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DYNAMIC STABILITY OF A THREE-LAYER BEAM – GENERALISATION OF THE SANDWICH STRUCTURE THEORY

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Abstract: The work focuses on the dynamic stability problem of a simply supported three-layer beam subjected to a pulsating axial force. Two analytical models of this beam are developed: one model takes into account the non-linear hypothesis of cross-section deformation, and the other takes into account the standard "broken line" hypothesis. Displacements, strains and stresses for each model are formulated in detail. Based on the Hamilton principle, equations of motion are determined for each of these models. These systems of two differential equations for each model are approximately solved with the consideration of the axial pulsating force, and the fundamental natural frequencies, critical forces and the Mathieu equation are determined. Detailed studies are performed for an exemplary family of beams. The stable and unstable regions are calculated for the three pulsating load cases. The values of fundamental natural frequencies and critical forces of exemplary beams calculated from two models are compared.

Key words: three-layer beam, fundamental natural frequency, unstable regions, mathematical modelling, shear effect

1. INTRODUCTION

Sandwich constructions initiated in the mid-twentieth century have been intensively improved and are being used in aerospace, automotive, railway and shipbuilding industries. There are many scientific and research works on stability and free vibrations of sandwich beams. Ray and Kar [1] presented the parametric instability of a three-layer symmetrical sandwich beam subjected to periodic axial loading. Nine different boundary conditions were considered. The influence of the shear parameter on static buckling loads was considered, and the influence of the shear parameter and the core thickness parameter on the areas of parametric instability was investigated. Yeh et al. [2] studied the problem of dynamic stability of a sandwich beam with an electrorheological liquid core subjected to an axial dynamic force. The influence of natural frequency, static buckling loads and loss factors on the dynamic stability of a sandwich beam was investigated. In addition, the areas of instability of the studied beam were calculated using the harmonic balance method and the finite element method. Yang et al. [3] used the finite element method to study the vibration and dynamic stability of a moving sandwich beam. It was assumed that the damping layer is linearly viscoelastic and almost incompressible. Taking into account the numerical results, it was shown that the forced damping layer stabilises the movable layered beam. Lin and Chen [4] used the finite element method to study the problems of dynamic stability of spinning pre-twisted sandwich beams with a limited damping layer, subjected to periodic axial loads. For a viscoelastic material, a complex representation of the modulus was used. The influence of the pre-twisted angles, spinning speed, core thickness, shear parameter, core loss factor and constraint layer stiffness on unstable regions was discussed. Many different plate theories were described by Carrera and Brischetto [5]. Bending and vibration of sandwich structures were assessed. The kinematics of the classical and other theories (higher order, zigzag, layered and mixed) were described. Reddy [6] reformulated the classical and shear theories of beam and plate deformations using non-local differential Eringen constitutive relations and non-linear von Karman strains. Theoretical studies described in the article can be used to determine the influence of geometric non-linearity and non-local constitutive relations on the bending response. Misiurek [7] studied the dynamic response of a finite, simply supported sandwich beam subjected to a force moving at a constant speed. The main goal of the work was to show that the aperiodic part of the solution can be presented in a closed form, instead of an infinite form (an infinite series). Based on the Tymoshenko beam theory, Chen et al. [8] studied the behaviour of non-linear natural vibrations of a porous sandwich beam deformable under the influence of shear. The beam consisted of two facing layers and a functionally graded porous core, which contains internal pores with different porosity distributions. The authors assumed that the elastic modulus and mass density change along the thickness direction in terms of porosity coefficients and mass density. Grygorowicz and Magnucka-Blandzi [9] described the static and dynamic stability of a simply supported sandwich beam with a metal foam core. Mathematical modelling of the problem was presented. The displacement field was formulated based on the broken line hypothesis and the assumed non-linear hypothesis. Using the Hamilton principle, the equations of motion were obtained. Critical loads, areas of instability, natural frequencies of the beam and static and dynamic equilibrium paths were calculated analytically and verified numerically. Kolakowski and Teter [10] reviewed papers on static and dynamic buckling and post-buckling behaviours of thin-walled structures. Based on the analytical-numerical method, the static buckling stresses, natural

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frequencies, equation coefficients describing the post-buckling equilibrium path and the dynamic response of the plate structure subjected to a compressive load and/or bending moment were determined. In addition, all buckling modes and post-buckling behaviour of thin-walled columns made of different materials were described. Sayyad and Ghugal [11] developed a unified shear deformation theory for the analysis of shear-deformable composite beams and plates. To account for the transverse shear deformation effect, the authors expanded the theory with different shape functions (parabolic (PSDT), trigonometric (TSDT), hyperbolic (HSDT) and exponential (ESDT)) in terms of thickness coordinates. To verify the accuracy and effectiveness of the authors' theory, the obtained results were compared with the exact elasticity solution and other higher order shear deformation theories. Sayyad and Ghugal [12] provided a critical literature review on bending, buckling and free vibration analysis of isotropic, laminated and shear-deformable beams based on equivalent theories (single-layer theories, layer-wise theories, zig-zag theories and exact elastic solution). In addition, the literature on finite element modelling of laminated and sandwich beams was reviewed based on classical and refined theories. Finally, for reference to researchers of the subject, the displacement fields of various equivalent single-layer and layer-wise theories were summarised. Based on the Grigolyuk-Chulkov hypothesis and the modified couple stress theory, Avrejcewicz et al. [13] developed a mathematical model of three-layer beams. The authors took into account the movements of the layers at the micro- and the nano-scale. Based on the Hamilton principle, the equations of motion and the boundary/initial conditions for the displacement of the beams were obtained. Smyczyński and Magnucka-Blandzi [14] considered the stability analysis of a simply supported five-layer beam. The beam consisted of two facings, a core and two tie layers between the facings and the core. Based on the Hamilton principle and the formulated non-linear hypothesis of beam cross-section deformation, a system of four stability equations was derived. Then, the system was approximately solved, and critical loads, free vibrations and areas of instability were determined. Sayyad and Ghugal [15] used the theory of trigonometric shear and normal strain to study the bending, buckling and vibration responses of shear-deformable composite laminated beams and sandwich beams. The main goal of this theory was to take into account the effects of transverse shear and normal strain. According to this theory, the shear stresses on the top and bottom surfaces were equal to zero (zero shear stress conditions were met without applying a shear correction factor). Numerical results for deflections, stresses, natural frequencies and critical buckling loads for isotropic, laminated and sandwich beams were presented. Magnucka-Blandzi and Magnucki [16] presented a mathematical model of a simply supported sandwich beam subjected to three-point bending. To describe the problem, the authors adopted the appropriate hypothesis of flat cross-section deformation. An important feature of this analytical beam model was that it included the shear effect of the facings and was reduced to a classic sandwich beam described by two differential equilibrium equations. Al-shujairi and Mollamahmutoglu [17] studied the dynamic stability of a functionally graded (FG) size-dependent layered microbeam subjected to parametric axial excitation. The authors considered various boundary conditions, including thermal effects. The material properties of the FG part of the multilayer microbeam varied depending on the thickness of the beam. The problem was solved numerically. The original contribution to the article was to determine the parametric instability regions of the FG microbeam under different boundary conditions and with different effects. In a review, Birman and Kardomateas [18] described contemporary trends in theoretical developments, innovative designs and modern applications of layered structures. Examples of problems faced by engineers and designers of sandwich structures were considered, including typical failures, responses to various loads, environmental effects and fire. Example applications of sandwich structures were concentrated in the aerospace, civil and marine engineering, electronics and biomedicine industries. Li et al. [19] discussed the non-linear amplitude–frequency response and the unstable boundary and the dynamic responses of an axially moving viscoelastic layer beam at low- and high-frequency fundamental resonances and compared them. Sayyad and Ghugal [20] presented a literature review on the modelling and analysis of functionally graded sandwich beams using the theory of elasticity, analytical methods and numerical methods based on classical and refined theories of shear deformation, citing 250 references. In addition, suggestions for future research into the analysis of functionally graded sandwich beams were made. Sayyad and Ghugal [21] presented an analysis of the static behaviour of curved FG sandwich beams. For the bending analysis of vertically curved beams, the sinusoidal beam theory was used, taking into account the influence of transverse normal stresses/strains. Sayyad and Avhad [22] presented closed form Navier-type solutions for static bending, elastic buckling and free vibration analysis of functionally graded (FG) symmetrical layered beams using the theory of deformation under hyperbolic shear. Eloy et al. [23] investigated numerically and experimentally sandwich panels with carbon–epoxy composite coatings and a magnetorheological elastomer honeycomb core. Based on the results, it was noticed that the honeycomb sandwich panel shifted the natural frequencies due to the increase of the induced magnetic field, especially for the shape of the first mode. Chen et al. [24] extended the model of a higher order shear-deformable mixed beam element with a rational distribution of shear stresses to the vibration analysis of functionally graded beams. The authors discussed the load–frequency relationship of functionally graded sandwich beams. The results showed that in addition to the axial force, the bending moment exerted a significant difference in the vibration frequency of the functionally graded beams. Tewelde and Krawczuk [25] presented a review of the literature on non-linear effects caused by the closure of cracks in the structure, i.e. beam and plate structures. After analysing various methods, the advantages, disadvantages and perspectives of a number of non-linear vibration methods for detecting structural damage were discussed. In addition, recommendations were made for future researchers.

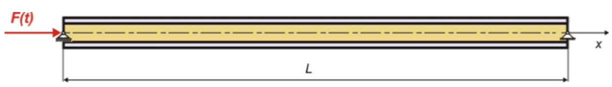

Fig. 1. Scheme of the beam subjected to a pulsating axial force

The subject of the study is a simply supported three-layer beam of length L, width b and total depth h subjected to a pulsating axial force F (Fig. 1). A novelty in this work is the individual hypothesis-theory of deformation of the plane cross section, which is assumed for beam modelling taking into account the shear effect in layers. The main purpose of the work is to analytically determine the fundamental natural frequencies, critical loads and unstable regions of this beam. This work is a continuation/development of the study presented in the proceedings of the

8th International Conference on Coupled Instabilities in Metal Structures [26].

2. MATHEMATICAL MODEL OF THE BEAM

A non-linear hypothesis of a planar cross-section deformation is assumed. This hypothesis takes into account the shear effect in each layer of this beam and is a generalisation of the standard "broken line" hypothesis. The deformation shape of the planar cross section is a curved line perpendicular to the outer surfaces of the beam (Fig. 2). Thus, this hypothesis satisfies the necessary condition of zeroing the shear stresses on these surfaces.

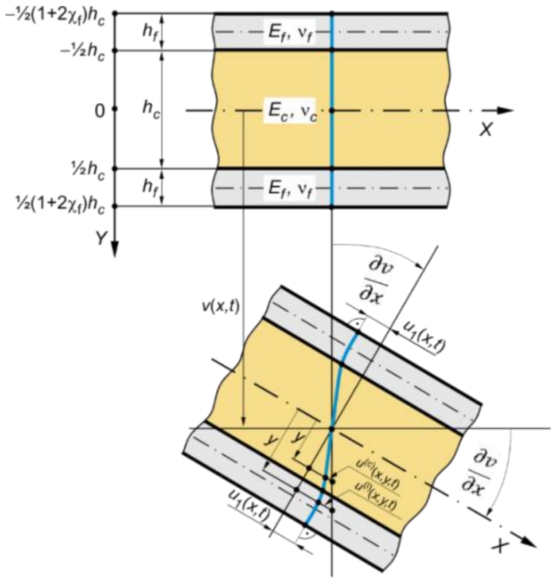

Fig. 2. Scheme of the deformation of a planar cross section of the three-layer beam – the non-linear hypothesis

The total height of the beam equals

$$h = 2h_f + h_c,$$

where h_f, h_c denote thicknesses of the outer layers and the middle layer (core), respectively.

Moreover, the following notation is introduced:

$\eta = y/h_c$ – dimensionless coordinate,
$\tilde{u}_1(x,t) = u_1(x,t)/h_c$ – dimensionless displacement,
$\chi_f = h_f/h_c$ – parameter,
$k_f \in \langle 0,1 \rangle$ – coefficient (real number),
$\beta_c \in \langle 0,1 \rangle$ – coefficient (real number),
E_f, E_c – Young modulus of facings and core,
ν_f, ν_c – Poisson ratio of facings and core,
ρ_f, ρ_c – mass densities of facings and core.

So, the total mass density of the beam is as follows:

$$\rho_b = \rho_c + 2\rho_f\chi_f.$$

Based on the assumed theory, longitudinal displacements are formulated separately for each layer, i.e. for

upper layer: $-\left(1+2\chi_f\right)/2 \le \eta \le -1/2$

$$u^{(u)}(x.y,t) = -h_c\left[\eta\frac{\partial v}{\partial x} + f_d^{(u)}(\eta)\tilde{u}_1(x,t)\right], \tag{1}$$

where

$$f_d^{(u)}(\eta) = \left\{-\left[3 - 4\left(\frac{\eta}{1+2\chi_f}\right)^2\right]\frac{\eta}{1+2\chi_f}\right\}^{k_f},$$

middle layer (core): $-1/2 \le \eta \le 1/2$

$$u^{(c)}(x,y,t) = -h_c\left[\eta\frac{\partial v}{\partial x} - 2f_d^{(c)}(\eta)\tilde{u}_1(x,t)\right], \tag{2}$$

where

$$f_d^{(c)}(\eta) = c_f\,\frac{3-4\beta_c\eta^2}{3-\beta_c}\eta, \quad c_f = \left[\frac{1+6\left(1+\chi_f\right)\chi_f}{\left(1+2\chi_f\right)^3}\right]^{k_f},$$

lower layer: $1/2 \le \eta \le \left(1+2\chi_f\right)/2$

$$u^{(l)}(x.y,t) = -h_c\left[\eta\frac{\partial v}{\partial x} - f_d^{(l)}(\eta)\tilde{u}_1(x,t)\right], \tag{3}$$

where

$$f_d^{(l)}(\eta) = \left\{\left[3 - 4\left(\frac{\eta}{1+2\chi_f}\right)^2\right]\frac{\eta}{1+2\chi_f}\right\}^{k_f}.$$

The non-linear deformation functions $f_d^{(u)}(\eta)$, $f_d^{(c)}(\eta)$ and $f_d^{(l)}(\eta)$ existing in these expressions are developed taking into account the conditions of continuity between layers and conditions of perpendicularity to the outer surfaces of the beam.

A linear relationship between strains and displacements is assumed, so the strains for each layer are as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{h_c\partial\eta}.$$

Then, taking into account the aforementioned expressions and expressions (1)–(3), the strains are determined.

Hence, the stresses are determined by using the following formulas:

upper layer: $-\left(1+2\chi_f\right)/2 \le \eta \le -1/2$

$$\sigma_x^{(u)} = E_f\cdot\varepsilon_x^{(u)}, \qquad \tau_{xy}^{(u)} = \frac{E_f}{2\left(1+\nu_f\right)}\cdot\gamma_{xy}^{(u)},$$

middle layer (core): $-1/2 \le \eta \le 1/2$

$$\sigma_x^{(c)} = E_c\cdot\varepsilon_x^{(c)}, \qquad \tau_{xy}^{(c)} = \frac{E_c}{2\left(1+\nu_c\right)}\cdot\gamma_{xy}^{(c)},$$

lower layer: $1/2 \le \eta \le (1+2\chi_f)/2$

$$\sigma_x^{(l)} = E_f\cdot\varepsilon_x^{(l)}, \qquad \tau_{xy}^{(l)} = \frac{E_f}{2\left(1+\nu_f\right)}\cdot\gamma_{xy}^{(l)}.$$

Then, the elastic strain energy

$$U_\varepsilon = \frac{1}{2}bh_c\cdot$$

$$\cdot\int_0^L\left\{E_f\int_{-\frac{1+2\chi_f}{2}}^{-\frac{1}{2}}\left\{\left[\varepsilon_x^{(u)}\right]^2 + \frac{1}{2\left(1+\nu_f\right)}\left[\gamma_{xy}^{(u)}\right]^2\right\}d\eta\, + \right.$$

$$+E_c\int_{-\frac{1}{2}}^{\frac{1}{2}}\left\{\left[\varepsilon_x^{(c)}\right]^2 + \frac{1}{2\left(1+\nu_c\right)}\left[\gamma_{xy}^{(c)}\right]^2\right\}d\eta\, +$$

$$\left. +E_f\int_{\frac{1}{2}}^{\frac{1+2\chi_f}{2}}\left\{\left[\varepsilon_x^{(l)}\right]^2 + \frac{1}{2\left(1+\nu_f\right)}\left[\gamma_{xy}^{(l)}\right]^2\right\}d\eta\right\}dx,$$

the kinetic energy

$$T = \frac{1}{2}\left(2\rho_f\chi_f + \rho_c\right)bh_c\int_0^L\left(\frac{\partial v}{\partial t}\right)^2 dx,$$

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the work of the load

$$W = \frac{1}{2} F \int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx$$

are derived.

Based on the Hamilton principle,

$$\delta \int_{t_1}^{t_2} [T - (U_\varepsilon - W)]\, dt = 0, \tag{4}$$

two differential equations of motion are obtained in the following form:

$$\begin{cases} \rho_b b h_c \frac{\partial^2 v}{\partial t^2} + E_c b h_c^3 \left(C_{vv} \frac{\partial^4 v}{\partial x^4} - C_{vu} \frac{\partial^3 \tilde{u}_1}{\partial x^3} \right) + \frac{\partial^2 v}{\partial x^2} F(t) = 0 \\ C_{vu} \frac{\partial^3 v}{\partial x^3} - C_{uu} \frac{\partial^2 \tilde{u}_1}{\partial x^2} + C_u \frac{\tilde{u}_1(x,t)}{h_c^2} = 0 \end{cases} \tag{5}$$

where C_{vv}, C_{vu}, C_{uu} and C_u are– dimensionless coefficients.

These coefficients are as follows:

$$C_{vv} = \frac{1}{12}\left[1 + 2e_f\left(3 + 6\chi_f + 4\chi_f^2\right)\chi_f\right],$$

$$C_{vu} = \frac{1}{10}\left[c_f \frac{5-\beta_c}{3-\beta_c} + 20 e_f I_1\right],$$

$$C_{uu} = 2e_f I_2 + \frac{c_f^2}{35}\left(105 - 42\beta_c + 5\beta_c^2\right),$$

$$C_u = \frac{e_f}{1+\nu_f} I_3 + \frac{6}{5(1+\nu_c)} c_f^2 \frac{15 - 10\beta_c + 3\beta_c^2}{(3-\beta_c)^2},$$

where

$$e_f = \frac{E_f}{E_c}, \quad I_1 = \int_{\frac{1}{2}}^{\frac{1}{2}+\chi_f} \eta f_d^{(l)}(\eta) d\eta,$$

$$I_2 = \int_{\frac{1}{2}}^{\frac{1}{2}+\chi_f} \left[f_d^{(l)}(\eta)\right]^2 d\eta,$$

$$I_3 = \int_{\frac{1}{2}}^{\frac{1}{2}+\chi_f} \left[\frac{d f_d^{(l)}}{d\eta}\right]^2 d\eta.$$

3. NATURAL FREQUENCY AND UNSTABLE REGIONS

The system of two differential equations (5) is approximately solved with the use of two assumed functions:

$$v(x,t) = v_a(t) \sin\left(\pi \frac{x}{L}\right), \; \tilde{u}_1(x,t) = \tilde{u}_{1a}(t) \cos\left(\pi \frac{x}{L}\right), \tag{6}$$

where $v_a(t)$ and $\tilde{u}_{1a}(t)$ are time-dependent functions.

These functions identically satisfy the boundary conditions:

$$v(0,t) = v(L,t) = 0, \quad \frac{d\tilde{u}_1}{dx}\Big|_{x=0} = \frac{d\tilde{u}_1}{dx}\Big|_{x=L} = 0.$$

The loading force – the pulsating force – is in the following form:

$$F(t) = F_m + F_a \cos(\theta t), \tag{7}$$

where F_m, F_a and θ are mean value, amplitude and frequency of the force, respectively. Substituting functions (6) and (7) into equation (5), and after a simply transformation, the Mathieu equation is obtained:

$$\frac{d^2 v_a}{dt^2} + \Omega^2 [1 - 2\mu \cos(\theta t)] v_a(t) = 0, \tag{8}$$

where

$v_a(t) = \bar{v}_a \sin(\omega t)$, v_a– deflection amplitude,

$$\Omega^2 = \omega^2 (1 - \alpha_m), \qquad \mu = \frac{1}{2} \cdot \frac{\alpha_a}{1 - \alpha_m},$$

$$\alpha_m = \frac{F_m}{F_{CR}}, \alpha_a = \frac{F_a}{F_{CR}},$$

$$\omega^2 = \left(\frac{\pi}{\lambda}\right)^4 \frac{10^{12}}{\left(1+2\chi_f\right)^2} (1 - C_{sv}) C_{vv} \frac{E_c}{\rho_b h^2},$$

$$F_{CR} = \left(\frac{\pi}{\lambda}\right)^2 (1 - C_{sv}) \frac{C_{vv}}{\left(1+2\chi_f\right)^3} E_c b h,$$

$$C_{sv} = max_{\beta_c, k_f} \left\{\frac{\pi^2}{C_{vv}} \cdot \frac{C_{vu}^2}{\pi^2 C_{uu} + \lambda_c^2 C_u}\right\}, \lambda_c = \frac{L}{h_c}.$$

Using the aforementioned notations, unstable regions can be described by the following inequalities:
the first unstable region

$$2\Omega\sqrt{1-\mu} \le \theta \le 2\Omega\sqrt{1+\mu}, \tag{9}$$

the second unstable region

$$\Omega\sqrt{1-2\mu^2} \le \theta \le \Omega\sqrt{1+\frac{1}{3}\mu^2}. \tag{10}$$

Sample Calculations

Detailed studies are carried out for an exemplary family of three-layer beams. Stable and unstable regions are calculated for the three load cases – pulsating forces.

The geometric dimensions of the beams are as follows:
$h = 20$ mm – total depth,
$L = 600$ mm – length,
$\lambda = L/h = 30$ – relative length,
and the following mechanical properties are considered:
$E_f = 65\,000$ MPa – Young modulus in facings,
$\nu_f = 0.33$ – Poisson ratio in facings,
$E_c = 1\,200$ MPa – Young modulus in the core,
$\nu_c = 0.3$ – Poisson ratio in the core,
$\rho_f = 2\,600\ \mathrm{kg/m^3}$ – mass density in facings,
$\rho_c = 350\ \mathrm{kg/m^3}$ – mass density in the core.

The fundamental natural frequency

$$\omega = \left(\frac{\pi}{\lambda}\right)^2 \frac{10^6}{1+2\chi_f} \sqrt{(1 - C_{sv}) C_{vv} \frac{E_c}{\rho_b h^2}}, \tag{11}$$

and the dimensionless critical force

$$\tilde{F}_{CR} = \frac{F_{CR}}{E_c b h} = \left(\frac{\pi}{\lambda}\right)^2 (1 - C_{sv}) \frac{C_{vv}}{\left(1+2\chi_f\right)^3}, \tag{12}$$

are calculated and specified in Tab. 1.

Tab. 1. Dimensionless coefficients β_c, k_f, shear coefficient C_{sv}, fundamental natural frequency ω and dimensionless critical force $\tilde{F}_{CR}$

χ_f	$\frac{1}{18}$	$\frac{2}{16}$	$\frac{3}{14}$	$\frac{4}{12}$	$\frac{5}{10}$
β_c	0.1097	0.06230	0.04127	0.02798	0.01847
k_f	0.07041	0.04187	0.03263	0.02862	0.02698
C_{sv}	0.03488	0.05871	0.07476	0.08358	0.08552
$\omega\ \left[\frac{1}{s}\right]$	881.8	976.3	987.5	969.9	941.2
$\tilde{F}_{CR}$	0.01359	0.02318	0.03038	0.03575	0.03971

Three load cases are taken into account (LC-1, LC-2, LC-3) – according to the parameter values given in Tab. 2.

Tab. 2. Dimensionless coefficients, natural frequency and dimensionless critical force ($\lambda = 30$)

Load case	LC-1	LC-2	LC-3
α_a	0.5	1.0	1.5
α_m	0.5	0.25	0.1
μ	$1/2$	$2/3$	$5/6$

Then, the following unstable regions are determined for the following:

– first load case LC-1 (Fig. 3)
  – the first unstable region (9):

$$\omega \leq \theta \leq \sqrt{3} \cdot \omega,$$

  – the second unstable region (10):

$$\frac{1}{2} \cdot \omega \leq \theta \leq \frac{1}{2}\sqrt{\frac{13}{6}} \cdot \omega,$$

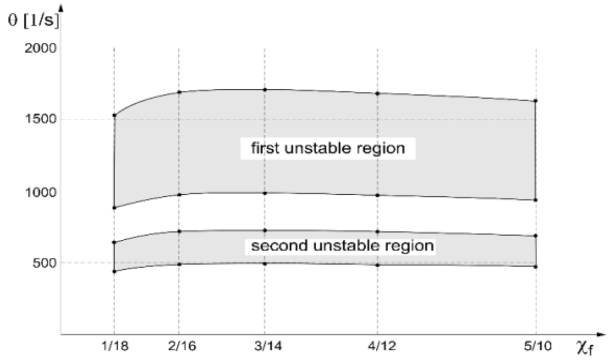

Fig. 3. Unstable regions for the first load case (LC-1)

– second load case LC-2 (Fig. 4)
  – the first unstable region (9):

$$\omega \leq \theta \leq \sqrt{5} \cdot \omega,$$

  – the second unstable region (10):

$$\frac{\sqrt{3}}{6} \cdot \omega \leq \theta \leq \frac{\sqrt{31}}{6} \cdot \omega,$$

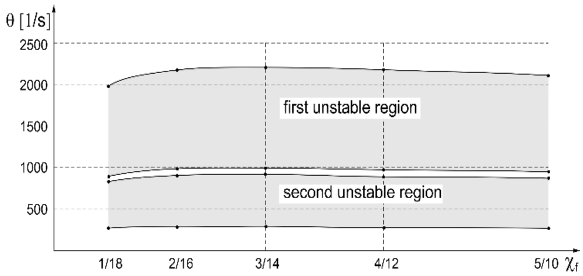

Fig. 4. Unstable regions for the second load case (LC-2)

– third load case LC-3 (Fig. 5)
  – the first unstable region (9)

$$\frac{\sqrt{15}}{5} \cdot \omega \leq \theta \leq \frac{\sqrt{165}}{5} \cdot \omega,$$

  – the second unstable region does not exist (10).

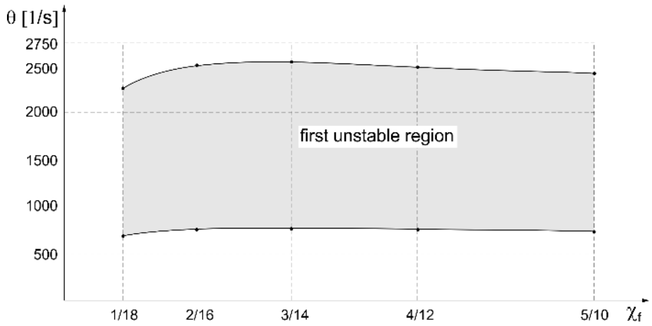

Fig. 5. Unstable regions for the third load case (LC-3)

4. CLASSICAL MODEL OF THE BEAM

The classical beam model is a particular case of the beam model presented in Section 2, i.e. it is a simplification of the mathematical model. Therefore, the deformation of the plane cross section of the sandwich beam, taking into account the "broken line" hypothesis-theory, is shown in Fig. 6.

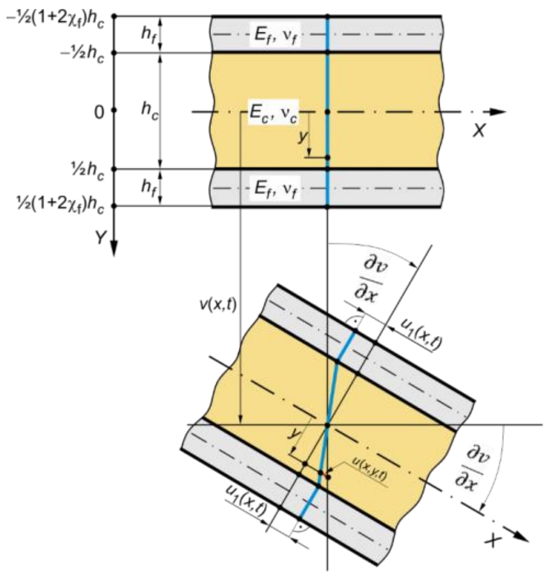

Fig. 6. Scheme of the deformation of a plane cross section of the beam – the standard "broken line" hypothesis-theory

Thus, the displacements, strains and stresses in the successive layers of the beam are as follows:

– upper layer: $-(1 + 2\chi_f)/2 \leq \eta \leq -1/2$

$$u^{(u)}(x, y, t) = -h_c\left[\eta\frac{\partial v}{\partial x} + \tilde{u}_1(x, t)\right] \quad (13)$$

$$\varepsilon_x^{(u)}(x, y, t) = -h_c\left[\eta\frac{\partial^2 v}{\partial x^2} + \frac{\partial \tilde{u}_1}{\partial x}\right], \quad \gamma_{xy}^{(u)}(x, y, t) = 0 \quad (14)$$

$$\sigma_x^{(u)}(x, y, t) = -E_f h_c\left[\eta\frac{\partial^2 v}{\partial x^2} + \frac{\partial \tilde{u}_1}{\partial x}\right], \quad \tau_{xy}^{(u)}(x, y, t) = 0 \quad (15)$$

– core: $-1/2 \leq \eta \leq 1/2$

$$u^{(c)}(x, y, t) = -h_c\eta\left[\frac{\partial v}{\partial x} - 2\tilde{u}_1(x, t)\right] \quad (16)$$

$$\varepsilon_x^{(c)}(x,y,t) = -h_c\eta\left[\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial \tilde{u}_1}{\partial x}\right], \gamma_{xy}^{(c)}(x,t) = 2\tilde{u}_1(x,t) \quad (17)$$

$$\sigma_x^{(c)}(x,y,t) = -E_c h_c \eta\left[\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial \tilde{u}_1}{\partial x}\right], \quad (18)$$

$$\tau_{xy}^{(c)}(x,t) = \frac{E_c}{1+\nu_c}\tilde{u}_1(x,t)$$

– lower layer: $1/2 \le \eta \le 1/2 + \chi_f$

$$u^{(l)}(x,y,t) = -h_c\left[\eta\frac{\partial v}{\partial x} - \tilde{u}_1(x,t)\right] \quad (19)$$

$$\varepsilon_x^{(l)}(x,y,t) = -h_c\left[\eta\frac{\partial^2 v}{\partial x^2} - \frac{\partial \tilde{u}_1}{\partial x}\right], \gamma_{xy}^{(l)}(x,y,t) = 0, \quad (20)$$

$$\sigma_x^{(l)}(x,y,t) = -E_f h_c\left[\eta\frac{\partial^2 v}{\partial x^2} - \frac{\partial \tilde{u}_1}{\partial x}\right], \tau_{xy}^{(l)}(x,y,t) = 0. \quad (21)$$

Based on the Hamilton principle (4), two differential equations of motion are obtained in form (5), where dimensionless coefficients of a sandwich beam are as follows:

$$C_{vv} = \frac{1}{12}\left[1 + 2e_f\chi_f\left(3 + 6\chi_f + 4\chi_f^2\right)\right],$$

$$C_{vu} = \frac{1}{6}\left[1 + 6e_f\chi_f\left(1 + \chi_f\right)\right],$$

$$C_{uu} = \frac{1}{3}\left(1 + 6e_f\chi_f\right),\ C_u = \frac{2}{1+\nu_c},\quad e_f = \frac{E_f}{E_c},\ \chi_f = \frac{h_f}{h_c}.$$

The fundamental natural frequency ω [1/s] and the dimensionless critical force $\tilde{F}_{CR}$ have identical forms (11) and (12), where the dimensionless coefficient is as follows:

$$C_{sv} = \frac{\pi^2}{C_{vv}} \cdot \frac{C_{vu}^2}{\pi^2 C_{uu} + \lambda_c^2 C_u}.$$

Detailed calculations are performed for sample data from section 4. The values of the dimensionless coefficient C_{sv}, the fundamental natural frequency ω [1/s] and the dimensionless critical force $\tilde{F}_{CR}$ are specified in Tab. 3.

Tab. 3. Results of analytical calculations of the exemplary beams

χ_f	$\frac{1}{18}$	$\frac{2}{16}$	$\frac{3}{14}$	$\frac{4}{12}$	$\frac{5}{10}$
C_{sv}	0.03534	0.05927	0.07519	0.08374	0.08531
$\omega\left[\frac{1}{s}\right]$	881.6	976.0	987.2	969.9	941.3
$\tilde{F}_{CR}$	0.01358	0.02316	0.03037	0.03574	0.03972

As a result of comparing the values of the fundamental natural frequencies ω [1/s] and the dimensionless critical forces $\tilde{F}_{CR}$ calculated based on the generalised model (Tab. 1) and the classical sandwich beam model (Tab. 3), it is easy to see that these differences are negligible. Thus, the unstable regions calculated with consideration of two models of sandwich beams are identical.

5. CONCLUSIONS

Summing up the research presented in this article, the following conclusions can be drawn:
– the assumed non-linear hypothesis-theory of deformation of a plane cross section of the beam takes into account the shear effect in the facings, so it is a generalisation of the "broken line" hypothesis, in which the shear effect in the facings is omitted,
– the influence of the shear effect in the beam facings on the values of the fundamental natural frequency ω and the critical load–force $\tilde{F}_{CR}$ is negligibly small, which is easy to see when comparing their values specified in Tabs 1 and 3,
– the influence of the three-layer beam structure, i.e. the ratio of the thickness of the facings to the thickness of the core (value of the parameter χ_f), on the values of the fundamental natural frequency ω and the critical load force $\tilde{F}_{CR}$ is significant (Tabs 1 and 3), which is graphically presented in Figs 7 and 8.

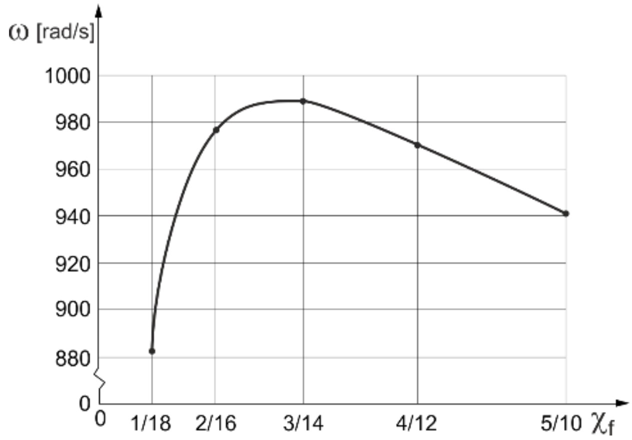

Fig. 7. Fundamental natural frequency

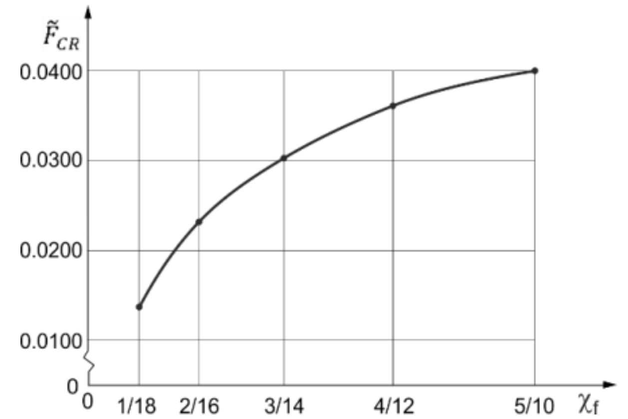

Fig. 8. Critical load–force

Therefore, based on the aforementioned, it can be concluded that when examining critical loads, fundamental natural frequencies and unstable regions, it is sufficient to apply the "broken line" hypothesis.

REFERENCES

1. Ray KR, Kar C. Parametric instability of a sandwich beam under various boundary conditions. Computers & Structures. 1995;55(5): 857-870.
2. Yeh J-Y, Chen L-W, Wang C-C. Dynamic stability of a sandwich beam with a constrained layer and electrorheological fluid core. Composite Structures. 2004;64(1):47-54.
3. Yang W-P, Chen L-W, Wang C-C. Vibration and dynamic stability of a traveling sandwich beam. Journal of Sound and Vibration. 2005;285(3):597-614.

4. Lin C-Y, Chen L-W. Dynamic stability of spinning pre-twisted sandwich beams with a constrained damping layer subjected to periodic axial loads. Composite Structures. 2005;70(3):275-286.
5. Carrera E, Brischetto S. A survey with numerical assessment of classical and refined theories for the analysis of sandwich plates. Applied Mechanics Reviews. 2009;62:01080-1-17
6. Reddy JN. Nonlocal nonlinear formulations for bending of classical and shear deformation theories of beams and plates. International Journal of Engineering Science. 2010;48:1507-1518.
7. Misiurek K, Śniady P. Vibrations of sandwich beam due to a moving force. Composite Structures. 2013;104:85-93.
8. Chen D, Kitipornchai S, Yang J. Nonlinear free vibration of shear deformable sandwich beam with a functionally graded porous core, Thin-Walled Structures. 2016;107:39-48.
9. Grygorowicz M, Magnucka-Blandzi E. Mathematical modeling for dynamic stability of sandwich beam with variable mechanical properties of core. Applied Mathematics and Mechanics. 2016;37(10):1361-1374.
10. Kolakowski Z, Teter A. Coupled static and dynamic buckling modelling of thin-walled structures in elastic range: Review of selected problems. Acta Mechanica et Automatica. 2016;10(2):141-149.
11. Sayyad AS, Ghugal YM. A unified shear deformation theory for the bending of isotropic, functionally graded, laminated and sandwich beams and plates. International Journal of Applied Mechanics. 2017;9(1):1750007.
12. Sayyad AS, Ghugal YM. Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature. Composite Structures. 2017;171:486-504.
13. Awrejcewicz J, Krysko VA, Pavlov SP, Zhigalov MV, Krysko AV. Mathematical model of a three-layer micro- and nano-beams based on the hypotheses of the Grigolyuk–Chulkov and the modified couple stress theory. International Journal of Solids and Structures. 2017;117:39-50.
14. Smyczynski M, Magnucka-Blandzi E. Stability and free vibrations of the three layer beam with two binding layers. Thin-Walled Structures. 2017;113:144-150.
15. Sayyad AS, Ghugal YM. Effect of thickness stretching on the static deformations, natural frequencies, and critical buckling loads of laminated composite and sandwich beams. Journal of the Brazilian Society of Mechanical Sciences and Engineering. 2018;40(6):No 296.
16. Magnucka-Blandzi E, Magnucki K. Mathematical modelling of a sandwich beam with consideration of the shear effect in the faces – three-point bending. Eighth International Conferece of Thin-Walled Structures – ICTWS 2018, Lisbon, Portugal, 24-27 July, 2018.
17. Al-shujairi M, Mollamahmutoğlu Ç. Dynamic stability of sandwich functionally graded micro-beam based on the nonlocal strain gradient theory with thermal effect. Composite Structures. 2018;201: 1018-1030.
18. Birman V, Kardomateas GA. Review of current trends in research and applications of sandwich structures. Composites Part B: Engineering . 2018;142:221-240.
19. Li YH, Dong YH, Qin Y, Lv HW. Nonlinear forced vibration and stability of an axially moving viscoelastic sandwich beam. International Journal of Mechanical Sciences. 2018;138-139:131-145.
20. Sayyad AS, Ghugal YM. Modeling and analysis of functionally graded sandwich beams: A review. Mechanics of Advanced Materials and Structures. 2019;26(21):1776-1795.
21. Sayyad AS, Ghugal YM. A sinusoidal beam theory for functionally graded sandwich curved beams. Composite Structures. 2019;226:111246.
22. Sayyad AS, Avhad PV. On static bending, elastic buckling and free vibration analysis of symmetric functionally graded sandwich beams. Journal of Solid Mechanics. 2019;11(1):166-180.
23. Eloy FS, Gomes GF, Ancelotti Jr. AC, Cunha Jr. SS, Bombard AJF, Junqueira DM. A numerical-experimental dynamic analysis of composite sandwich beam with magnetorheological elastomer honeycomb core. Composite Structures. 2019;209:242-257.
24. Chen S, Geng R, Li W. Vibration analysis of functionally graded beams using a higher-order shear deformable beam model with rational shear stress distribution. Composite Structures. 2021;277:114586.
25. Tewelde SA, Krawczuk M. Nonlinear vibration analysis of beam and plate with closed crack: A review. Acta Mechanica et Automatica. 2022;16(3):274-285.
26. Magnucki K., Magnucka-Blandzi E. Dynamic stability of a three-layer beam – Generalization of the sandwich structures theory. Proceedings of the 8th International Conference on Coupled Instabilities in Metal Structures, Lodz University of Technology, Poland, July 12-14, 2021.

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