{\frac{\left(z_3^3 + c_2z_3^2 + c_1z_3 + c_0\right)\left(z_3 + \mu\right)}{z_3\left(z_1 - z_3\right)\left(z_2 - z_3\right)\left(z_3 - z_4\right)}\right\}\exp\left(z_3t\right) \\ & - \left\{\frac{\left(z_4^3 + c_2z_4^2 + c_1z_4 + c_0\right)\left(z_4 + \mu\right)}{z_4\left(z_4 - z_1\right)\left(z_4 - z_2\right)\left(z_3 - z_4\right)}\right\}\exp\left(z_4t\right) \end{aligned} \tag{63}$$

Where: $c_2 = \left(\lambda_1 + \lambda_m + \alpha + \mu_1 + \mu_2\right), c_1 = \left(\lambda_1\mu_2 + \mu_1\mu_2 + \alpha\mu_2 + \lambda_m\mu_1 + \alpha\mu_1\right), c_0 = \alpha\mu_1\mu_2$

and z_1, z_2 are two roots of the equation $\left(s^2 + s\left(\lambda + \alpha + \mu\right) + \alpha\mu\right) = 0$ and z_3, z_4 are two roots of the equation $\left(s^2 + s\left(\lambda_1 + \lambda_m + \mu_1 + \mu_2\right) + \left(\lambda_1\mu_2 + \mu_1\mu_2 + \lambda_m\mu_1\right)\right) = 0.$

8.2. Availability of the system for model-2

Proceeding in similar way as in model-1, the expressions for $p_0(s)$ is same as that of defined in equation (58) for model-1and remaining expressions are:

$$p_1(s) = \frac{J(s)}{I(s)} \tag{64}$$

Where:

$$J(s) = \frac{\alpha\left(s + \mu_1\right)\left(s + \mu_2\right)\left(s + \mu_3\right)\left(s + \lambda_2\right)}{\left(s + \lambda_1 + \lambda_m\right)\left(s + \mu_1\right)\left(s + \mu_2\right)\left(s + \mu_3\right) - \mu_2\lambda_m\left(s + \mu_1\right)\left(s + \mu_3\right)\left(s + \lambda_2\right) - \lambda_1\lambda_2\mu_1\mu_3\left(s + \mu_2\right)} \tag{65}$$

And $I(s)$ is same as defined in equation (59):

$$p_5(s) = \frac{J(s)K(s)}{I(s)} \tag{66}$$

Where $K(s) = \dfrac{\mu_1\lambda_1}{\left(s + \mu_1\right)\left(s + \lambda_2\right)}$ (67)

$$p_{up}(s) = p_0(s) + p_1(s) + p_5(s)$$

$$= \frac{\left(s^5 + b_4s^4 + b_3s^3 + b_2s^2 + b_1s + b_0\right)\left(s + \mu\right)}{s\left(s^2 + s\left(\lambda + \alpha + \mu\right) + \alpha\mu\right)\left(s^4 + a_3s^3 + a_2s^2 + a_1s + a_0\right)} \tag{68}$$

Where $b_4 = \left(\lambda_1 + \lambda_2 + \lambda_m + \alpha + \mu_1 + \mu_2 + \mu_3\right)$,

$$b_3 = \begin{pmatrix} \lambda_1\mu_1 + \lambda_1\mu_2 + \lambda_2\mu_2 + \lambda_1\lambda_2 + \lambda_2\mu_1 + \lambda_2\mu_3 + \lambda_1\mu_3 + \lambda_m\mu_1 + \lambda_m\mu_3 \\ +\lambda_m\lambda_2 + \mu_1\mu_3 + \mu_1\mu_2 + \mu_2\mu_3 + \alpha\mu_1 + \alpha\mu_2 + \alpha\mu_3 + \alpha\lambda_2 \end{pmatrix}$$

$$b_2 = \begin{pmatrix} \lambda_1\mu_1\mu_3 + \lambda_2\mu_1\mu_3 + \lambda_2\mu_1\mu_2 + \lambda_2\mu_2\mu_3 + \lambda_1\mu_1\mu_2 + \lambda_1\mu_2\mu_3 + \lambda_m\mu_1\mu_3 + \lambda_1\lambda_2\mu_1 + \lambda_1\lambda_2\mu_3 + \lambda_1\lambda_2\mu_2 \\ +\lambda_m\lambda_2\mu_1 + \lambda_m\lambda_2\mu_3 + \mu_1\mu_2\mu_3 + \alpha\mu_1\mu_2 + \alpha\mu_1\mu_3 + \alpha\mu_3\mu_2 + \alpha\lambda_2\mu_1 + \alpha\lambda_2\mu_3 + \alpha\lambda_2\mu_2 + \alpha\lambda_1\mu_1 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} \lambda_1\mu_1\mu_2\mu_3 + \lambda_1\lambda_2\mu_1\mu_2 + \lambda_1\lambda_2\mu_3\mu_2 + \lambda_m\lambda_2\mu_1\mu_3 + \alpha\lambda_2\mu_1\mu_3 \\ +\alpha\lambda_2\mu_1\mu_2 + \alpha\lambda_2\mu_2\mu_3 + \alpha\lambda_1\mu_1\mu_3 + \alpha\lambda_1\mu_1\mu_2 + \alpha\mu_1\mu_2\mu_3 \end{pmatrix}$$

$b_0 = \alpha\lambda_1\mu_1\mu_2\mu_3 + \alpha\lambda_2\mu_1\mu_2\mu_3$,

$$a_3 = \left(\lambda_1 + \lambda_2 + \lambda_m + \mu_1 + \mu_2 + \mu_3\right)$$

$$a_2 = \begin{pmatrix} \lambda_1\mu_1 + \lambda_1\mu_3 + \lambda_1\lambda_2 + \lambda_2\mu_1 + \lambda_2\mu_2 + \lambda_1\mu_2 + \lambda_m\mu_1 \\ +\lambda_m\mu_3 + \lambda_2\mu_3 + \lambda_m\lambda_2 + \mu_1\mu_3 + \mu_1\mu_2 + \mu_2\mu_3 \end{pmatrix},$$

$$a_1 = \begin{pmatrix} \lambda_1\mu_1\mu_3 + \lambda_2\mu_1\mu_3 + \lambda_2\mu_1\mu_2 + \lambda_2\mu_2\mu_3 + \lambda_1\mu_1\mu_2 + \lambda_1\mu_2\mu_3 + \lambda_1\lambda_2\mu_1 \\ +\lambda_1\lambda_2\mu_3 + \lambda_1\lambda_2\mu_2 + \lambda_m\mu_1\mu_3 + +\lambda_m\lambda_2\mu_1 + \lambda_m\lambda_2\mu_3 + \mu_1\mu_2\mu_3 \end{pmatrix}$$

and $a_0 = \left(\lambda_1\mu_1\mu_2\mu_3 + \lambda_1\lambda_2\mu_1\mu_2 + \lambda_1\lambda_2\mu_3\mu_2 + \lambda_m\lambda_2\mu_1\mu_3\right)$

Taking inverse Laplace transform of equation (68), we get:

$$\begin{aligned} p_{up}(t) = {} & \frac{b_0\mu}{z_1z_2z_3z_4z_5z_6} + \left\{\frac{\left(z_1^5 + b_4z_1^4 + b_3z_1^3 + b_2z_1^2 + b_1z_1 + b_0\right)\left(z_1 + \mu\right)}{z_1\left(z_1 - z_2\right)\left(z_1 - z_3\right)\left(z_1 - z_4\right)\left(z_1 - z_5\right)\left(z_1 - z_6\right)}\right\}\exp\left(z_1t\right) \\ & + \left\{\frac{\left(z_2^5 + b_4z_2^4 + b_3z_2^3 + b_2z_2^2 + b_1z_2 + b_0\right)\left(z_2 + \mu\right)}{z_2\left(z_2 - z_1\right)\left(z_2 - z_3\right)\left(z_2 - z_4\right)\left(z_2 - z_5\right)\left(z_2 - z_6\right)}\right\}\exp\left(z_2t\right) \\ & + \left\{\frac{\left(z_3^5 + b_4z_3^4 + b_3z_3^3 + b_2z_3^2 + b_1z_3 + b_0\right)\left(z_3 + \mu\right)}{z_3\left(z_3 - z_1\right)\left(z_3 - z_2\right)\left(z_3 - z_4\right)\left(z_3 - z_5\right)\left(z_3 - z_6\right)}\right\}\exp\left(z_3t\right) \\ & + \left\{\frac{\left(z_4^5 + b_4z_4^4 + b_3z_4^3 + b_2z_4^2 + b_1z_4 + b_0\right)\left(z_4 + \mu\right)}{z_4\left(z_4 - z_1\right)\left(z_4 - z_2\right)\left(z_4 - z_3\right)\left(z_4 - z_5\right)\left(z_4 - z_6\right)}\right\}\exp\left(z_4t\right) \\ & + \left\{\frac{\left(z_5^5 + b_4z_5^4 + b_3z_5^3 + b_2z_5^2 + b_1z_5 + b_0\right)\left(z_5 + \mu\right)}{z_5\left(z_5 - z_1\right)\left(z_5 - z_2\right)\left(z_5 - z_3\right)\left(z_5 - z_4\right)\left(z_5 - z_6\right)}\right\}\exp\left(z_5t\right) \\ & + \left\{\frac{\left(z_6^5 + b_4z_6^4 + b_3z_6^3 + b_2z_6^2 + b_1z_6 + b_0\right)\left(z_6 + \mu\right)}{z_6\left(z_6 - z_1\right)\left(z_6 - z_2\right)\left(z_6 - z_3\right)\left(z_6 - z_4\right)\left(z_6 - z_5\right)}\right\}\exp\left(z_6t\right) \end{aligned} \tag{69}$$

z_1 and z_2 are roots of the equation $\left(s^2 + s\left(\lambda + \alpha + \mu\right) + \alpha\mu\right) = 0$ and z_3, z_4, z_5 and z_6 are roots of the equation $\left(s^4 + a_3s^3 + a_2s^2 + a_1s + a_0\right) = 0$.

9. Profit analysis of the User

Suppose that the warranty period of the system is (0, w). Since the repairman is always available with the system, therefore beyond warranty period, it remains busy during the interval (w, t). Let K_1 be the revenue per unit time and K_2 be the repair cost per unit time, then the expected profits $H_1(t)$ and $H_2(t)$ for model-1 and 2 during the interval (0, t) are given by

For model-1

Using equation (63), we get the expected profit $H_1(t)$ as:

$$H_1(t) = K_1\int_0^t p_{up}(t)dt - K_2\left(t-w\right)$$

$$H_1(t) = K_1\left\{\begin{array}{l} \dfrac{c_0\mu t}{z_1z_2z_3z_4} - \left\{\dfrac{\left(z_1^3+c_2z_1^2+c_1z_1+c_0\right)\left(z_1+\mu\right)}{z_1^2\left(z_1-z_2\right)\left(z_1-z_3\right)\left(z_1-z_4\right)}\right\}\left(1-e^{z_1t}\right) \\ +\left\{\dfrac{\left(z_2^3+c_2z_2^2+c_1z_2+c_0\right)\left(z_2+\mu\right)}{z_2^2\left(z_1-z_2\right)\left(z_2-z_3\right)\left(z_2-z_4\right)}\right\}\left(1-e^{z_2t}\right) \\ -\left\{\dfrac{\left(z_3^3+c_2z_3^2+c_1z_3+c_0\right)\left(z_3+\mu\right)}{z_3^2\left(z_1-z_3\right)\left(z_2-z_3\right)\left(z_3-z_4\right)}\right\}\left(1-e^{z_3t}\right) \\ +\left\{\dfrac{\left(z_4^3+c_2z_4^2+c_1z_4+c_0\right)\left(z_4+\mu\right)}{z_4^2\left(z_4-z_1\right)\left(z_4-z_2\right)\left(z_3-z_4\right)}\right\}\left(1-e^{z_4t}\right) \end{array}\right\} - K_2(t-w) \quad (68)$$

For model-2

Using equation (69), we get the expected profit $H_2(t)$ as:

$$H_2(t) = K_1\int_0^t p_{up}(t)dt - K_2\left(t-w\right)$$

$$H_2(t) = K_1\left\{\begin{array}{l} \dfrac{b_0\mu t}{z_1z_2z_3z_4z_5z_6} + \left\{\dfrac{\left(z_1^5+b_4z_1^4+b_3z_1^3+b_2z_1^2+b_1z_1+b_0\right)\left(z_1+\mu\right)}{z_1^2\left(z_1-z_2\right)\left(z_1-z_3\right)\left(z_1-z_4\right)\left(z_1-z_5\right)\left(z_1-z_6\right)}\right\}\left(e^{z_1t}-1\right) \\ +\left\{\dfrac{\left(z_2^5+b_4z_2^4+b_3z_2^3+b_2z_2^2+b_1z_2+b_0\right)\left(z_2+\mu\right)}{z_2^2\left(z_2-z_1\right)\left(z_2-z_3\right)\left(z_2-z_4\right)\left(z_2-z_5\right)\left(z_2-z_6\right)}\right\}\left(e^{z_2t}-1\right) \\ +\left\{\dfrac{\left(z_3^5+b_4z_3^4+b_3z_3^3+b_2z_3^2+b_1z_3+b_0\right)\left(z_3+\mu\right)}{z_3^2\left(z_3-z_1\right)\left(z_3-z_2\right)\left(z_3-z_4\right)\left(z_3-z_5\right)\left(z_3-z_6\right)}\right\}\left(e^{z_3t}-1\right) \\ +\left\{\dfrac{\left(z_4^5+b_4z_4^4+b_3z_4^3+b_2z_4^2+b_1z_4+b_0\right)\left(z_4+\mu\right)}{z_4^2\left(z_4-z_1\right)\left(z_4-z_2\right)\left(z_4-z_3\right)\left(z_4-z_5\right)\left(z_4-z_6\right)}\right\}\left(e^{z_4t}-1\right) \\ +\left\{\dfrac{\left(z_5^5+b_4z_5^4+b_3z_5^3+b_2z_5^2+b_1z_5+b_0\right)\left(z_5+\mu\right)}{z_5^2\left(z_5-z_1\right)\left(z_5-z_2\right)\left(z_5-z_3\right)\left(z_5-z_4\right)\left(z_5-z_6\right)}\right\}\left(e^{z_5t}-1\right) \\ +\left\{\dfrac{\left(z_6^5+b_4z_6^4+b_3z_6^3+b_2z_6^2+b_1z_6+b_0\right)\left(z_6+\mu\right)}{z_6^2\left(z_6-z_1\right)\left(z_6-z_2\right)\left(z_6-z_3\right)\left(z_6-z_4\right)\left(z_6-z_5\right)}\right\}\left(e^{z_6t}-1\right) \end{array}\right\} - K_2(t-w) \quad (69)$$

10.Numerical analysis

Table 1. Effect of failure rates (λ and λ_1), transition rate (λ_m) and transition rate of completion of warranty (α) on Reliability of the system (R (t))

Time (t)	λ_1=0.02, α=0.003, λ_m=0.04	λ_1=0.02, α=0.003, λ_m=0.04	λ=0.01, α=0.003, λ_m=0.04	λ=0.01, λ_1=0.02, λ_m=0.04	λ=0.01, λ_1=0.02, α=0.003
	R(t) (for λ=0.01)	R(t) (for λ=0.02)	R(t) (for λ_1=0.03)	R(t) (for α=0.005)	R(t) (for λ_m=0.05)
10	0.899114	0.814457	0.898175	0.895363	0.898175
11	0.889088	0.797518	0.888004	0.884676	0.888004
12	0.8791	0.780872	0.877867	0.873994	0.877867
13	0.869154	0.764518	0.867771	0.863327	0.867771
14	0.859254	0.748454	0.857722	0.852681	0.857722
15	0.849405	0.732679	0.847724	0.842066	0.847724
16	0.83961	0.717189	0.837782	0.831487	0.837782
17	0.829873	0.701984	0.827901	0.820952	0.827901

Table 2. Effect of repair cost (K_2), PM rate (μ_2) and transition rate by which unit goes for PM (λ_m) on expected profit ($H_1(t)$)

Time (t)	λ=0.01, μ_1=0.1, λ_1=0.02, μ_2=0.3, α=0.003, μ=0.2, λ_m=0.04, K_1=500, W=3	λ=0.01, μ_1=0.1, λ_1=0.02, μ_2=0.3, α=0.003, μ=0.2, λ_m=0.04, K_1=500, W=3	λ=0.01, μ_1=0.1, λ_1=0.02, μ=0.2, α=0.003, W=3, λ_m=0.04, K_1=500, K_2=150	λ=0.01, μ_1=0.1, λ_1=0.02, μ_2=0.3, α=0.003, W=3, μ=0.2, K_1=500, K_2=150
	$H_1(t)$ (For K_2=150)	$H_1(t)$ (For K_2=100)	$H_1(t)$ (For μ_2=0.4)	$H_1(t)$ (For λ_m=0.03)
10	3807.814	4157.814	3808.29	3809.311
11	4135.365	4535.365	4136.008	4137.224
12	4462.205	4912.205	4463.044	4464.464
13	4788.424	5288.424	4789.488	4791.118
14	5114.094	5664.094	5115.411	5117.259
15	5439.274	6039.274	5440.873	5442.942
16	5764.01	6414.01	5765.921	5768.216
17	6088.342	6788.342	6090.592	6093.117

Table 3. Effect of repair cost (K_2), PM rate (μ_2), failure rate of degraded unit (λ_2), transition rate by which unit goes for PM (λ_m) and replacement rate of failed degraded unit (μ_3) on expected profit ($H_2(t)$)

Time (t)	λ=0.01, λ_1=0.02, λ_2=0.03, λ_m=0.04, α=0.003, μ=0.2, μ_1=0.1, μ_2=0.3, μ_3=0.4, W=3, K_1=500	λ=0.01, λ_1=0.02, λ_2=0.03, λ_m=0.04, α=0.003, μ=0.2, μ_1=0.1, μ_2=0.3, μ_3=0.4, W=3, K_1=500	λ=0.01, λ_1=0.02, λ_2=0.03, λ_m=0.04, α=0.003, μ=0.2, μ_1=0.1, μ_3=0.4, W=3, K_1=500, K_2=150	λ=0.01, λ_1=0.02, λ_m=0.04, α=0.003, μ=0.2, μ_1=0.1, μ_1=0.3, μ_3=0.4, W=3, K_1=500, K_2=150	λ=0.01, λ_1=0.02, λ_2=0.03, α=0.003, μ=0.2, μ_1=0.1, μ_2=0.3, μ_3=0.4, W=3, K_1=500, K_2=150	λ=0.01, λ_1=0.02, λ_2=0.03, λ_m=0.04, α=0.003, μ=0.2, μ_1=0.1, μ_2=0.3, W=3, K_1=500, K_2=150
	$H_2(t)$ (For K_2=150)	$H_2(t)$ (For K_2=100)	$H_2(t)$ (For μ_2=0.4)	$H_2(t)$ (For λ_2=0.02)	$H_2(t)$ (For λ_m=0.03)	$H_2(t)$ (For μ_3=0.5)
10	3804.924	4154.924	3805.578	3805.106	3806.008	3804.95
11	4132.078	4532.078	4132.946	4132.232	4133.431	4132.119
12	4458.548	4908.548	4459.667	4458.66	4460.201	4458.61
13	4784.439	5284.439	4785.845	4784.488	4786.421	4784.527
14	5109.836	5659.836	5111.57	5109.851	5112.178	5109.957
15	5434.816	6034.816	5436.915	5434.971	5437.547	5434.976
16	5759.44	6409.44	5761.946	5759.456	5762.59	5759.647
17	6083.764	6783.764	6086.717	6083.809	6087.363	6084.026

Table 4. Expected profit difference ($H_1(t)$-$H_2(t)$)

Time (t)	λ=0.01, λ_1=0.02, λ_2=0.03, λ_m=0.04, α=0.003, μ=0.2, μ_1=0.1, μ_2=0.3, μ_3=0.4		
	K_1	W	$H_1(t)$-$H_2(t)$
10	500	3	2.890131
11	500	3	3.287116
12	500	3	3.657437
13	500	3	3.985432
14	500	3	4.258423
15	500	3	4.457927
16	500	3	4.570277
17	500	3	4.577749

11. Interpretation and conclusion

Table 1 shows that reliability for both the models is same. It is found that reliability of the system decreases with the increase of failure rates (λ and λ_1), transition rate by which unit goes for PM (λ_m) and transition rate of completion of warranty (α) with respect to time and for fixed values of other parameters. Tables 2 and 3 highlight the behaviour of expected profit for model-1 and 2. From table 2, it is observed that expected profit $H_1(t)$ increases with the decrease of repair cost (K_2) and transition rate by which unit goes for PM (λ_m) while with the increase of PM rate (μ_2) with respect to time. From Table 3, it is analyzed that expected profit $H_2(t)$ increases with the increase of PM rate (μ_2) and replacement rate of failed degraded unit (μ_3) while with the decrease of failure rate (λ_2), transition rate by which unit goes for PM (λ_m) and repair cost (K_2) with respect to time. Table 4 shows the expected profit difference ($H_1(t)$-$H_2(t)$>0) which goes on increasing with respect to time. This implies that model-1 is profitable over model-2.

Hence study reveals that after getting PM beyond warranty, a system in which unit works with reduced capacity after its repair will not be economically beneficial to use.

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