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MAJID JAMAL-OMIDI[1], *MAHDI SHAYANMEHR*[1], *SAEID SAZESH*[2]

A FUNDAMENTAL STUDY ON THE FREE VIBRATION OF GEOMETRICAL NONLINEAR CANTLEVER BEAM USING AN EXACT SOLUTION AND EXPERIMENTAL INVESTIGATION

Two fundamental challenges in investigation of nonlinear behavior of cantilever beam are the reliability of developed theory in facing with the reality and selecting the proper assumptions for solving the theory-provided equation. In this study, one of the most applicable theory and assumption for analyzing the nonlinear behavior of the cantilever beam is examined analytically and experimentally. The theory is concerned with the slender inextensible cantilever beam with large deformation nonlinearity, and the assumption is using the first-mode discretization in dealing with the partial differential equation provided by the theory. In the analytical study, firstly the equation of motion is derived based on the theory of large deformable inextensible beam. Then, the partial differential equation of motion is discretized using the Galerkin method via the assumption of the first mode. An exact solution to the obtained nonlinear ordinary differential equation is developed, because the available semi analytical and approximated methods, due to their limitations, are not always sufficiently reliable. Finally, an experiment set-up is developed to measure the nonlinear frequency of oscillations of an aluminum beam within a domain of initial displacement. The results show that the proposed analytical method has excellent convergence with experimental data.

1. Introduction

Slender and long beam is one of the important models for investigation of the vibration behavior of engineering structures such as airplane wing, long bridge and helicopter blades [1]]. In the case of small deformation, classic linear theories

[1] *Department of Aerospace Engineering, Space Research Institute, Malek-Ashtar University of Technology, Tehran, Iran. Emails: j_omidi@mut.ac.ir, mahdishayanmehr@mut.ac.ir*
[2] *Department of New Sciences and Technologies, University of Tehran, Tehran, Iran. Email: sazesh@ut.ac.ir*

would suffice for modelling of these beams. But, in the case of large deflections, for accurate modeling several nonlinearities need to be included. Nonlinear inertia and curvature, shear deformation, Poisson effects and warping are the most important factors of nonlinearities that affect the equation of motion [2]. According to the equations of motion formulated by Crespo da Silva and Glynn [3], in many researches one assumes that, in the case of inextensible long slender beams with large deflections, warping, shear deformations, and Poisson effects are negligible [4–7]. Crespo da Silva and Glynn also showed that the generally neglected nonlinear terms due to curvature are of the same order as the nonlinear terms due to inertia, and that the curvature terms may have a significant influence on the response of the system [8].

In developing the theory of inextensible long slender beam, some nonlinear terms are neglected and some nonlinear terms are taken into account, therefore this theory should be examined to determine the domain of validity with respect to deformation.

Another issue lies in the extracting nonlinear ordinary differential equation (ODE) from the partial differential equation (PDE) provided by the theory of nonlinear long inextensible beam. For this purpose, different approximations based on different numbers of mode shapes are utilized. Nevertheless, the first mode discretization via the Galerkin method is one of the traditional approaches to study the nonlinear behavior of the aforementioned beams [4, 9–13].

The assumption of using the first mode for discretizing the PDE is also needed to be evaluated experimentally to obtain the domain of validity of this assumption.

For solving the derived nonlinear ODE, several approximate methods, such as perturbation method [14–16] and harmonic balance method (HBM) [17–20] are used. These methods usually have some limitations because of their approximate nature. There are some disadvantages in the perturbation method. Generally, all perturbation methods are based on small parameter (ε) and, where the small parameter becomes larger, the perturbation method is no longer valid [21]. Some typical numerical examples show that the error is increasing if the parameter ε becomes larger [22]. The limitations of the perturbation method were also pointed out in [23]. In the case of HBM, there are some limitations which are clearly defined in [24], such as that the higher harmonic terms should have small amplitudes. The mentioned limitations of approximate methods can significantly affects the accuracy of the response obtained from the applied solving technique.

Based on the literature, there is a need to evaluate the theory and the first-mode assumption for obtaining the domain of displacement where the aforementioned theory and assumption are applicable. The remaining important issue is the lack of accuracy of the approximate solving technique in dealing with the nonlinear equation of motion. Therefore, developing an exact solution to the nonlinear equation of motion would provide a better deduction.

In the present study, firstly the equations of motion of the geometrical nonlinear cantilever beams are derived based on the variation approach and Hamilton principle. Next, the Galerkin method with the first-mode assumption is used to discretize the PDE to the nonlinear ODE. An exact solution for the nonlinear ODE is developed to calculate the nonlinear frequency of oscillation. In the following, an experimental set-up with infrared sensor is built to detect the nonlinear frequency of an aluminum beam under large-amplitude vibration. The comparison between the analytical and experimental results clearly indicates the domain of oscillation amplitudes where the theory and the first-mode assumption can be accurately applied. This theory and assumption is used to different modelling in various areas including nanomechanical resonators [25], noncontacting atomic force microscopes [26], beams with lumped mass [27], NEMS and MEMS [28], microcantilevers [29], etc. Therefore, the result of the study can be used for a vast domain of applications which deal with the inextensible geometrical nonlinear cantilever beam with homogenous material.

2. Theoretical

Based on the literature review, it is shown that Crespo da Silva and Glynn theory [3] is the most applicable idea in the case of slender inextensible beam with large deformation nonlinearity. Also it is shown that first-mode discretization is a proper assumption for deriving the nonlinear ordinary differential equation of motion. So at first, based on the mentioned theory, the nonlinear equation is derived for the flexural motion of beams with large deformation. Then, the first mode assumption based on Galerkin method is selected for discretization PDE to the nonlinear ODE. Finally, an exact solution is developed in order to calculate the nonlinear frequency of oscillation for cantilever beams.

2.1. Equation of Motion

In the first step of theoretical study, Crespo da Silva and Glynn beam theory is utilized to derive the equation of motion for an elastic, homogenous, cantilever, long and slender beam by assuming the flexural motion (Fig. 1).

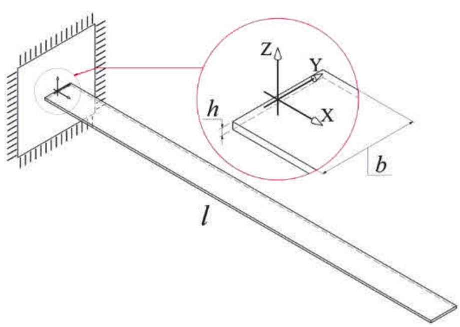

Fig. 1. A schematic of a long slender beam

A schematic of large-deflection of cantilever beam is shown in Fig. 2. In this figure, ρ is the radius of curvature, θ is angle of rotation, $w(x,t)$ is vertical displacement, $u(x,t)$ is axial displacement, ds is related to beam length direction and dx is related to x direction, l is the length of beam, z is axes in vertical direction, σ_{xx} is the stress, ε_{xx} is the strain, E is the Young's modulus, V is the potential energy and A is the cross section of beam.

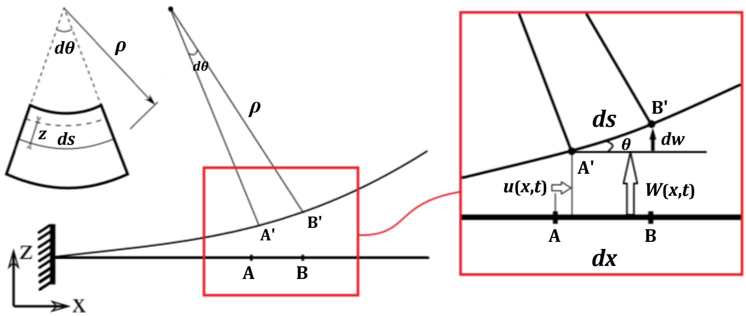

Fig. 2. Large deflection of cantilever beam

Based on the selected theory, it is assumed that the beam's neutral axis is inextensional and ds is equal to dx. So, the flexural displacement for a cantilever beam (see Fig. 2) is given by the following equation:

$$\begin{aligned} &\sin(\theta) = \frac{dw}{ds} = \frac{dw}{dx}\,; \qquad \theta = \sin^{-1}(w'); \\ &\rho d\theta = ds = dx \quad \rightarrow \quad \frac{1}{\rho} = \frac{d\theta}{dx} = \theta' = \frac{\mathrm{w}''}{\sqrt{1-w'^2}}\,. \end{aligned} \tag{1}$$

According to Eq. (1) the kinetic and potential energy of the element can be obtained as follows:

Potential energy:

$$V = \frac{1}{2}\int\limits_{v} \sigma_{xx}\varepsilon_{xx}\,\mathrm{d}v, \tag{2}$$

where

$$\varepsilon_{xx} = \frac{\Delta l}{l} = \frac{(\rho - z)d\theta - \rho d\theta}{\rho d\theta} = \frac{-z}{\rho} = \frac{-zw''}{\sqrt{1-w'^2}}\,. \tag{3}$$

A large deformation of a structure does not necessarily mean the presence of large strains. Under large rigid-body rotations, cantilever beams undergo large deformations but strains remain small. Even when the rigid-body rotations are small, deformations will still be large for long beams. With respect to a coordinate system co-rotated with the rigid-body movement, the relative displacements are small and the problem becomes linearly elastic [2]. So, the potential energy can be derived as:

$$V = \frac{1}{2}\int_A\int_0^l E\varepsilon_{xx}^2\,\mathrm{d}A\,\mathrm{d}x = \frac{1}{2}\int_A\int_0^l E\frac{z^2w''^2}{1-w'^2}\,\mathrm{d}x =$$
$$= \frac{1}{2}\int_0^l EIw''^2\left(1+w'^2\right)\mathrm{d}x \quad\Rightarrow\quad V = \frac{1}{2}\int_0^l EI\left(w''^2+w''^2w'^2\right)\mathrm{d}x. \tag{4}$$

Kinetic energy:

$$T = \frac{1}{2}\int_0^l m\left(\dot{u}^2+\dot{w}^2\right)\mathrm{d}x, \tag{5}$$

where

$$\begin{aligned} du &= dx - ds(\cos(\theta)) = dx - dx(\cos(\theta)) = dx(1-cos(\theta)),\\ \frac{du}{dx} &= 1-\cos(\theta) = 1-\sqrt{1-\sin^2(\theta)} = 1-\sqrt{1-w'^2} \cong \frac{1}{2}w'^2 \\ &\rightarrow\quad u = \int_0^x \frac{w'^2}{2}\,\mathrm{d}x. \end{aligned} \tag{6}$$

So,

$$T = \frac{1}{2}\int_0^l m\left[(\dot{w}^2) + \left(\frac{\partial}{\partial t}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)^2\right]\mathrm{d}x. \tag{7}$$

Using the Hamiltonian principle the following equation is obtained:

$$\int_{t_1}^{t_2}\delta(V-T)\,\mathrm{d}t = 0 \quad\Rightarrow\quad \int_{t_1}^{t_2}(\delta V-\delta T)\,\mathrm{d}t = 0. \tag{8}$$

For $\int_{t_1}^{t_2}(\delta V)\,\mathrm{d}t$,

$$\begin{aligned} \delta V &= \frac{1}{2}\delta\int_0^l EI\left(w''^2+w''^2w'^2\right)\mathrm{d}x \\ &= \int_0^l \frac{EI}{2}\left(\delta w''^2\right)\mathrm{d}x + \int_0^l \frac{EI}{2}\left(\delta\left(w''^2w'^2\right)\right)\mathrm{d}x. \end{aligned} \tag{9}$$

Also, for $\int_0^l \frac{EI}{2}\left(\delta w''^2\right)\,\mathrm{d}x$, by integration by parts:

$$\int_0^l \frac{EI}{2}\left(\delta w''^2\right)\,\mathrm{d}x = \int_0^l EIw''(\delta w'')\,\mathrm{d}x = EI\left[w''\delta w'\Big|_0^l - \int_0^l w'''(\delta w')\,\mathrm{d}x\right]$$

$$= EI\left[w''\delta w'\Big|_0^l - w'''\delta w\Big|_0^l + \int_0^l w^{(IV)}(\delta w)\,\mathrm{d}x\right]. \tag{10}$$

Based on initial condition,

$$\begin{gathered} w(0,t) = w'(0,t) = w''(l,t) = w'''(l,t) = 0 \\ \Rightarrow \quad 0 - 0 + EI\int_0^l w^{(IV)}(\delta w)\,\mathrm{d}x. \end{gathered} \tag{11}$$

For term $\int_0^l \frac{EI}{2}\left(\delta(w''^2 w'^2)\right)\,\mathrm{d}x$ and the same approach:

$$\int_0^l EIw''w'(\delta(w''w'))\,\mathrm{d}x = \int_0^l EIw''w'(w'\delta w'' + w''\delta w')\,\mathrm{d}x$$

$$= \left[\int_0^l EIw''^2 w'(\delta w')\,\mathrm{d}x\right] + \left[\int_0^l EIw''w'^2(\delta w'')\,\mathrm{d}x\right]$$

$$= \left[EIw''^2 w'(\delta w')\Big|_0^l - \int_0^l EI(w''^2 w')'(\delta w)dx\right] +$$

$$\left[EI(w''w'^2)\delta w'\Big|_0^l - \int_0^l EI(w''w'^2)'(\delta w')\,\mathrm{d}x - \right.$$

$$\left. EI(w''w'^2)'\delta w\Big|_0^l + \int_0^l EI(w''w'^2)''(\delta w)\,\mathrm{d}x\right]. \tag{12}$$

Based on boundary condition,

$$w(0,t) = w'(0,t) = w''(l,t) = w'''(l,t) = 0 \;\Rightarrow\; EI\int_0^l ((w''w')'w')'\,\delta w\,\mathrm{d}x. \tag{13}$$

So,

$$\int_{t_1}^{t_2} (\delta V)\,\mathrm{d}t = \int_{t_1}^{t_2} \left(\int_0^l \left(w^{(IV)} + \frac{1}{2}\left((w''w')'w'\right)' \right) \delta w\,\mathrm{d}x \right) \mathrm{d}t. \tag{14}$$

For $\int_{t_1}^{t_2} (\delta T)\,\mathrm{d}t$, where

$$\begin{aligned}\delta T &= \frac{1}{2}\delta \int_0^l m\left[(\dot{w}^2) + \left(\frac{\delta}{\delta t}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x \right) \right)^2 \right] \mathrm{d}x \\ &= \frac{1}{2}\int_0^l (m\delta\dot{w}^2)\,\mathrm{d}t + \frac{1}{2}\int_0^l m\delta\left[\frac{\delta}{\delta t}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x \right) \right]^2 \mathrm{d}x.\end{aligned} \tag{15}$$

So,

$$\begin{aligned}\int_{t_1}^{t_2} \delta T\,\mathrm{d}t &= \int_{t_1}^{t_2} \left[\frac{1}{2}\int_0^l (m\delta\dot{w}^2)\,\mathrm{d}x + \frac{1}{2}\int_0^l m\delta\left(\frac{\partial}{\partial t}\left(\int_0^l \frac{w'^2}{2}\,\mathrm{d}x \right) \right)^2 \mathrm{d}x \right] \mathrm{d}t = \\ &\int_{t_1}^{t_2} \left(\frac{1}{2}\int_0^l (m\delta\dot{w}^2)\,\mathrm{d}x \right) \mathrm{d}t + \int_{t_1}^{t_2} \left[\frac{1}{2}\int_0^l m\delta\left(\frac{\partial}{\partial t}\left(\int_0^l \frac{w'^2}{2}\,\mathrm{d}x \right) \right)^2 \mathrm{d}x \right] \mathrm{d}t.\end{aligned} \tag{16}$$

For term $\int_{t_1}^{t_2} \left(\frac{1}{2}\int_0^l \left(m\delta\dot{w}^2 \right) \mathrm{d}x \right) \mathrm{d}t$,

$$\begin{aligned}\int_{t_1}^{t_2} \left[\frac{1}{2}\int_0^l (m\delta\dot{w}^2)\,\mathrm{d}x \right] \mathrm{d}t &= m\int_0^l \int_{t_1}^{t_2} \left((\dot{w}\delta\dot{w})\,\mathrm{d}t \right) \mathrm{d}x \\ &= m\int_{t_1}^{t_2} \left[\dot{w}\delta w\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\ddot{w}\delta w)\,\mathrm{d}t \right] \mathrm{d}x,\end{aligned} \tag{17}$$

where

$$\delta w_{@t_1\&t_2} = 0 \quad \Rightarrow \quad m\int_0^1 \int_{t_1}^{t_2} \left((\ddot{w}\delta w)\,\mathrm{d}t \right) \mathrm{d}x. \tag{18}$$

For term $\int_{t_1}^{t_2} \frac{1}{2} \int\limits_0^l m\delta \left[\frac{\partial}{\partial t}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]^2 \mathrm{d}x\,\mathrm{d}t$,

$$
\begin{aligned}
&\int\limits_{t_1}^{t_2} \frac{1}{2} \int\limits_0^l m\delta\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]^2 \mathrm{d}x\,\mathrm{d}t \\
&= \int\limits_{t_1}^{t_2}\int\limits_0^l m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\delta\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\mathrm{d}x\,\mathrm{d}t \\
&= \int\limits_{t_1}^{t_2}\int\limits_0^l m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\left[\frac{\partial}{\partial t}\left(\int\limits_0^x w'\delta w'\,\mathrm{d}x\right)\right]\mathrm{d}x\,\mathrm{d}t \\
&= \int\limits_{t_1}^{t_2}\int\limits_0^l m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\left[\frac{\partial}{\partial t}\left(w'\delta w\Big|_0^x - \int\limits_0^x w''\delta w\,\mathrm{d}x\right)\right]\mathrm{d}x\,\mathrm{d}t - \\
&\quad \int\limits_{t_1}^{t_2}\int\limits_0^l m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\left[\frac{\partial}{\partial t}(w'\delta w)\right]\mathrm{d}x\,\mathrm{d}t - \\
&\quad \int\limits_{t_1}^{t_2}\int\limits_0^l m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\left[\frac{\partial}{\partial t}\left(\int\limits_0^x w''\delta w\,\mathrm{d}x\right)\right]\mathrm{d}x\,\mathrm{d}t \\
&= \int\limits_0^l\int\limits_{t_1}^{t_2} m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\left[\frac{\partial}{\partial t}(w'\delta w)\right]\mathrm{d}x\,\mathrm{d}t - \\
&\quad \int\limits_0^l\int\limits_{t_1}^{t_2} m\left[\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right]\left[\frac{\partial}{\partial t}\left(\int\limits_0^x w''\delta w\,\mathrm{d}x\right)\right]\mathrm{d}x\,\mathrm{d}t \\
&= \left\{\left[\int\limits_0^l m\left(\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)\left(w'\delta w\Big|_{t_1}^{t_2}\right)\right] - \right. \\
&\quad \left.\left[\int\limits_0^l\int\limits_{t_1}^{t_2} m\left(\frac{\partial^2}{\partial t^2}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)(w'\delta w)\,\mathrm{d}t\,\mathrm{d}x\right]\right\} - \\
&\quad \left\{\left[\int\limits_0^l m\left(\frac{\partial}{\partial t}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)\left(\int\limits_0^x w''\delta w\,\mathrm{d}x\Big|_{t_1}^{t_2}\right)\right] - \right. \\
&\quad \left.\left[\int\limits_0^l\int\limits_{t_1}^{t_2} m\left(\frac{\partial^2}{\partial t^2}\left(\int\limits_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)\left(\int\limits_0^x w''\delta w\,\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}x\right]\right\},
\end{aligned}
\tag{19}
$$

where

$$\delta w_{@t_1\&t_2} = 0 \quad \Rightarrow$$
$$-\left[\int_{t_1}^{t_2}\int_0^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)(w'\delta w)\,\mathrm{d}t\,\mathrm{d}x\right]$$
$$+\left[\int_{t_1}^{t_2}\int_0^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)\left(\int_0^x w''\delta w\,\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}x\right]. \tag{20}$$

The Eq. (22) can be simplified using the following equation,

$$\int_0^l g(x)\left(\int_0^x f(x)\delta w\,\mathrm{d}x\right)\mathrm{d}x = \int_0^l \left(\int_x^l g(x)\,\mathrm{d}x\right) f(x)\delta w\,\mathrm{d}x. \tag{21}$$

So,

$$-\left[\int_{t_1}^{t_2}\int_0^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)(w'\delta w)\,\mathrm{d}t\,\mathrm{d}x\right] +$$
$$\left[\int_{t_1}^{t_2}\int_0^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)\left(\int_0^x w''\delta w\,\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}x\right]$$
$$= -\left[\int_{t_1}^{t_2}\int_0^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w^2}{2}\,\mathrm{d}x\right)\right)(w'\delta w)\,\mathrm{d}t\,\mathrm{d}x\right] +$$
$$\left[\int_{t_1}^{t_2}\int_0^l g(x)\left(\int_0^x f(x)\delta w\,\mathrm{d}x\right)\mathrm{d}t\,\mathrm{d}x\right], \tag{22}$$

where

$$g(x) = m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right); \qquad f(x) = w''. \tag{23}$$

Consequently:

$$\left[\int_{t_1}^{t_2}\int_0^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)(w'\delta w)\,\mathrm{d}t\,\mathrm{d}x\right] + \left[\int_{t_1}^{t_2}\int_0^l\int_x^l m\left(\frac{\partial^2}{\partial t^2}\left(\int_0^x \frac{w'^2}{2}\,\mathrm{d}x\right)\right)w''\partial w\,\mathrm{d}t\,\mathrm{d}x\right] = -\int_{t_1}^{t_2}\int_0^l \frac{1}{2}m\left\{w'\int_l^x\left[\frac{\partial^2}{\partial t^2}\left(\int_0^x w'^2\,\mathrm{d}x\right)\right]\right\}'\delta w\,\mathrm{d}x\,\mathrm{d}t. \tag{24}$$

Finally:

$$m\ddot{w} + EI\xi w^{(IV)} = EI\,[w'(w'w'')']' - \frac{1}{2}m\left\{w'\int_l^x\left(\frac{\partial^2}{\partial t^2}\int_0^x w'^2\,\mathrm{d}x\right)\right\}'. \tag{25}$$

2.2. First-mode discretization

In this section, Galerkin method via assumption of the first-mode shape in the case of Cantilever beam is used for converting the derived PDE (Eq. (22)) to ODE. In this discretization method, one essentially postulates the system response in the form

$$w(x,t) = \varphi(x)q(t), \tag{26}$$

where $\varphi(x)$ and $q(t)$ are the first-mode shape and generalized coordinate of the system, respectively. The mentioned mode shape of the cantilever beam can be written as:

$$\begin{gathered}\cos(\beta x) - \cosh(\beta x) + \alpha_1\left[\sin(\beta x) - \sinh(\beta x)\right],\\ \alpha_1 = \frac{\cos(\beta l) + \cosh(\beta l)}{\sin(\beta l) + \sinh(\beta l)}, \qquad \beta l = 1.8751.\end{gathered} \tag{27}$$

Here, Eqs. (26) and (27) are applied to solve Eq. (25) and the ODE is derived as

$$m[1 + a_1 q^2(t)]\ddot{q}(t) + a_2 q(t)\dot{q}^2(t) + k_1 q(t) + k_2 q^3(t) = 0. \tag{28}$$

Which,

$$a_1 = \frac{4.6}{l^2}, \qquad a_2 = \frac{4.6m}{l^2}, \qquad k_1 = \frac{12.36EI}{l^4}, \qquad k_2 = \frac{40.4EI}{l^6}. \tag{29}$$

2.3. Exact solution for the equation of motion to calculate the nonlinear frequency of oscillation

In the final section of the theoretical investigation, an exact solution is developed based on the obtained nonlinear ODE of cantilever beam (Eq. (28)). The developed solution presents the nonlinear frequency of oscillation for a long slender beam. At first, the Eq. (28) is multiplied by $\dot{q}(t)$,

$$m\ddot{q}(t)\dot{q}(t) + a_2\left(q^2(t)\ddot{q}(t)\dot{q}(t) + \dot{q}^3(t)q(t)\right) + k_1\dot{q}(t)q(t) + k_2\dot{q}(t)q^3(t) = 0 \tag{30}$$

or

$$m\ddot{q}(t)\dot{q}(t) + \frac{a_2}{2}\frac{d(\dot{q}^2(t)q^2(t))}{dt} + k_1\dot{q}(t)q(t) + k_2\dot{q}(t)q^3(t) = 0. \tag{31}$$

Then, by integrating, the following equation is derived.

$$m\frac{\dot{q}^2(t)}{2} + \frac{a_2}{2}\dot{q}^2(t)q^2(t) + \frac{k_1}{2}q^2(t) + \frac{k_2}{4}q^4(t) = cte\,. \tag{32}$$

In Eq. (32), cte term is a constant value and can be obtained from initial condition of motion (initial velocity and displacement). Similarly, Eq. (32) can be rewritten as

$$\dot{q}^2(t)B(q(t)) + A(q(t)) = cte\,, \tag{33}$$

where

$$B(q(t)) = \frac{m}{2} + \frac{a_2}{2}q^2(t), \qquad A(q(t)) = \frac{k_1}{2}q^2(t) + \frac{k_2}{4}q^4(t). \tag{34}$$

So,

$$\frac{d(q(t))}{dt} = \sqrt{\frac{A(q(t_0)) - A(q(t))}{B(q(t))}} \quad\Rightarrow\quad dt = \frac{d(q(t))}{\sqrt{\dfrac{A(q(t_0)) - A(q(t))}{B(q(t))}}}\,. \tag{35}$$

Integrating Eq. (35), one obtains the period of oscillation, written as

$$T = \int_0^t \mathrm{d}t = 4\int_0^{q(t_0)} \frac{d(q(t))}{\sqrt{\dfrac{A(q(t_0)) - A(q(t))}{B(q(t))}}}\,. \tag{36}$$

Based on Eq. (36), the nonlinear frequency equation of system can be obtained as follows,

$$\omega = \frac{2\pi}{\displaystyle\int_0^{q(t_0)} \frac{d(q(t))}{\sqrt{\dfrac{A(q(t_0)) - A(q(t))}{B(q(t))}}}}\,. \tag{37}$$

3. Experimental

In the previous section, an exact solution for prediction of nonlinear free vibration of long slender beams was developed. For verifying the result of this solution, an accurate and sensitive experimental test is needed.

3.1. Test set-up

In the experimental process and settings of the test, there are some noises and errors that decrease the accuracy of the test. These noises and errors are classified in three main groups, mechanical and electrical noises and errors and computational errors.

The important sources of noises and errors in mechanical structure are the loose-fitting of fixture, transforming the shakes and tremors from the base to the electrical equipment and specimen. The most important challenges for electrical part are recording all the data without any loss or wrong data. Also, in the case of long slender beams, the common contact sensor is the source of errors. Finally, computational errors could occur during receiving data, translating data to decimal format and analyzing of data correctness with respect to noise in previous noise groups.

In order to overcome these defects, a sensitive and accurate set-up was designed and prepared. This set-up has 3 different parts, mechanical, electrical and computational ones. The test set-up is shown in Fig. 3.

The mechanical part consists of a strong adjustable clamp (fixture) applied in order to fix the specimen for vibration test in various lengths. Also, it locates the electrical equipment with respect to slender beam. The structure is designed with high rigidity to operate the vibration test without any shake. The housing of electrical equipment is isolated from the main structure to avoid transforming the mechanical noises.

The most important advantage of the electrical part of this set-up is the application of a non-contact sensor. As noted above, in the case of slender beams, this kind of sensor is suitable for accurate detecting. So, the optical non-contacted sensor with a source of light is utilized. In this system, a noise could occur in sensing the light beam when this beam is not perpendicular to the slender beam and also when light beams converge to the sensor detecting angle. So, the selected optical sensor must have sufficient response time and small detecting angle. Also, a lens is required in order to improve the quality of detection. So, the HY870 infrared sensor is selected, which has the response time between 4–20 μs and the frame of this sensor has a narrow slot with 0.5 millimeter width in order to minimize the detecting angle. Finally, the electrical part includes a driver for BIOS, and for controlling the sensor, recording the data and sending them to the computational unit. It must be noted that, in order to reduce the mechanical noise from base structure,

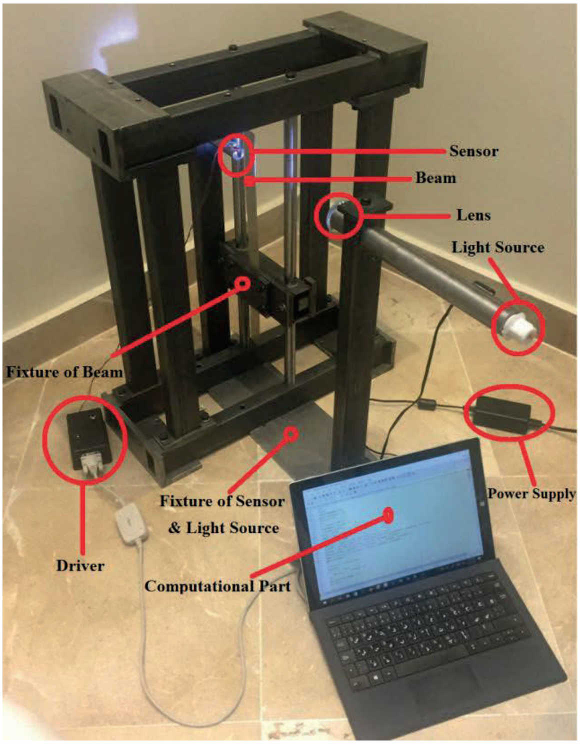

Fig. 3. Test set-up

the sensor and the light beam are separately located perpendicularly to the slender beam, opposed to the beam when it is in rest.

In the computational part, all the received data in hexadecimal format, as well as some additional data for indexing and packaging, must be processed. These processes include receiving data, checking the package of received data, translating the format of all raw data from hexadecimal to decimal format and summing up all times of each period to calculate frequencies of oscillation. Also, the computational part detects if there exist incorrect data due to previous noises or errors of communication between the driver and the computational unit. At the end, the computational part draws the frequency diagram with respect to the number of frequencies until rest.

3.2. Test Procedure

For vibrational test of slender long beams, homogenous aluminum (2024 T3) samples were selected. The geometric and mechanical properties of the specimen are shown in Fig. 4 and Table 1.

After fixing the sample on the set-up, a certain initial displacement is applied on the beam and the frequencies of oscillation for each period from the start to

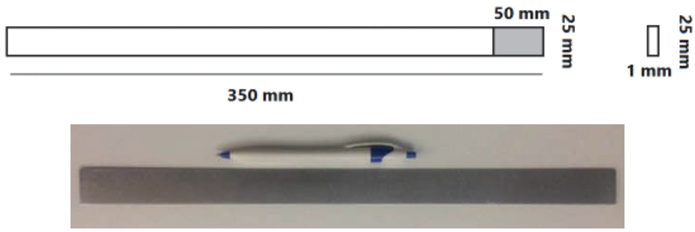

Fig. 4. Geometry and dimensions of the test specimen

Table 1.

Mechanical property of AL 2024 T3

E (GPa)	G (GPa)	ν	ρ (gr/cm^3)
73.1	27.5	0.33	2.78

the stop are recorded. This process is repeated several times for different initial displacement.

4. Results and discussion

In this section, at first the natural frequency predicted by each of the theoretical and the experimental approaches for small displacement is reported. The summary of the results is shown in Table 2.

Table 2.

Theoretical and experimental natural frequencies of cantilever beam

Method	Natural frequency, ω_0 (Hz)
Theoretical	9.20
Experimental	
1	8.99
2	8.99
3	8.98
4	8.99
5	8.97
Average	8.984

After that, the large deflection is applied for evaluation of free oscillation of the samples. As an example, the results of four experimental investigations are shown in Fig. 5.

The results of Fig. 5 show that, for large displacements, the values of initial frequencies of oscillations are higher than the natural frequency. By continuing the oscillations, the frequencies have been decreased and finally converge to the natural frequency of the beam. This process occurred for all of the tests based on the different initial excitations.

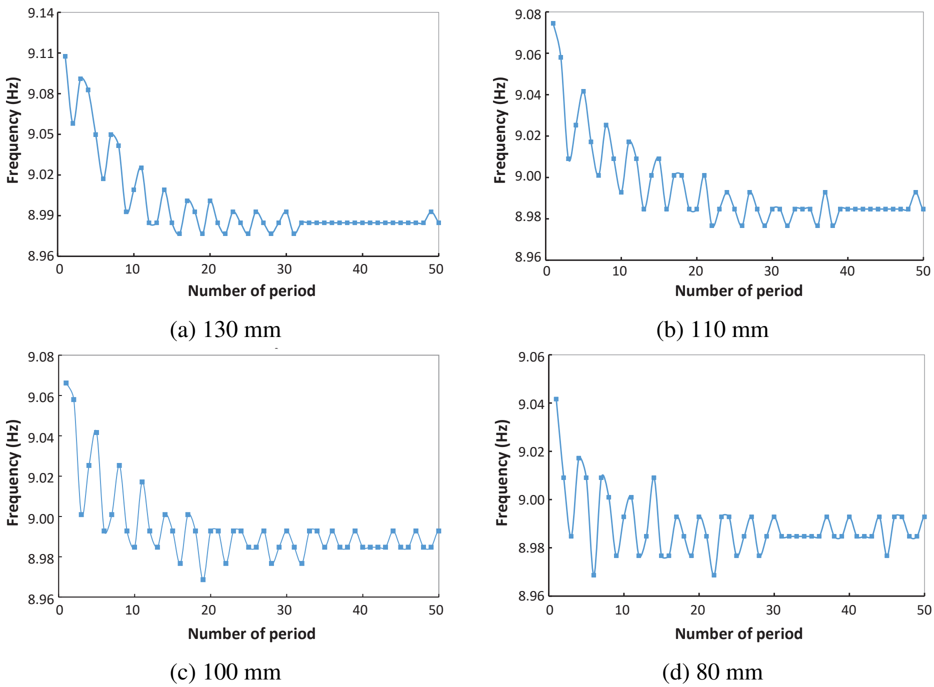

Fig. 5. The frequencies of free oscillation of cantilever beam with length 300 mm under different initial amplitudes

Next, the non-dimensional frequencies of free oscillation based on the initial deflection of oscillation (normalized with the length of beam) are plotted in Fig. 6. It should be noted that these non-dimensional frequencies are normalized with the

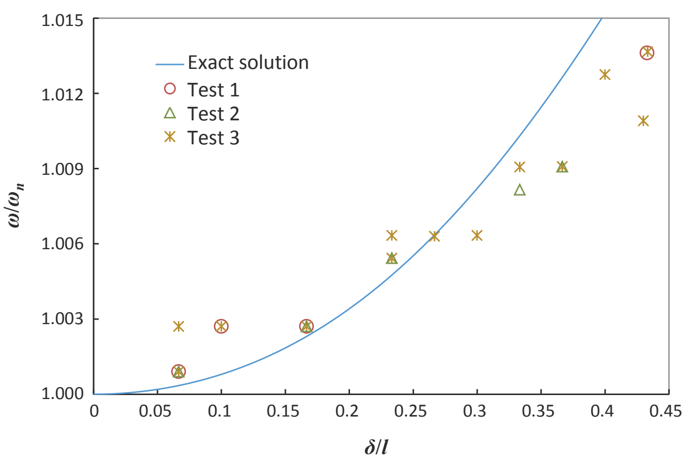

Fig. 6. Non dimensional frequencies of free oscillation for cantilever aluminum beam under initial large deflections

natural frequency of the models, and these results are based on the initial oscillation of system.

The results of Fig. 6 show that, for initial displacement smaller than one third of the beam length, the differences between theoretically developed approach and experimental result are less than 3%. Also, these tests show that the differences and errors for higher range of initial excitations have been increased.

5. Conclusion

In this paper, based on the two fundamental challenges in investigation of nonlinear behavior of cantilever beams, the theoretical and experimental studies are carried out. As the first challenge, for selecting the proper theory, based on the past studies we concluded that Crespo da Silva and Glynn theory is the most acceptable in the case of cantilever beams. This theory can be used for deriving the equation of motion for inextensible long slender beams with large deflections.

After deriving the equation of motion, selecting the proper assumptions is the most important challenge. Therefore, based on the traditional approach, the PDE of motion is discretized using the Galerkin method via the assumption of the first mode. Finally, because of the lack of accuracy of the approximate methods, an exact solution is developed in order to solve the nonlinear ODE and subsequently calculate the nonlinear frequency of oscillation.

In the following, an experimental test is established for examining the validity domain of the mentioned theoretical approach. For the experimental test, a sensitive non-contact set-up is designed and prepared. This set-up records all the frequencies of free vibration for a long slender aluminum sample with high accuracy.

The comparison between the results shows that the theoretical natural frequency differs by nearly 2.5% from the tests. Subsequently, the nonlinear investigations show that the results of theoretical approach and experimental tests differ by less than 3% based on initial excitations up to 33% of beam length. Also, larger initial displacements lead to greater differences and errors. In summary, it can be concluded that the developed theoretical approach is valid for initial displacements from zero to one third of the beam length.

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