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EXAMPLES OF FRACTAL OBJECTS GENERATED AS THE UNION OF TERMS OF A SEQUENCE OF SETS USING THE IFS METHOD

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Abstract: The Iterated Function System is a commonly used method of generating fractals. Iterating Hutchinson's operator, which is a set (union) of contraction mappings, leads to the construction of an attractor which is most commonly a fractal. Fractals obtained in this way have the property that their area or volume decreases as the iteration proceeds. Using the analogy of a geometric series, a modification of this method of fractal generation is proposed, based on the union of a sequence of sets, which enables the construction of geometric objects with an increasing volume, while preserving the fundamental fractal nature of the object.

Keywords: fractal, iterated function system, Hutchinson's operator

1 Classical fractals

The majority of classical fractals are constructed using an algorithm for removing fragments of a starting set, called the initiator. Repeated iterations of this procedure lead to a reduction in the area or volume of the set. The Sierpiński triangle and carpet are examples of two dimensional sets with zero area. The corresponding three dimensional sets with zero volume are the Sierpiński pyramid and the Menger sponge, respectively.

In the case of the Sierpiński pyramid, the initiator is a regular tetrahedron with sides of length *a*. The midpoints of the edges of this tetrahedron form the corners of a regular octahedron with sides of length *a*/2, which we remove from the original tetrahedron in the first stage of construction. The remaining set, which is called the generator, consists of four regular tetrahedrons with sides of length *a*/2. Repeating this procedure on each of the resulting tetrahedrons for *n* steps leads to the construction of the Sierpiński pyramid, whose volume tends to zero as *n* tends to infinity.

In the case of the Menger sponge, the initiator is a cube with sides of length *a*, which we initially divide into 27 contiguous cubes with sides of length *a*/3. In the first step of the construction, we remove the most central of these 27 cubes, together with the six cubes which share a common face with it. Thus, in the first step leading to the construction of the generator, seven cubes are removed and 20 remain. Repeating this procedure on each of the resulting cubes for *n* steps leads to the construction of the Menger cube, whose volume tends to zero as *n* tends to infinity.

2 The IFS Method

Independently of the descriptive procedure, the above objects can be generated using the iterated function system (IFS). The procedure for generating a fractal using the IFS can be described as follows:
Consider a given subset A of a complete metric space X = (X, d) contained in the space R^3, together with a Hutchinson operator W, which is the union of affine transformations [2]

$$W = \omega_1 \cup \omega_2 \cup \ldots \cup \omega_n. \quad (1)$$

From the properties of affine transformations, the result of the superposition of affine transformations is itself an affine transformation [1].

In this article we confine our considerations to homothety, translations and rotations. The superposition of such transformations can be presented in the following form:

$$\omega = \mu_s \circ \tau_{x,y,z} \circ \rho_{\varphi x, \varphi y, \varphi z}, \tag{2}$$

where μ_s is a homothety of scale factor s with centre at the origin O (if $|s|<1$ then this is a contraction), $\tau_{x,y,z}$ is a translation by the vector $[t_x , t_y , t_z]$ and $\rho_{\varphi x, \varphi y, \varphi z}$ is a rotation of angle φ_x with respect to the x-axis, φ_y with respect to the y-axis and φ_z with respect to the z-axis.

The order in which the transformations are performed is not arbitrary. In the examples considered here, it is assumed that the set A is first scaled by a factor s, then translated and finally rotated.

The effect of the Hutchinson operator on the set A can be described as the union of the results of transformations $\omega_1, \omega_2, \ldots, \omega_n$ on the set A:

$$W(A) = \omega_1(A) \cup \omega_2(A) \cup \ldots \cup \omega_n(A) =$$
$$= \mu_1(A) \circ \tau_1(A) \circ \rho_1(A) \cup \mu_2(A) \circ \tau_2(A) \circ \rho_2(A) \cup \ldots \cup \mu_n(A) \circ \tau_n(A) \circ \rho_n(A). \tag{3}$$

The initial set A_0 is transformed iteratively as follows:

$$A_1 = W(A_0),$$
$$A_2 = W(A_1) = W(W(A_0)),$$
$$A_{k+1} = W(A_k) \text{ for } k = 0, 1, 2,\ldots n. \tag{4}$$

If $\omega_1, \omega_2, \ldots, \omega_n$ are contraction mappings, iterating the Hutchinson operator W(A) creates a sequence of sets which tend in the limit to A_n, which is an attractor [4], i.e. fixed point [3], of the transformation W(A)

$$W(A_n) = A_n. \tag{5}$$

Example 1

Suppose the initial set A_0 is a tetrahedron with sides of length *a* with classical orientation in R^3 (see Fig. 1a), i.e. the geometric centre of the tetrahedron, defined to be the centre of the circumscribed sphere, is at the origin.

The Hutchinson operator W is the union of four transformations $\omega_1, \omega_2, \omega_3, \omega_4$, whose parameters are given in Table 1.

The attractor, which satisfies $W(A_n) = A_n$, is the Sierpiński pyramid.

Table 1

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/2	0	0	$a\sqrt{6}/8$	0	0	0
ω_2	1/2	0	$a\sqrt{3}/6$	$a\sqrt{6}/24$	0	0	0
ω_3	1/2	$a/4$	$-a\sqrt{3}/12$	$a\sqrt{6}/24$	0	0	0
ω_4	1/2	$-a/4$	$-a\sqrt{3}/12$	$a\sqrt{6}/24$	0	0	0

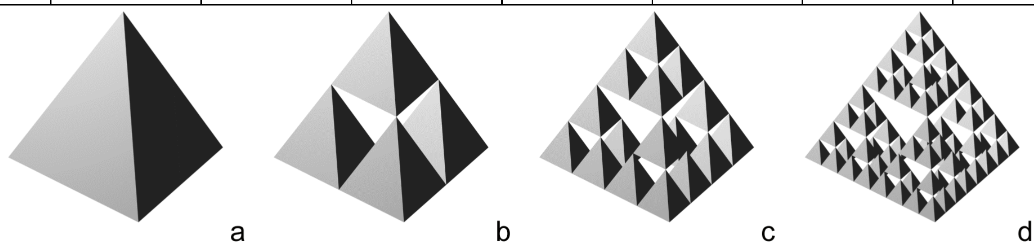

Figure 1: Successive steps in the construction of the Sierpiński pyramid

Example 2

The initial set B_0 is a cube with sides of length *b*. The Hutchinson operator W is the union of twenty transformations, ω_1, ω_2, …, ω_{20}, whose parameters are given in Table 2. The centre of the cube is at the origin. The attractor, which satisfies $W(B_n) = B_n$, is the Menger cube.

Table 2

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/3	*b*/3	0	*b*/3	0	0	0
ω_2	1/3	*b*/3	*b*/3	*b*/3	0	0	0
ω_3	1/3	0	*b*/3	*b*/3	0	0	0
ω_4	1/3	-*b*/3	*b*/3	*b*/3	0	0	0
ω_5	1/3	-*b*/3	0	*b*/3	0	0	0
ω_6	1/3	-*b*/3	-*b*/3	*b*/3	0	0	0
ω_7	1/3	0	-*b*/3	*b*/3	0	0	0
ω_8	1/3	*b*/3	-*b*/3	*b*/3	0	0	0
ω_9	1/3	*b*/3	*b*/3	0	0	0	0
ω_{10}	1/3	-*b*/3	*b*/3	0	0	0	0
ω_{11}	1/3	-*b*/3	-*b*/3	0	0	0	0
ω_{12}	1/3	*b*/3	-*a*/3	0	0	0	0
ω_{13}	1/3	*b*/3	0	-*b*/3	0	0	0
ω_{14}	1/3	*b*/3	*b*/3	-*b*/3	0	0	0
ω_{15}	1/3	0	*b*/3	-*b*/3	0	0	0
ω_{16}	1/3	-*b*/3	*b*/3	-*b*/3	0	0	0
ω_{17}	1/3	-*b*/3	0	-*b*/3	0	0	0
ω_{18}	1/3	-*b*/3	-*b*/3	-*b*/3	0	0	0
ω_{19}	1/3	0	-*b*/3	-*b*/3	0	0	0
ω_{20}	1/3	*b*/3	-*b*/3	-*b*/3	0	0	0

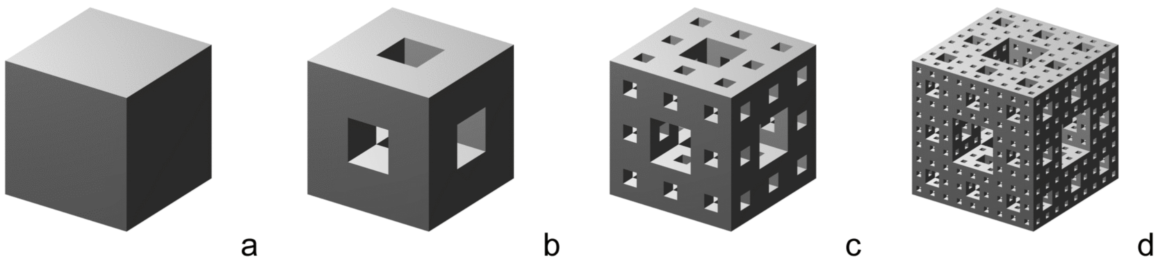

Figure 2: Sucessive steps in the construction of the Menger cube

3 Modified method of generating fractal objects using an iterated function system

A minor modification the IFS method enables us to generate fractal sets, whose volume increases as the iterative process progresses. As when the IFS method is used, a subset A of the complete metric space X = (X, d) contained in R^3 is transformed using a Hutchinson operator which is a union of contraction mappings

$$W(C) = \omega_1(C) \cup \omega_2(C) \cup \ldots \cup \omega_n(C). \quad (6)$$

Similarly to the previous examples, these contraction mappings are limited to the following transformations: homothety (of absolute scale factor s≤1), translations and rotations.

Iterating the Hutchinson operator W(A) generates the following sequence of sets:

$$C_{k+1} = W(C_k) \text{ for } k = 0, 1, \ldots, n. \quad (7)$$

The modification proposed is analogous to the concept of a geometric series and is based on summing the sets in this sequence.

$$UC_n = C_0 \cup C_1 \cup \ldots \cup C_n. \quad (8)$$

The limiting set of this sequence shows self-similarity properties – the most characteristic trait of fractals.

Example 3

The initial set C_0 is a regular octahedron with sides of length $c = a/2$, located in such a way that it is the figure removed from a tetrahedron with sides of length a at the first stage of generating the Sierpiński pyramid according to the descriptive procedure given above (see Fig. 3a).

The corresponding Hutchinson operator W(C) is the union of four transformations ω_1, ω_2, ω_3, ω_4, whose parameters are given in Table 3. The geometric centre of this octahedron, taken to be the centre of the circumscribed sphere, is at the origin.

Iterating the operator W(C) generates a sequence of sets, whose union UC_n, given by

$$UC_n = C_0 \cup C_1 \cup \ldots \cup C_n \quad \text{where } C_{k+1}=W(C_k) \text{ for } k = 0,1,\ldots,n \quad (9)$$

is the complement of the Sierpiński pyramid in the space of the tetrahedron with sides of length a, that is to say the set of the elements removed from the tetrahedron during the process generating the Sierpiński pyramid according to the descriptive procedure. We have

$$UC_n = A_n^c \quad (10)$$

Table 3

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/2	0	0	$c\sqrt{6}/4$	0	0	0
ω_2	1/2	0	$c\sqrt{3}/3$	$c\sqrt{6}/12$	0	0	0
ω_3	1/2	$c/2$	$-c\sqrt{3}/6$	$c\sqrt{6}/12$	0	0	0
ω_4	1/2	$-c/2$	$-c\sqrt{3}/6$	$c\sqrt{6}/12$	0	0	0

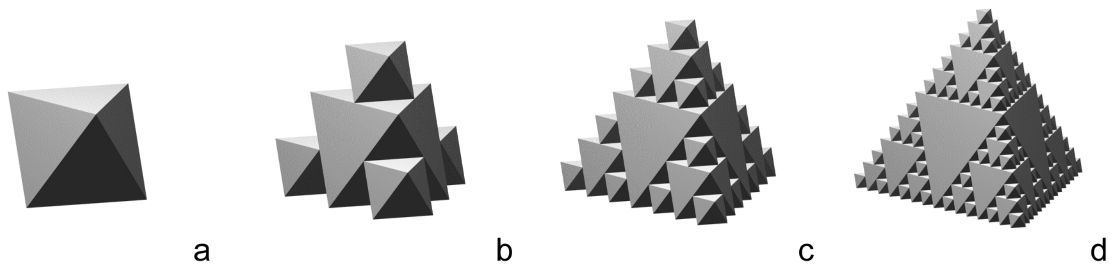

Figure 3: Successive steps in the construction of the complement to the Sierpiński pyramid $UC_n = A_n^c$

Example 4

In this example there are two phases in the generation of the limiting set. In the first phase the initial set D_0 is a cube with sides of length $d = b/3$ (Fig. 4a) and the Hutchinson operator T is the union of six transformations ψ_1, ψ_2, …, ψ_6, whose parameters are given in Table 4. We have

$$T(D) = \psi_1 \cup \psi_2 \cup \ldots \cup \psi_6. \quad (11)$$

The result of applying the operator $T(D_0)$ is the object $D_1 = E_0$, which is the initiator for the operator W(E) in the second phase of generating the limiting set. We have

$$T(D_0) = D_1 = E_0. \quad (12)$$

The centre of the cube D_0, as well as the geometric centre of the set E_0 and the fixed point of the homothety, is located at the origin.

Table 4

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ψ_1	1	*d*	0	0	0	0	0
ψ_2	1	-*d*	0	0	0	0	0
ψ_3	1	0	*d*	0	0	0	0
ψ_4	1	0	-*d*	0	0	0	0
ψ_5	1	0	0	*d*	0	0	0
ψ_6	1	0	0	-*d*	0	0	0

The Hutchinson operator W(E) is the union of twenty transformations $\omega_1, \omega_2, \ldots, \omega_{20}$, whose parameters are given in Table 5. We have

$$W(E) = \omega_1(E) \cup \omega_2(E) \cup \ldots \cup \omega_{20}(E). \qquad (13)$$

Table 5

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/3	*d*	0	*d*	0	0	0
ω_2	1/3	*d*	*d*	*d*	0	0	0
ω_3	1/3	0	*d*	*d*	0	0	0
ω_4	1/3	-*d*	*d*	*d*	0	0	0
ω_5	1/3	-*d*	0	*d*	0	0	0
ω_6	1/3	-*d*	-*d*	*d*	0	0	0
ω_7	1/3	0	-*d*	*d*	0	0	0
ω_8	1/3	*d*	-*d*	*d*	0	0	0
ω_9	1/3	*d*	*d*	0	0	0	0
ω_{10}	1/3	-*d*	*d*	0	0	0	0
ω_{11}	1/3	-*d*	-*d*	0	0	0	0
ω_{12}	1/3	*d*	-*d*	0	0	0	0
ω_{13}	1/3	*d*	0	-*d*	0	0	0
ω_{14}	1/3	*d*	*d*	-*d*	0	0	0
ω_{15}	1/3	0	*d*	-*d*	0	0	0
ω_{16}	1/3	-*d*	*d*	-*d*	0	0	0
ω_{17}	1/3	-*d*	0	-*d*	0	0	0
ω_{18}	1/3	-*d*	-*d*	-*d*	0	0	0
ω_{19}	1/3	0	-*d*	-*d*	0	0	0
ω_{20}	1/3	*d*	-*d*	-*d*	0	0	0

Iterating the Hutchinson operator W(E) generates a sequence of sets, whose union is given by

$$UE_n = E_0 \cup E_1 \cup \ldots \cup E_n, \text{ where } E_{k+1}=W(E_k). \qquad (14)$$

This is the complement of the Menger cube within the space of a cube with sides of length *b* and centre at the origin, because the sets in this union correspond exactly to the elements removed from such a cube during the construction of the Menger cube according to the descriptive procedure. We have

$$UE_n = B_n^{\,c}. \qquad (15)$$

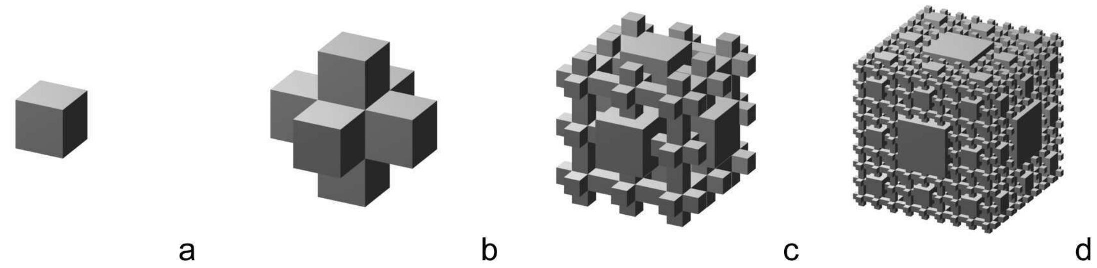

Figure 4: Successive steps in the construction of the complement to the Menger cube $UE_n = B_n^c$

Example 5

In this example the construction of the set is again done in two phases. The initial set A_0 is a tetrahedron with sides of length *a* (Fig. 5a). This set is transformed using ψ, which is the superposition of a homothety of scale factor 1, translation and rotation, as follows:

$$\psi_1(A_0) = (\mu_1 \circ \tau_1 \circ \rho_1)(A_0) = A_1. \quad (16)$$

Table 6

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ψ_1	1	0	0	$a\sqrt{6}/6$	0	180^o	180^o

The union of the sets A_0 and A_1 is the initiator F_0 of the operator W in the second phase of generating the limiting set. We have

$$F_0 = A_0 \cup A_1. \quad (17)$$

The Hutchinson operator W(F) is the result of superposing four transformations ω_1, ω_2, ω_3, ω_4, whose parameters are given in Table 7. We have

$$W(F) = \omega_1(F) \cup \omega_2(F) \cup \omega_3(F) \cup \omega_4(F). \quad (18)$$

The geometric centre of F_0, taken to be the centre of the circumscribed sphere, is at the origin.

Table 7

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/2	0	0	$a\sqrt{6}/8$	0	0	0
ω_2	1/2	0	$a\sqrt{3}/6$	$a\sqrt{6}/24$	0	0	0
ω_3	1/2	$a/4$	$-a\sqrt{3}/12$	$a\sqrt{6}/24$	0	0	0
ω_4	1/2	$-a/4$	$-a\sqrt{3}/12$	$a\sqrt{6}/24$	0	0	0

Iterating the Hutchinson operator W(F) generates a sequence of sets, whose union can be called the additive Sierpiński pyramid. We have

$$UF_n = F_0 \cup F_1 \cup \ldots \cup F_n \text{ where } F_{k+1} = W(F_k) \quad (19)$$

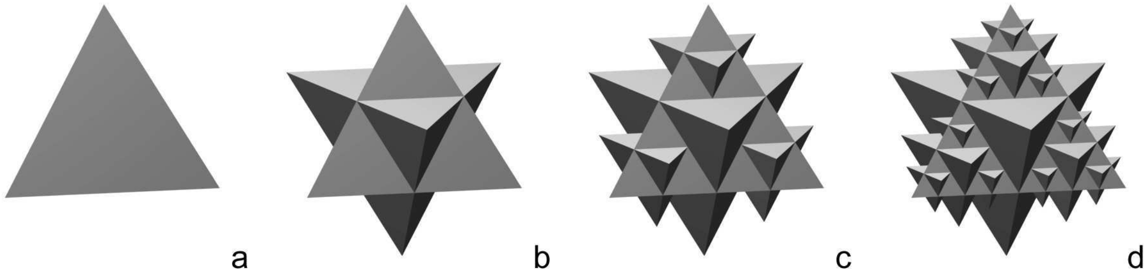

Figure 5: Successive steps in the construction of the additive Sierpiński pyramid

Example 6
The initial set B_0 is a cube with sides of length *b* (see Fig. 6a).
The operator W(B) is the union of twenty six transformations, whose parameters are given in Table 8. The centre of the cube is at the origin.

Table 8

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/3	*a*/3	0	*a*/3	0	0	0
ω_2	1/3	*a*/3	*a*/3	*a*/3	0	0	0
ω_3	1/3	0	*a*/3	*a*/3	0	0	0
ω_4	1/3	-*a*/3	*a*/3	*a*/3	0	0	0
ω_5	1/3	-*a*/3	0	*a*/3	0	0	0
ω_6	1/3	-*a*/3	-*a*/3	*a*/3	0	0	0
ω_7	1/3	0	-*a*/3	*a*/3	0	0	0
ω_8	1/3	*a*/3	-*a*/3	*a*/3	0	0	0
ω_9	1/3	*a*/3	*a*/3	0	0	0	0
ω_{10}	1/3	-*a*/3	*a*/3	0	0	0	0
ω_{11}	1/3	-*a*/3	-*a*/3	0	0	0	0
ω_{12}	1/3	*a*/3	-*a*/3	0	0	0	0
ω_{13}	1/3	*a*/3	0	-*a*/3	0	0	0
ω_{14}	1/3	*a*/3	*a*/3	-*a*/3	0	0	0
ω_{15}	1/3	0	*a*/3	-*a*/3	0	0	0
ω_{16}	1/3	-*a*/3	*a*/3	-*a*/3	0	0	0
ω_{17}	1/3	-*a*/3	0	-*a*/3	0	0	0
ω_{18}	1/3	-*a*/3	-*a*/3	-*a*/3	0	0	0
ω_{19}	1/3	0	-*a*/3	-*a*/3	0	0	0
ω_{20}	1/3	*a*/3	-*a*/3	-*a*/3	0	0	0
ω_{21}	1/3	2*a*/3	0	0	0	0	0
ω_{22}	1/3	-2*a*/3	0	0	0	0	0
ω_{23}	1/3	0	2*a*/3	0	0	0	0
ω_{24}	1/3	0	-2*a*/3	0	0	0	0
ω_{25}	1/3	0	0	2*a*/3	0	0	0
ω_{26}	1/3	0	0	-2*a*/3	0	0	0

Iterating the opearator W(B) generates a sequence of sets, whose union can be called the additive Menger cube. We have

$$UB_n = B_0 \cup B_1 \cup \ldots \cup B_n \quad \text{where } B_{k+1} = W(B_k) \qquad (20)$$

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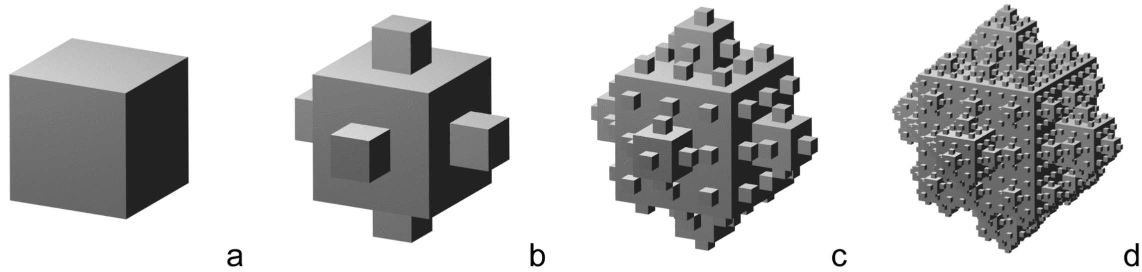

Figure 6: Successive steps in the construction of the additive Menger cube

Example 7

The initial set G_0 is a regular icosahedron with sides of length *d*. The Hutchinson operator is the union of twelve transformations, whose parameters are given in Table 10. The centre of the icosahedron is at the origin.

Table 9

	μ_s	$\tau_{x,y,z}$			$\rho_{\varphi x, \varphi y, \varphi z}$		
	s	t_x	t_y	t_z	φ_x	φ_y	φ_z
ω_1	1/3	0	0	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	0	180^o
ω_2	1/3	0	0	$-\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	0	180^o
ω_3	1/3	0	0	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	0
ω_4	1/3	0	0	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	72^o
ω_5	1/3	0	0	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	144^o
ω_6	1/3	0	0	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	-72^o
ω_7	1/3	0	0	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	-144^o
ω_8	1/3	0	0	$-\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	0
ω_9	1/3	0	0	$-\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	72^o
ω_{10}	1/3	0	0	$-\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	144^o
ω_{11}	1/3	0	0	$-\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	-72^o
ω_{12}	1/3	0	0	$-\frac{a}{4}\sqrt{10+2\sqrt{5}}$	0	63^o	-144^o

Iterating the Hutchinson operator W(H) generates the following sequence of sets:

$$H_{k+1}= W(H_k) \text{ for } k = 0, 1, 2, \ldots, n. \quad (21)$$

The union of the first four members of this sequence is illustrated in Figure 9. We have

$$UH_4 = H_1 \cup H_2 \cup H_3 \cup H_4. \quad (22)$$

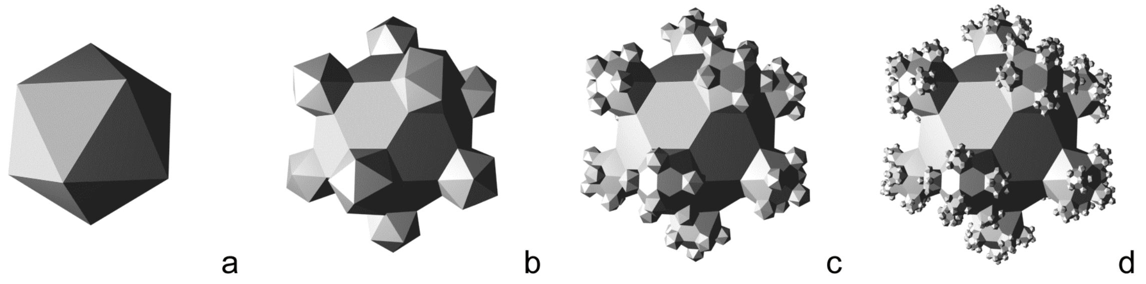

Figure 9: Successive partial unions of the sequence of sets H_n

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PRZYKŁADY OBIEKTÓW O CECHACH FRAKTALNYCH GENEROWANYCH JAKO SUMA WYRAZÓW CIĄGU ZBIORÓW W SYSTEMIE FUNKCJI ITEROWANYCH

Podstawową metodą generowania fraktali jest System Funkcji Iterowanych, w którym iterowanie operatora Hutchinsona będącego zbiorem (sumą) kontrakcji prowadzi do konstrukcji atraktora będącego najczęściej fraktalem. Fraktale uzyskiwane tą metodą wykazują cechy zmniejszania powierzchni lub objętości w miarę wzrostu liczby kroków iteracji. W analogii do szeregu geometrycznego proponowana modyfikacja sposobu generowania polegająca na sumowaniu wyrazów ciągu zbiorów pozwala na konstruowanie obiektów geometrycznych o rosnącym parametrze objętości przy zachowaniu podstawowych właściwości obiektów fraktalnych.

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