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HILBERT–SCHMIDTNESS OF WEIGHTED COMPOSITION OPERATORS AND THEIR DIFFERENCES ON HARDY SPACES

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Abstract. Let u and φ be two analytic functions on the unit disk $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. A weighted composition operator uC_φ induced by u and φ is defined on H^2, the Hardy space of $\mathbb{D}$, by $uC_\varphi f := u \cdot f \circ \varphi$ for every f in H^2. We obtain sufficient conditions for Hilbert–Schmidtness of uC_φ on H^2 in terms of function-theoretic properties of u and φ. Moreover, we characterize Hilbert–Schmidt difference of two weighted composition operators on H^2.

Keywords: weighted composition operators, Hardy spaces, compact operators, Hilbert–Schmidt operators.

Mathematics Subject Classification: 47B33, 30H10.

1. INTRODUCTION

Let $\mathbb{D}$ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and $\mathbb{T}$ be the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The Hardy space H^p, where $1 \leq p < \infty$, of $\mathbb{D}$ consists of all analytic functions f on $\mathbb{D}$ such that

$$\sup_{0\leq r<1} \int_0^{2\pi} |f(re^{i\theta})|^p dm < \infty,$$

where m is the normalized Lebesgue measure on $\mathbb{T}$, i.e. $dm := d\theta/2\pi$. We define H^∞ to be the set of all bounded and analytic functions on $\mathbb{D}$.

In the sequel, we write $L^p = L^p(m)$ and denote the norms of H^p and L^p by $\|\cdot\|_p$. If $f \in H^p$ for $1 \leq p \leq \infty$, its radial limit

$$\hat{f}(e^{i\theta}) := \lim_{r\to 1^-} f(re^{i\theta})$$

exists m-a.e. on $\mathbb{T}$ and $\hat{f} \in L^p$ with $\|\hat{f}\|_p = \|f\|_p$.

The extension of f to

$$\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\},$$

also denoted by f, is defined such that $f|_{\mathbb{T}} = \hat{f}$. Readers may consult [8] for a more comprehensive introduction of Hardy spaces.

In particular, H^2 is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g\rangle := \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, dm \quad \text{for every } f, g \in H^2.$$

The standard orthonormal basis for H^2 is $\{1, z, z^2, \ldots\}$. If w is an arbitrary point in $\mathbb{D}$, then

$$\langle f, k_w\rangle = f(w) \quad \text{for all } f \in H^2,$$

where $k_w(z) = 1/(1 - \overline{w}z)$ is the reproducing kernel corresponding to the point evaluation functional on H^2 at $z = w$. Moreover, we have $\|k_w\|_2 = 1/\sqrt{1 - |w|^2}$.

Let u and φ be two analytic functions on $\mathbb{D}$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. They induce a *weighted composition operator* uC_φ from H^2 into the linear space of all analytic functions on $\mathbb{D}$ by

$$uC_\varphi(f)(z) := u(z)f(\varphi(z)) \quad \text{for every } f \in H^2 \text{ and } z \in \mathbb{D}.$$

When $u \equiv 1$ (resp. $\varphi(z) = z$), the corresponding operator, denoted by C_φ (resp. M_u), is known as a *composition operator* with the symbol φ (resp. a *multiplication operator* with the weight u). By [7, p. 117], the operator C_φ is always bounded. However, this is not necessarily true for weighted composition operators. If uC_φ maps H^2 into itself, an appeal to the closed graph theorem yields its boundedness. In this case, we say uC_φ is a weighted composition operator on H^2. It can also be shown that

$$uC_\varphi^* k_w = \overline{u(w)}k_{\varphi(w)} \quad \text{for every } w \in \mathbb{D}.$$

Weighted composition operators arise from the study of modular structures of von Neumann algebras in quantum field theories. Given a von Neumann algebra $\mathcal{N}$ with a cyclic and separating vector Ω on a Hilbert space H, a Hilbert space $\tilde{H}$ is obtained from the domain of the modular operator of $(\mathcal{N}, \Omega)$ equipped with the graph norm. Moreover, an irreducible component of $\tilde{H}$ can be identified with the Hardy space $H^2(\mathbb{S})$, where $\mathbb{S}$ is a strip region in $\mathbb{C}$ such that the real axis is in the interior of $\mathbb{S}$. Associated with this setting is the operator $M_w : H^2(\mathbb{S}) \to L^2(\mathbb{R})$, where w is inner on $\mathbb{S}$. By mapping $\mathbb{S}$ conformally onto $\mathbb{D}$, we may consider M_w as a Carleson embedding operator, which can be estimated by a special class of weighted composition operators with the lens maps as the symbols. Further details and information about these topics can be found in [4, 11] and [12].

During the past two decades, there has been an extensive study of various properties of weighted composition operators on Hardy spaces, including their boundedness and compactness. In this paper, we investigate Hilbert–Schmidt weighted maps on H^2 and characterize Hilbert–Schmidt differences of weighted composition operators on H^2.

Hilbert–Schmidt operators are of particular importance in practice. In fact, a bounded linear operator L on L^2 is Hilbert–Schmidt if and only if it takes the form of an integral operator, i.e. $(Lf)(\theta) = \int_{\mathbb{T}} K(\theta,\sigma)f(\sigma)\,dm$ for some kernel function K in $L^2(\mathbb{T}\times\mathbb{T})$ [6, p. 267]. These integral operators play a crucial role in the theory of boundary-value problems in mathematical physics. For example, one may transform a boundary-value problem to an integral equation which is solved by numerical quadrature formulae [14, Chapter 12].

2. HILBERT–SCHMIDT WEIGHTED COMPOSITION OPERATORS

Let H be a separable Hilbert space and $L : H \to H$ be a bounded linear operator. Recall that L is said to be *Hilbert–Schmidt* if $\sum_{n=0}^{\infty} \|Le_n\|_H^2 < \infty$ for some orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of H [7, p. 144]. In fact, the value of this sum is independent of the choice of an orthonormal basis. It is also known that every Hilbert–Schmidt operator is compact, but a compact operator may not be Hilbert–Schmidt.

We first show that the existence of a non-zero Hilbert–Schmidt weighted operator uC_φ on H^2 is possible only if $|\varphi| < 1$ m-a.e. on $\mathbb{T}$.

Proposition 2.1. *Let uC_φ be a Hilbert–Schmidt weighted composition operator on H^2. If $m(\{\theta \in [0,2\pi] : |\varphi(e^{i\theta})| = 1\}) > 0$, then uC_φ is the zero operator.*

Proof. Since uC_φ is Hilbert–Schmidt, we have

$$\sum_{n=0}^{\infty} \|uC_\varphi z^n\|_2^2 < \infty.$$

Put $S := \{\theta \in [0,2\pi] : |\varphi(e^{i\theta})| = 1\}$. With

$$\sum_{n=0}^{\infty} \|uC_\varphi z^n\|_2^2 = \sum_{n=0}^{\infty} \int_0^{2\pi} |u(e^{i\theta})|^2 |\varphi(e^{i\theta})|^{2n}\,dm \ge \sum_{n=0}^{\infty} \int_S |u(e^{i\theta})|^2\,dm,$$

it follows that

$$\sum_{n=0}^{\infty} \int_S |u(e^{i\theta})|^2\,dm < \infty,$$

which holds only if $\int_S |u(e^{i\theta})|^2\,dm = 0$. This, together with the assumption $m(S) > 0$, implies that $u = 0$ m-a.e. on S. Hence $u \equiv 0$ on $\mathbb{D}$. □

When φ is an automorphism of $\mathbb{D}$, it follows immediately from the above proposition that there is no Hilbert–Schmidt weighted composition operator on H^2 except the zero operator.

Matache [13, Theorem 9] showed that a weighted composition operator uC_φ is Hilbert–Schmidt on H^2 if and only if

$$\int_0^{2\pi} \frac{|u(e^{i\theta})|^2}{1-|\varphi(e^{i\theta})|^2}\,dm < \infty. \tag{2.1}$$

In particular, the composition operator C_φ is Hilbert–Schmidt on H^2 if and only if $1/\sqrt{1-|\varphi|^2} \in L^2$. Other than the result of Matache, more explicit characterizations for Hilbert–Schmidt weighted composition operators on H^2 are lacking in the literature. In view of this, we present function-theoretic conditions on u and φ that guarantee the weighted operator uC_φ is Hilbert–Schmidt on H^2. Our result is also illustrated with examples.

Theorem 2.2. *Let uC_φ be a weighted composition operator on H^2. If*

(i) *φ is continuous on $\mathbb{T}$,*
(ii) *the set $S := \{\theta \in [0, 2\pi] : |\varphi(e^{i\theta})| = 1\}$ is finite, and*
(iii) *$\lim_{\theta \to \theta_i} |u(e^{i\theta})|^2/(1 - |\varphi(e^{i\theta})|^2)$ exists (as a real number) for every $\theta_i \in S$,*

then uC_φ is Hilbert–Schmidt.

Proof. Write $S = \{\theta_1, \theta_2, \ldots, \theta_n\}$. Since $\lim_{\theta \to \theta_i} |u(e^{i\theta})|^2/(1 - |\varphi(e^{i\theta})|^2)$ exists for each $i \in \{1, 2, \ldots, n\}$, we may choose a sufficiently small $\delta_i > 0$ for which $|u(e^{i\theta})|^2/(1 - |\varphi(e^{i\theta})|^2)$ is bounded on $(\theta_i - \delta_i, \theta_i + \delta_i) \setminus \{\theta_i\}$. Moreover, the finiteness of S ensures that there is some $M > 0$ such that

$$\frac{|u(e^{i\theta})|^2}{1 - |\varphi(e^{i\theta})|^2} \le M \quad \text{on } E,$$

where

$$E = \bigcup_{i=1}^{n} (\theta_i - \delta_i, \theta_i + \delta_i) \setminus \{\theta_i\}.$$

Note that $|\varphi(e^{i\theta})| < 1$ on the compact set

$$F := [0, 2\pi] \setminus \bigcup_{i=1}^{n} (\theta_i - \delta_i, \theta_i + \delta_i).$$

By the continuity of φ on F, there exists a constant $0 \le \alpha < 1$ for which

$$|\varphi(e^{i\theta})| \le \alpha \quad \text{on } F.$$

Now,

$$\begin{aligned} \int_0^{2\pi} \frac{|u(e^{i\theta})|^2}{1 - |\varphi(e^{i\theta})|^2}\, dm &= \int_E \frac{|u(e^{i\theta})|^2}{1 - |\varphi(e^{i\theta})|^2}\, dm + \int_F \frac{|u(e^{i\theta})|^2}{1 - |\varphi(e^{i\theta})|^2}\, dm \\ &\le M m(E) + \frac{1}{1-\alpha^2} \int_F |u(e^{i\theta})|^2\, dm \\ &\le M + \frac{1}{1-\alpha^2} \int_0^{2\pi} |u(e^{i\theta})|^2\, dm. \end{aligned}$$

With $u = uC_\varphi 1 \in H^2$, we see that $\int_0^{2\pi} |u(e^{i\theta})|^2/(1 - |\varphi(e^{i\theta})|^2)\, dm < \infty$. Hence uC_φ is Hilbert–Schmidt. $\square$

Example 2.3. Let $u(z) = z + 1$ and $\varphi(z) = (z - 1)/2$. Then φ is continuous on $\overline{\mathbb{D}}$ and $|\varphi(e^{i\theta})| = 1$ at $\theta = \pi$ only. Moreover,

$$\lim_{\theta\to\pi} |u(e^{i\theta})|^2/(1 - |\varphi(e^{i\theta})|^2) = \lim_{\theta\to\pi} (2 + 2\cos\theta)/[(1 + \cos\theta)/2] = 4.$$

By Theorem 2.2, uC_φ is Hilbert–Schmidt on H^2.

Example 2.4. Let $u(z) = (1 - z)^{1/4}$ and $\varphi(z) = 1 - (1 - z)^{1/2}$. Note that φ is continuous on $\overline{\mathbb{D}}$ and

$$(1 - e^{i\theta})^{\frac{1}{2}} = |1 - e^{i\theta}|^{\frac{1}{2}} e^{i(\frac{\theta}{4} - \frac{\pi}{4})}.$$

Then

$$\begin{aligned}
&1 - |\varphi(e^{i\theta})|^2 \\
&= 2\,\Re(1 - e^{i\theta})^{\frac{1}{2}} - |1 - e^{i\theta}| \\
&= |1 - e^{i\theta}|^{\frac{1}{2}} \left[2\cos\left(\frac{\theta}{4} - \frac{\pi}{4}\right) - |1 - e^{i\theta}|^{\frac{1}{2}}\right] \\
&= |1 - e^{i\theta}|^{\frac{1}{2}} \left[\sqrt{2}\cos\frac{\theta}{4} + \sqrt{2}\sin\frac{\theta}{4} - \sqrt{2}\left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}}\right] \\
&= \sqrt{2}|1 - e^{i\theta}|^{\frac{1}{2}} \left[\left(\left(\cos\frac{\theta}{4}\right)^{\frac{1}{2}} - \left(\sin\frac{\theta}{4}\right)^{\frac{1}{2}}\right)^2 + (\sqrt{2} - 1)\left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}}\right].
\end{aligned}$$

Since the function in the square bracket of the last equality is always positive on $[0, 2\pi]$, it follows that $|\varphi(e^{i\theta})| = 1$ at $\theta = 0$ only. Furthermore,

$$\lim_{\theta\to 0^+} \frac{|u(e^{i\theta})|^2}{1 - |\varphi(e^{i\theta})|^2} = \lim_{\theta\to 0^+} \frac{1}{\sqrt{2}\left[\left((\cos\frac{\theta}{4})^{\frac{1}{2}} - (\sin\frac{\theta}{4})^{\frac{1}{2}}\right)^2 + (\sqrt{2} - 1)(\sin\frac{\theta}{2})^{\frac{1}{2}}\right]} = \frac{1}{\sqrt{2}}.$$

An appeal to Theorem 2.2 shows that uC_φ is Hilbert–Schmidt on H^2.

We remark that the conditions (i)–(iii) in Theorem 2.2 are not necessary for Hilbert–Schmidtness of uC_φ, as shown in the following example.

Example 2.5. Let $\varphi(z) = (z + 1)/2$. Then $|\varphi(e^{i\theta})| = 1$ at $\theta = 0$ only. Consider the function $f : [0, 2\pi] \to \mathbb{R}$ defined by

$$f(\theta) := \begin{cases} 1 + 1/\sqrt{\theta} & \text{if } 0 < \theta \le 2\pi, \\ 0 & \text{if } \theta = 0. \end{cases}$$

Note that $\log f > 0$ on $(0, 2\pi]$. Using integration by parts, we have

$$\begin{aligned}\int_0^{2\pi} \log f\, dm &= \lim_{\epsilon\to 0^+} \int_\epsilon^{2\pi} \log\left(1+\frac{1}{\sqrt{\theta}}\right) dm \\ &= \lim_{\epsilon\to 0^+}\left(\frac{1}{2\pi}\left[\theta\log\left(1+\frac{1}{\sqrt{\theta}}\right)\right]_\epsilon^{2\pi} + \frac{1}{2}\int_\epsilon^{2\pi}\frac{1}{1+\sqrt{\theta}}\, dm\right) \\ &= \log\left(1+\frac{1}{\sqrt{2\pi}}\right) + \frac{1}{\sqrt{2\pi}} - \frac{1}{2\pi}\log(1+\sqrt{2\pi}) < \infty,\end{aligned}$$

i.e. $\log f$ is integrable. By [10, p. 53], there exists a non-zero function g in H^2 such that $f = |g|^2$. Now, we take $u(z) = (z-1)g(z)$. Since $|e^{i\theta}-1|^2 + |e^{i\theta}+1|^2 = 4$, we have

$$\begin{aligned}\int_0^{2\pi} \frac{|u(e^{i\theta})|^2}{1-|\varphi(e^{i\theta})|^2}\, dm &= 4\int_0^{2\pi}\frac{|e^{i\theta}-1|^2 f}{4-|e^{i\theta}+1|^2}\, dm \\ &= 4\int_0^{2\pi}\left(1+\frac{1}{\sqrt{\theta}}\right) dm \\ &= 4\left(1+\sqrt{\frac{2}{\pi}}\right) < \infty.\end{aligned}$$

Hence uC_φ is Hilbert–Schmidt. However,

$$\lim_{\theta\to 0^+}\frac{|u(e^{i\theta})|^2}{1-|\varphi(e^{i\theta})|^2} = 4\lim_{\theta\to 0^+}(1+1/\sqrt{\theta})$$

does not exist.

3. HILBERT–SCHMIDT DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS

Let v and ψ be two analytic functions on $\mathbb{D}$ such that $\psi(\mathbb{D}) \subset \mathbb{D}$. Berkson [2] showed that if $\varphi \neq \psi$, then

$$\|C_\varphi - C_\psi\| \geq \sqrt{\frac{m(E)}{2}},$$

where $E := \{\theta \in [0, 2\pi] : |\varphi(e^{i\theta})| = 1\}$. Shapiro and Sundberg [15, Corollary 2.4] improved the lower bound of this estimate and showed that if $\varphi \neq \psi$, then

$$\|C_\varphi - C_\psi\| \geq \sqrt{\frac{m(E)+m(F)}{2}},$$

where $F := \{\theta \in [0, 2\pi] : |\psi(e^{i\theta})| = 1\}$. These striking results motivate the study of topological structure of the space of composition operators on H^2 (endowed with the operator norm metric). Readers may consult [3, 9] and [15] for research on this topic. A question which emerges from such study is to investigate properties of the difference of two (weighted) composition operators. Let $\phi(z) = (\varphi(z) - \psi(z))/(1 - \overline{\varphi(z)}\psi(z))$. Then it follows from [7, p. 339] that $C_\varphi - C_\psi$ is Hilbert–Schmidt on H^2 if and only if

$$\int_0^{2\pi} \rho^2 \frac{1 - |\varphi\psi|^2}{(1 - |\varphi|^2)(1 - |\psi|^2)}\, dm < \infty,$$

where $\rho(z) = |\phi(z)|$ is the pseudo-hyperbolic distance between $\varphi(z)$ and $\psi(z)$ for $z \in \mathbb{D}$. Other sufficient conditions for $C_\varphi - C_\psi$ to be Hilbert–Schmidt are in [1, Theorems 3.1 and 3.2]. In this section, we characterize the Hilbert–Schmidt difference of $uC_\varphi - vC_\psi$ on H^2. Let us recall two useful identities for easy reference:

$$1 - \left|\frac{w - z}{1 - \overline{w}z}\right|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - \overline{w}z|^2} \quad \text{and} \quad 1 - \overline{w}\left(\frac{w - z}{1 - \overline{w}z}\right) = \frac{1 - |w|^2}{1 - \overline{w}z} \tag{3.1}$$

for every $w, z \in \mathbb{D}$.

In what follows, we further assume $|\varphi|$, $|\psi| < 1$ m-a.e. on $\mathbb{T}$.

Theorem 3.1. *Let uC_φ and vC_ψ be two weighted composition operators on H^2. Then the following statements are equivalent.*

(i) *The operator $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on H^2.*

(ii) $\dfrac{\rho u}{\sqrt{1 - |\varphi|^2}}, \dfrac{v}{\sqrt{1 - |\psi|^2}} - \dfrac{u\sqrt{1 - |\psi|^2}}{1 - \overline{\psi}\varphi} \in L^2,$

(iii) $\dfrac{\rho v}{\sqrt{1 - |\psi|^2}}, \dfrac{u}{\sqrt{1 - |\varphi|^2}} - \dfrac{v\sqrt{1 - |\varphi|^2}}{1 - \overline{\varphi}\psi} \in L^2.$

Proof. We first establish the equivalence of (i) and (iii). To this end, we compute $\sum_{n=0}^{\infty} \|(uC_\varphi - vC_\psi)z^n\|_2^2$:

$$\begin{aligned}
\sum_{n=0}^{\infty} \|(uC_\varphi - vC_\psi)z^n\|_2^2 &= \sum_{n=0}^{\infty} \|u\varphi^n - v\psi^n\|_2^2 \\
&= \sum_{n=0}^{\infty} \int_0^{2\pi} |u\varphi^n - v\psi^n|^2\, dm \\
&= \sum_{n=0}^{\infty} \int_0^{2\pi} [|u|^2|\varphi|^{2n} + |v|^2|\psi|^{2n} - 2\Re(\overline{u}v(\overline{\varphi}\psi)^n)]\, dm.
\end{aligned}$$

Interchanging the summation and the integral signs in the third equality is legitimate because the terms $|u\varphi^n - v\psi^n|^2$ are all non-negative. Then

$$\begin{aligned}
&\sum_{n=0}^{\infty} \|(uC_\varphi - vC_\psi)z^n\|_2^2 \\
&= \int_0^{2\pi} \sum_{n=0}^{\infty} [|u|^2|\varphi|^{2n} + |v|^2|\psi|^{2n} - 2\Re(\overline{u}v(\overline{\varphi}\psi)^n)]\, dm \\
&= \int_0^{2\pi} \left[|u|^2 \sum_{n=0}^{\infty} |\varphi|^{2n} + |v|^2 \sum_{n=0}^{\infty} |\psi|^{2n} - 2\Re\left(\overline{u}v \sum_{n=0}^{\infty} (\overline{\varphi}\psi)^n \right) \right] dm \\
&= \int_0^{2\pi} \left[\frac{|u|^2}{1-|\varphi|^2} + \frac{|v|^2}{1-|\psi|^2} - 2\Re\left(\frac{\overline{u}v}{1-\overline{\varphi}\psi} \right) \right] dm.
\end{aligned}$$

With $\phi = (\varphi - \psi)/(1 - \overline{\varphi}\psi)$, rearranging terms gives $\psi = (\varphi - \phi)/(1 - \overline{\varphi}\phi)$. Applying the identities in (3.1), we have

$$1 - |\psi|^2 = \frac{(1-|\varphi|^2)(1-\rho^2)}{|1-\overline{\varphi}\phi|^2} \quad \text{and} \quad 1 - \overline{\varphi}\psi = \frac{1-|\varphi|^2}{1-\overline{\varphi}\phi}.$$

Now,

$$\begin{aligned}
&\frac{|u|^2}{1-|\varphi|^2} + \frac{|v|^2}{1-|\psi|^2} - 2\Re\left(\frac{\overline{u}v}{1-\overline{\varphi}\psi} \right) \\
&= \frac{|u|^2}{1-|\varphi|^2} + \frac{|v|^2|1-\overline{\varphi}\phi|^2}{(1-|\varphi|^2)(1-\rho^2)} - 2\Re\left(\frac{\overline{u}v(1-\overline{\varphi}\phi)}{1-|\varphi|^2} \right) \\
&= \frac{1}{1-|\varphi|^2} \left[|u|^2 + \frac{|v|^2|1-\overline{\varphi}\phi|^2}{1-\rho^2} - 2\Re(\overline{u}v(1-\overline{\varphi}\phi)) \right] \\
&= \frac{1}{1-|\varphi|^2} \left[|u - v(1-\overline{\varphi}\phi)|^2 - |v|^2|1-\overline{\varphi}\phi|^2 + \frac{|v|^2|1-\overline{\varphi}\phi|^2}{1-\rho^2} \right] \\
&= \frac{1}{1-|\varphi|^2} \left[|u - v(1-\overline{\varphi}\phi)|^2 + \frac{\rho^2|v|^2|1-\overline{\varphi}\phi|^2}{1-\rho^2} \right] \\
&= \frac{1}{1-|\varphi|^2} \left[|u - v(1-\overline{\varphi}\phi)|^2 + \frac{\rho^2|v|^2(1-|\varphi|^2)}{1-|\psi|^2} \right] \\
&= \frac{\rho^2|v|^2}{1-|\psi|^2} + \frac{|u - v(1-\overline{\varphi}\phi)|^2}{1-|\varphi|^2} \\
&= \frac{\rho^2|v|^2}{1-|\psi|^2} + \left| \frac{u}{\sqrt{1-|\varphi|^2}} - \frac{v\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} \right|^2.
\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} \|(uC_\varphi - vC_\psi)z^n\|_2^2 = \int_0^{2\pi} \left(\frac{\rho^2|v|^2}{1-|\psi|^2} + \left| \frac{u}{\sqrt{1-|\varphi|^2}} - \frac{v\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} \right|^2 \right) dm.$$

Hence $uC_\varphi - vC_\psi$ is Hilbert–Schmidt, i.e. $\sum_{n=0}^{\infty} \|(uC_\varphi - vC_\psi)z^n\|_2^2 < \infty$, if and only if

$$\frac{\rho v}{\sqrt{1-|\psi|^2}}, \frac{u}{\sqrt{1-|\varphi|^2}} - \frac{v\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} \in L^2.$$

Note that $|(\psi - \varphi)/(1-\overline{\psi}\varphi)| = \rho$. Upon switching the roles of u and v and the roles of φ and ψ, we also have

$$\sum_{n=0}^{\infty} \|(vC_\psi - uC_\varphi)z^n\|_2^2 = \int_0^{2\pi} \left(\frac{\rho^2|u|^2}{1-|\varphi|^2} + \left| \frac{v}{\sqrt{1-|\psi|^2}} - \frac{u\sqrt{1-|\psi|^2}}{1-\overline{\psi}\varphi} \right|^2 \right) dm.$$

This shows that (i) and (ii) are equivalent. □

By putting $v = 0$ in the above theorem, we obtain Matache's characterization in (2.1) for Hilbert–Schmidtness of a weighted composition operator on H^2. Theorem 3.1 also yields other interesting consequences.

Corollary 3.2. *The operator $C_\varphi + C_\psi$ is Hilbert–Schmidt on H^2 if and only if both C_φ and C_ψ are Hilbert–Schmidt on H^2.*

Proof. If C_φ and C_ψ are Hilbert–Schmidt on H^2, then so is $C_\varphi + C_\psi$. Conversely, assume that $C_\varphi + C_\psi$ is Hilbert–Schmidt on H^2. By taking $u = 1$ and $v = -1$ in Theorem 3.1, we have $\int_0^{2\pi} \rho^2/(1-|\psi|^2)\, dm < \infty$. With

$$\frac{1}{\sqrt{1-|\varphi|^2}} + \frac{\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} = \frac{\overline{\varphi}\phi}{\sqrt{1-|\varphi|^2}} + \frac{2\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi}$$

and the fact that $|\varphi| < 1$, we have

$$\begin{aligned}
\frac{1-\rho^2}{1-|\psi|^2} &= \frac{1-|\varphi|^2}{|1-\overline{\varphi}\psi|^2} \\
&= \frac{1}{4}\left| \frac{\overline{\varphi}\phi}{\sqrt{1-|\varphi|^2}} + \frac{2\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} - \frac{\overline{\varphi}\phi}{\sqrt{1-|\varphi|^2}} \right|^2 \\
&= \frac{1}{4}\left| \frac{1}{\sqrt{1-|\varphi|^2}} + \frac{\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} - \frac{\overline{\varphi}\phi}{\sqrt{1-|\varphi|^2}} \right|^2 \\
&\le \frac{1}{2}\left(\left| \frac{1}{\sqrt{1-|\varphi|^2}} + \frac{\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi} \right|^2 + \frac{\rho^2}{1-|\varphi|^2} \right).
\end{aligned}$$

Since both $1/\sqrt{1-|\varphi|^2} + \sqrt{1-|\varphi|^2}/(1-\overline{\varphi}\psi)$ and $\rho/\sqrt{1-|\varphi|^2}$ are in L^2, it follows that $\int_0^{2\pi}(1-\rho^2)/(1-|\psi|^2)\,dm < \infty$. Hence $\int_0^{2\pi} 1/(1-|\psi|^2)\,dm < \infty$, i.e. C_ψ is Hilbert–Schmidt on H^2. Consequently, $C_\varphi = C_\varphi + C_\psi - C_\psi$ is also Hilbert–Schmidt. □

Corollary 3.3. *The operator $C_\varphi - C_\psi$ is Hilbert–Schmidt on H^2 if and only if*

$$\frac{\rho}{\sqrt{1-|\varphi|^2}},\ \frac{\rho}{\sqrt{1-|\psi|^2}} \in L^2.$$

Proof. When $u = v = 1$,

$$\left|\frac{u}{\sqrt{1-|\varphi|^2}} - \frac{v\sqrt{1-|\varphi|^2}}{1-\overline{\varphi}\psi}\right|^2 = \rho^2\frac{|\varphi|^2}{1-|\varphi|^2}$$

and

$$\left|\frac{v}{\sqrt{1-|\psi|^2}} - \frac{u\sqrt{1-|\psi|^2}}{1-\overline{\psi}\varphi}\right|^2 = \rho^2\frac{|\psi|^2}{1-|\psi|^2}.$$

These, together with Theorem 3.1, yield the desired result. □

This characterization for Hilbert–Schmidt difference of two composition operators is the limiting version of [5, Corollary 3.7]. The following result, which follows from Theorem 3.1 and Corollary 3.3, is the limiting case of [5, Theorem 3.10].

Corollary 3.4. *Suppose that $a, b \in \mathbb{C}\setminus\{0\}$ and C_φ, C_ψ are not Hilbert–Schmidt on H^2. Then $aC_\varphi + bC_\psi$ is Hilbert–Schmidt on H^2 if and only if $a + b = 0$ and $C_\varphi - C_\psi$ is Hilbert–Schmidt on H^2.*

Proof. If $a + b = 0$ and $C_\varphi - C_\psi$ is Hilbert–Schmidt on H^2, then so is $aC_\varphi + bC_\psi = a(C_\varphi - C_\psi)$. Conversely, assume $aC_\varphi + bC_\psi$ is Hilbert–Schmidt on H^2. Taking $u = a$ and $v = -b$ in Theorem 3.1, we see that both $\rho/\sqrt{1-|\varphi|^2}$ and $\rho/\sqrt{1-|\psi|^2}$ are in L^2 (since $a, b \neq 0$). By Corollary 3.3, $C_\varphi - C_\psi$ is Hilbert–Schmidt on H^2. Hence $(a+b)C_\varphi = aC_\varphi + bC_\psi + b(C_\varphi - C_\psi)$ is also Hilbert–Schmidt. This, however, is possible only if $a + b = 0$. □

The following example is modified from an exercise in [7, p. 344].

Example 3.5. Let a and b be constants such that $0 < a \le 1/2$ and $b > 0$. Take $u(z) = v(z) = z$, $\varphi(z) = az + 1 - a$ and $\psi(z) = \varphi(z) + t(z-1)^b$, where t is a positive constant small enough for which $\psi(\mathbb{D}) \subset \mathbb{D}$.

(a) If $0 < b \le 2$, then $uC_\varphi - vC_\psi$ is not compact (and thus not Hilbert–Schmidt) on H^2.
(b) If $b > 5/2$, then $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on H^2.

We first prove (a). Let $f_n = \sqrt{1-|z_n|^2}k_{z_n}$, where $\{z_n\}_{n=1}^{\infty}$ is a sequence such that $|1-z_n|^2 = 1-|z_n|^2$ and $z_n \to 1$. Then $\|f_n\|_2 = 1$ and

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)^* f_n\|_2^2 \\
&= (1-|z_n|^2)|z_n|^2 \|k_{\varphi(z_n)} - k_{\psi(z_n)}\|_2^2 \\
&= (1-|z_n|^2)|z_n|^2 (\|k_{\varphi(z_n)}\|_2^2 + \|k_{\psi(z_n)}\|_2^2 - 2\Re\langle k_{\varphi(z_n)}, k_{\psi(z_n)}\rangle) \\
&= (1-|z_n|^2)|z_n|^2 \left[\frac{1}{1-|\varphi(z_n)|^2} + \frac{1}{1-|\psi(z_n)|^2} - 2\Re\left(\frac{1}{1-\overline{\varphi(z_n)}\psi(z_n)}\right)\right].
\end{aligned}$$

Since

$$\begin{aligned}
1-|\varphi(z_n)|^2 &= 1-|az_n+1-a|^2 \\
&= 2a-a^2-a^2|z_n|^2 - 2a(1-a)\Re(z_n) \\
&= a(1-a)|1-z_n|^2 + a(1-|z_n|^2) \\
&= (2a-a^2)(1-|z_n|^2)
\end{aligned}$$

and

$$\begin{aligned}
\Re\left(\frac{1}{1-\overline{\varphi(z_n)}\psi(z_n)}\right) &\le \frac{1}{|1-|\varphi(z_n)|^2 - t\overline{\varphi(z_n)}(z-1)^b|} \\
&\le \frac{1}{|(2a-a^2)(1-|z_n|^2) - t|az_n+1-a||z_n-1|^b|},
\end{aligned}$$

it follows that

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)^* f_n\|_2^2 \\
&\ge \left[\frac{1}{2a-a^2} - \frac{2(1-|z_n|^2)^{1-b/2}}{|(2a-a^2)(1-|z_n|^2)^{1-b/2} - t|az_n+1-a||}\right]|z_n|^2.
\end{aligned}$$

With $0 < b \le 2$, the sequence $\{\|(uC_\varphi - vC_\psi)^* f_n\|_2\}_{n=1}^{\infty}$ is thus bounded away from zero for all sufficiently large n. This shows that $(uC_\varphi - vC_\psi)^*$ (and hence $uC_\varphi - vC_\psi$) is not compact [6, p. 174].

It remains to prove (b). Note that the image of $\mathbb{T}$ under φ is the circle

$$\{z \in \mathbb{C} : |z-1|^2 + |z-(1-2a)|^2 = 4a^2\}.$$

Since $\Re(\varphi(e^{i\theta})) \le 1$, we have

$$\begin{aligned}
a^2|e^{i\theta}-1|^2 = |\varphi(e^{i\theta})-1|^2 &= 4a^2 - |\varphi(e^{i\theta}) - (1-2a)|^2 \\
&= 4a-1-|\varphi(e^{i\theta})|^2 + 2(1-2a)\Re(\varphi(e^{i\theta})) \le 1-|\varphi(e^{i\theta})|^2.
\end{aligned}$$

With $b > 2$, we also have

$$\begin{aligned}
1-|\psi(e^{i\theta})|^2 &\ge 1-|\psi(e^{i\theta})| \ge 1-|\varphi(e^{i\theta})| - t|e^{i\theta}-1|^b \\
&\ge \frac{1-|\varphi(e^{i\theta})|^2}{2} - t|e^{i\theta}-1|^b \ge \frac{a^2}{2}|e^{i\theta}-1|^2 - t|e^{i\theta}-1|^b \\
&= |e^{i\theta}-1|^2\left(\frac{a^2}{2} - t|e^{i\theta}-1|^{b-2}\right) \ge c|e^{i\theta}-1|^2,
\end{aligned}$$

where $c = a^2/2 - 2^{b-2}t > 0$ (for small t). Moreover,

$$|1 - \overline{\varphi(e^{i\theta})}\psi(e^{i\theta})| \geq 1 - |\psi(e^{i\theta})| \geq c|e^{i\theta} - 1|^2.$$

Consequently,

$$\rho^2 = \left|\frac{\varphi(e^{i\theta}) - \psi(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})}\psi(e^{i\theta})}\right|^2 \leq \left(\frac{t}{c}\right)^2 |e^{i\theta} - 1|^{2b-4}.$$

If $b > 5/2$, then

$$\int_0^{2\pi} \frac{\rho^2|u|^2}{1 - |\varphi|^2}\, dm = \int_0^{2\pi} \frac{\rho^2}{1 - |\varphi|^2}\, dm \leq \left(\frac{t}{ac}\right)^2 \int_0^{2\pi} |e^{i\theta} - 1|^{2b-6}\, dm < \infty$$

and

$$\begin{aligned} \int_0^{2\pi} \left|\frac{v}{\sqrt{1 - |\psi|^2}} - \frac{u\sqrt{1 - |\psi|^2}}{1 - \overline{\psi}\varphi}\right|^2 dm &= \int_0^{2\pi} \frac{\rho^2|\psi|^2}{1 - |\psi|^2}\, dm \\ &\leq \int_0^{2\pi} \frac{\rho^2}{1 - |\psi|^2}\, dm < \infty. \end{aligned}$$

In light of Theorem 3.1, $uC_\varphi - vC_\psi$ is Hilbert–Schmidt on H^2.

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