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ANALYSIS OF AXIALLY LOADED TAPERED BEAMS WITH GENERAL END RESTRAINTS ON TWO-PARAMETER FOUNDATION

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The stability and free vibration of axially-loaded tapered beams with elastic end restraints resting on two-parameter foundations are studied using the differential quadrature method (DQM). The governing differential equation is discretized at sampling points, and then the boundary conditions due to elastic end restraints are implemented and substituted into the governing differential equation yielding a system of homogeneous algebraic equations. The equivalent two-parameter eigenvalue problem is obtained and solved for critical loads in the static case and for natural frequencies in the dynamic case. The obtained solutions are found compatible with those obtained from other techniques. The influences of different parameters on the critical loads and natural frequencies are investigated.

Key words: tapered beams, elastic end restraints, two-parameter foundation, differential quadrature method

1. Introduction

Tapered beams are used in numerous engineering applications to obtain optimum designs, and centrally tapered beam-columns are frequently used in many structures. Indeed, in practical situations, the conventional idealized end conditions (pinned – clamped) may lead to models that misrepresent the actual situations where movements and/or rotations may occur due to the weakening of the supporting structure at the beam ends. In other situations, the end translation and/or rotational displacement is expected due to the flexibility of the connections at the beam ends as in the framed structures. However, the analytical treatments of differential equations governing the behavior of these elements are intractable while the numerical techniques offer a tractable alternative. Closed forms and analytical solutions for simple cases of prismatic and non-prismatic beams may be found in literature. Ruta (1999) used the Chebychev series to obtain solutions for non-prismatic beam vibration. Asymptotic perturbation was used by Maccari (1999) to analyze the nonlinear dynamics of continuous systems. Taha (2012) investigated the nonlinear vibration of initially stressed prismatic beams resting on an elastic foundation by employing the elliptic integrals. Taha and Abohadima (2008) studied the free vibration of non-uniform beams resting on an elastic foundation using the Bessel functions. Sato (1980) reported the transverse vibration of linearly tapered beams using the Ritz method. Numerical methods were used such as the FEM by Naidu *et al.* (1995), the differential transform methods by Banerjee *et al.* (2006), Seval Çatal *et al.* (2008) and Ho and Chen (1998) and the differential quadrature method by Shu (2000), Bert *et al.* (1994) and Essam (2012) to study different configurations of such elements.

The free vibration of tapered beams with nonlinear elastic restraints was studied by Naidu *et al.* (2001) using the FEM, and the effect of tapering ratio and end restraints were analyzed.

Closed form solutions for the case of axially compressed centrally tapered beams resting on a two parameter foundation are not found in literature. Indeed, considering the foundation and external axial compression increases the number of significant parameters of the equivalent mathematical model which leads to an intractable model for analytical methods. Further, the implementation of end conditions in DQM comprises some difficulties in the numerical stability and accuracy analysis, and many algorithms are suggested in literature to overcome these complications in the case of conventional boundary conditions (Shu, 2000).

In the present work, the stability and vibration behavior of an axially compressed centrally tapered beam-column with elastic end restraints resting on a two-parameter foundation will be investigated. The implementation of elastic end restraints (including conventional end conditions) in the application of DQM is suggested and verified. In addition, in the present work the DQM is used to extend Naidu *et al.* (2001) FEM work to consider the influences of the two-parameter foundation and external axial loading.

The governing equations are formulated in a dimensionless form and discretized over the studied domain. The boundary conditions due to elastic end restraints are implemented and substituted into the governing equations yielding a system of homogeneous algebraic equations. Using the two-parameter eigenvalue analysis, the critical loads in the static case and the natural frequencies for the dynamic case are obtained. The obtained solutions are verified and then used to investigate the significance of different parameters related to the studied model on the stability and natural frequencies of the beam.

2. Formulation of the problem

2.1. Vibration equation

The equation of free vibration of the centrally tapered axially-loaded beam of length L resting on a two-parameter foundation shown in Fig. 1, is given as

$$\frac{\partial^2}{\partial X^2}\Big(EI(X)\frac{\partial^2 Y}{\partial X^2}\Big) + (P_o - k_2)\frac{\partial^2 Y}{\partial X^2} + \rho A(X)\frac{\partial^2 Y}{\partial t^2} + k_1 Y(X) = 0 \tag{2.1}$$

where X $(0 \leqslant X \leqslant L)$ is the distance along the beam; t is time; $I(X)$ is the moment of inertia of the beam cross section at a distance X; ρ is the beam mass density per unit volume; E is the beam modulus of elasticity; $A(X)$ is the area of the beam cross section at X; $Y(X,t)$ is the beam lateral displacement; P_o is the external axial load acting on the beam; k_1 and k_2 are the foundation stiffnesses per unit length of the beam. Using dimensionless variables $x = X/L$ $(0 \leqslant x \leqslant 1)$ and $y = Y/L$, Eq. (2.1) can be expressed in the dimensionless form as

$$\frac{\partial^2}{\partial x^2}\Big(\frac{EI(x)}{L^3}\frac{\partial^2 y}{\partial x^2}\Big) + \frac{P_o - k_2)}{L}\frac{\partial^2 y}{\partial x^2} + \rho L A(x)\frac{\partial^2 y}{\partial t^2} + k_1 L y(x) = 0 \tag{2.2}$$

The solution to Eq. (2.2) depends on the boundary conditions at the beam ends.

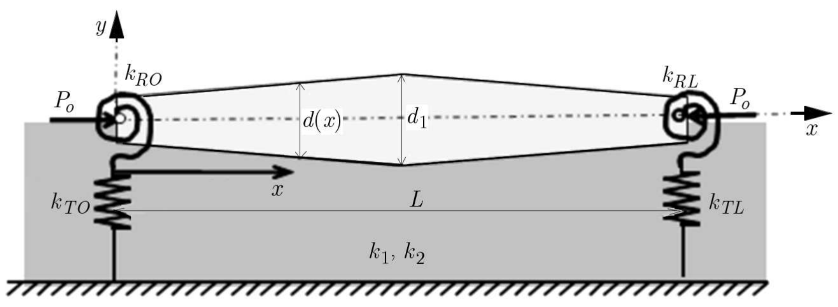

Fig. 1. An axially-loaded centrally tapered beam-column with elastic end restraints resting on a two-parameter foundation

2.2. Boundary conditions

The boundary conditions due to elastic restraints are:
— at $x = 0$

$$k_{T0}y(0,t) = -\frac{\partial}{\partial x}\Big(\frac{EI_o}{L^3}\frac{\partial^2 y(0,t)}{\partial x^2}\Big) \qquad k_{R0}\frac{\partial y(0,t)}{\partial x} = \frac{EI_o}{L}\frac{\partial^2 y(0,t)}{\partial x^2} \tag{2.3}$$

— at $x = 1$

$$k_{TL}y(1,t) = \frac{\partial}{\partial x}\Big(\frac{EI_o}{L^3}\frac{\partial^2 y(1,t)}{\partial x^2}\Big) \qquad k_{RL}\frac{\partial y(1,t)}{\partial x} = -\frac{EI_o}{L}\frac{\partial^2 y(1,t)}{\partial x^2} \tag{2.4}$$

where k_{T0} and k_{TL} are the elastic stiffnesses against lateral displacements at $x = 0, 1$, respectively, k_{R0} and k_{RL} are the elastic stiffnesses against rotation at $x = 0, 1$, respectively and I_o is the moment of inertia of the beam cross section at $x = 0, 1$.

2.3. Mode functions

The solution to Eq. (2.2) using the separation of variables method can be assumed as

$$y(x,t) = \phi(x)\psi(t) \tag{2.5}$$

where $\phi(x)$ is the mode function, $\psi(t)$ is a function representing the variation of lateral displacement with time. Substituting Eq. (2.5) into Eq. (2.2), then Eq. (2.2) can be separated into

$$\begin{aligned} &\frac{d^2}{dx^2}\Big(I(x)\frac{d^2\phi}{dx^2}\Big) + \frac{(P_o - k_2)L^2}{E}\frac{d^2\phi}{dx^2} + \Big(\frac{k_1 L^4 - \rho L^4 A(x)\omega^2}{E}\Big)\phi(x) = 0 \\ &\frac{d^2\psi}{dt^2} + \omega^2\psi(t) = 0 \end{aligned} \tag{2.6}$$

where ω is the separation constant. The solution to Eq. $(2.6)_2$ is

$$\psi(t) = A\sin(\omega t) + B\cos(\omega\, t) \tag{2.7}$$

where A and B are constants obtained from the initial conditions and ω is the natural frequency of the lateral vibration of the beam.

The general solution to Eq. $(2.6)_1$ depends on the distribution of section geometry along the beam. Figure 1 shows the case of a symmetric tapered beam, where the depth of the beam increases lineally from d_o at $x = 0$ to d_1 at $x = 0.5$, then decreases linearly form d_1 at $x = 0.5$ to d_o at $x = 1$, while the width of the beam b is assumed constant, then

$$d(x) = d_o\eta(x) \tag{2.8}$$

where

$$\eta(x) = \begin{cases} 1 + 2x(\alpha - 1) & \text{for} \quad 0 \leqslant x \leqslant 0.5 \\ 1 + 2(1 - x)(\alpha - 1) & \text{for} \quad 0.5 \leqslant x \leqslant 1.0 \end{cases} \tag{2.9}$$

and $\alpha = d_1/d_o$ is the tapering ratio.

Using the distribution of section geometry expressed in Eqs. (2.9), the distribution of the area and moment of inertia of the beam cross section along the beam are given as

$$A(x) = A_o\eta(x) \qquad I(x) = I_o\eta^3(x) \tag{2.10}$$

where A_o and I_o are the area and the second moment of area of the beam cross section at $x = 0$, respectively.

Equation $(2.6)_1$ may be rewritten as

$$\frac{d^4\phi}{dx^4} + \beta(x)\frac{d^3\phi}{dx^3} + [\eta_1(x) + \eta_2(x)P_o]\frac{d^2\phi}{dx^2} + [\xi_1(x) - \xi_2(x)\omega^2]\phi(x) = 0 \tag{2.11}$$

where

$$\begin{array}{lll} \beta(x) = \dfrac{2}{I(x)}\dfrac{dI(x)}{dx} & \eta_1(x) = \dfrac{1}{I(x)}\dfrac{d^2I(x)}{dx^2} - \dfrac{k_2L^2}{EI(x)} & \\ \eta_2(x) = \dfrac{L^2}{EI(x)} & \xi_1(x) = \dfrac{k_1L^4}{EI(x)} & \xi_2(x) = \dfrac{\rho AL^4}{EI(x)} \end{array} \tag{2.12}$$

The boundary condition in terms of the boundary values of the mode function may be expressed at $x = 0$ as

$$k_{T0}\phi(0) = -\frac{EI_o}{L^3}[\phi''' + 6(\alpha - 1)\phi''] \qquad k_{R0}\phi'(0) = \frac{EI_o}{L}\phi''(0) \tag{2.13}$$

Also, the boundary conditions at $x = 1$ can be expressed as

$$k_{TL}\phi(1) = \frac{EI_o}{L^3}[\phi''' + 6(\alpha - 1)\phi''] \qquad k_{RL}\phi'(1) = -\frac{EI_o}{L}\phi''(1) \tag{2.14}$$

3. Solution of the problem

3.1. Differential quadrature method (DQM)

In the differential quadrature method (DQM), the solution domain is discretized into N sampling points and the derivatives of a function $f(x_i)$ at any point are approximated by a weighted linear summation of all functional values $f(x_j)$ at the other points as (Shu, 2000)

$$\left.\frac{d^m f(x)}{dx^m}\right|_{X_i} \approx \sum_{j=1}^{N} C_{i,j}^{(m)} f(x_j)s \qquad i = 1,\ldots,N \qquad m = 1,\ldots,M \tag{3.1}$$

where M is the order of the highest derivative in the governing equation, $f(x_j)$ is the functional value at the point $x = x_j$ and $C_{i,j}^{(m)}$ is the weighting coefficient relating the functional value at $x = x_j$ to the m-derivative of the function $f(x)$ at $x = x_i$. To obtain the weighting coefficients, many polynomials with different base functions are used to approximate the functional values. Using the Lagrange interpolation formula, the functional value at the point x can be approximated by all the functional values $f(x_k)$ $(k = 1,\ldots,N)$ as

$$f(x) \approx \sum_{k=1}^{N} \frac{L(x)}{(x - x_k)L_1(x_k)} f(x_k) \tag{3.2}$$

where

$$L(x) = \prod_{j=1}^{N}(x - x_j) \qquad L_1(x_k) = \prod_{k=1}^{N}(x_i - x_k) \qquad i,k = 1,\ldots,N \tag{3.3}$$

Substitution of Eq. (3.2) into Eq. (3.1) yields the weighting coefficients of the first derivative as (Shu, 2000)

$$C_{i,j}^{(1)} = \begin{cases} \dfrac{L_1(x_i)}{(x_i - x_j)L_1(x_j)} & \text{for} \quad i \neq j \quad \wedge \quad i,j = 1,N \\ -\sum\limits_{j=1,j\neq i}^{N} C_{i,j}^{(1)} & \text{for} \quad i = j \quad \wedge \quad i,j = 1,N \end{cases} \tag{3.4}$$

Differentiation of Eq. (3.1) yields the weighting coefficients of the m-th order derivative in terms of the weighting coefficients of $(m-1)$-th order derivative as

$$C_{i,k}^{(m)} = \sum_{k=1}^{N} C_{i,k}^{(1)} C_{i,k}^{(m-1)} \qquad i,k=1,\dots,N \qquad m=1,\dots,M \tag{3.5}$$

The accuracy and stability of the obtained solution using numerical methods as DQM and FEM are affected by both the number and the distribution of discretization points. In boundary value problems, it is well known that irregular distributions of the sampling points with closer dimensions near the boundaries yield stable and accurate results. One of the frequently used distributions for mesh points generation is the normalized Gauss-Chebychev-Lobatto distribution given as

$$x_i = \frac{1}{2}\Big[1-\cos\Big(\frac{i-1}{N-1}\pi\Big)\Big] \qquad i=1,\dots,N \tag{3.6}$$

3.2. Discretization of the boundary conditions

The boundary conditions can be discretized as:
— at $x=0$

$$\overline{K}_{T0}\phi_1 = -\sum_{k=1}^{N} C_{1,k}^{(3)}\phi_k - 6(\alpha-1)\sum_{k=1}^{N} C_{1,k}^{(2)}\phi_k \qquad \overline{K}_{R0}\sum_{k=1}^{N} C_{1,k}^{(1)}\phi_k = \sum_{k=1}^{N} C_{1,k}^{(2)}\phi_k \tag{3.7}$$

— at $x=1$

$$\overline{K}_{TL}\phi_N = \sum_{k=1}^{N} C_{N,k}^{(3)}\phi_k + 6(\alpha-1)\sum_{k=1}^{N} C_{N,k}^{(2)}\phi_k \qquad \overline{K}_{RL}\sum_{k=1}^{N} C_{N,k}^{(1)}\phi_k = -\sum_{k=1}^{N} C_{N,k}^{(2)}\phi_k \tag{3.8}$$

where N is the number of the sampling points and the elastic restraints stiffness parameters $\overline{K}_{T0}$, $\overline{K}_{TL}$, $\overline{K}_{R0}$ and $\overline{K}_{RL}$ are defined as

$$\overline{K}_{T0} = \frac{k_{T0}L^3}{EI_0} \qquad \overline{K}_{TL} = \frac{k_{TL}L^3}{EI_0} \qquad \overline{K}_{R0} = \frac{k_{R0}L}{EI_0} \qquad \overline{K}_{RL} = \frac{k_{RL}L}{EI_0} \tag{3.9}$$

Using Eqs. (3.7) and (3.8) and employed the common matrices operations, expressions for the boundary unknown functional values ϕ_1, ϕ_2, ϕ_{N-1} and ϕ_N in terms of the other functional values ϕ_i, $(i=3,\ N-2)$ can be obtained as

$$\phi_1 = \sum_{i=3}^{N-2} D_{1,i}\phi_i \qquad \phi_2 = \sum_{i=3}^{N-2} D_{2,i}\phi_i \qquad \phi_{N-1} = \sum_{i=3}^{N-2} D_{N-1,i}\phi_i \qquad \phi_N = \sum_{i=3}^{N-2} D_{N,i}\phi_i \tag{3.10}$$

where $D_{i,j}$ are numerical coefficients obtained from the DQM weighting coefficients and matrices operations and ϕ_i are the required unknowns.

3.3. Discretization of the governing equation

The governing equation of the beam vibration; Eq. (2.11); can be discretized at the sampling point x_i as

$$\begin{aligned}\sum_{k=1}^{N} C_{i,k}^{(4)}\phi_k + \beta(x_i)\sum_{k=1}^{N} C_{i,k}^{(3)}\phi_k + [\eta_1(x_i)+\eta_2(x_i)P_o]\sum_{k=1}^{N} C_{i,k}^{(2)}\phi_k \\ + [\xi_1(x_i)-\xi_2(x_i)\omega^2]\phi_i = 0\end{aligned} \tag{3.11}$$

Substituting the boundary functional values expressed in Eq. (3.10) into Eq. (3.11), one obtains

$$\sum_{k=3}^{N-2}\Big[C_{i,k}^{(4)}+\beta_i C_{i,k}^{(3)}+(\eta_{1,i}+\eta_{2,i}P_o)C_{i,k}^{(2)}+\delta_{i,k}(\xi_{1,k}-\xi_{2,k}\omega^2)\Big]\phi_k=0 \tag{3.12}$$

where $\delta_{i,j}$ is the Kronecker delta ($\delta_{i,j}=1$ for $i=j$ and $\delta_{i,j}=0$ for $i\neq j$).

Application of Eq. (3.12) at the sampling points x_i, $i=3,\dots,N-2$ yields a system of $N-4$ homogeneous equations in $N-4$ unknown functional values of the mode function (ϕ_i; $i=3,N-2$) with the two parameters (P_o and ω). Using the equivalent two-parameter eigenvalue problem, the critical axial load P_{cr} and the natural frequencies ω_n are obtained. In addition, the functional values of the dimensionless lateral displacement at different locations along the beam can be obtained and used to illustrate the mode functions.

3.4. Verification of the present solution

For the case of conventional end conditions (pinned-clamped), the results of both the fundamental stability parameter λ_b and the fundamental frequency parameter λ, calculated from the present analysis, are compared with those obtained from closed form solution and FEM and are found to be compatible (Essam, 2012). The fundamental stability parameter λ_b and the fundamental frequency parameter λ are defined as

$$\lambda_b^2=\frac{P_{cr}L^2}{EI_o} \qquad\qquad \lambda^4=\frac{\rho A_o L^4\omega_1^2}{EI_o} \tag{3.13}$$

where P_{cr} is the critical load defined as the lowest value of the external axial load after which the beam loses its stability (also called buckling or Euler's load). For the case of elastic end restraints, the values of the frequency parameter λ for tapered beams, obtained from the present solution, are presented against those obtained from the FEM (Naidu *et al.*, 2001) for different values of the tapering ratio α in Table 1. The values shown in the table indicate close agreement between the results obtained from the two approaches for small values of the tapering ratio α. There are some deviations for large values of α because in the FEM results, the beam is divided into a limited number of prismatic elements of equal length, which may yield less accurate values.

Table 1. Values of the frequency parameter λ for the tapered columns

End restraints stiffness				Tapering ratio $\alpha=d_1/d_o$						Analysis
$\overline{K}_{T0}$	$\overline{K}_{TL}$	$\overline{K}_{R0}$	$\overline{K}_{RL}$	1.0	1.1	1.2	1.3	1.4	1.5	
1E5	1E5	0	0	3.141	3.248	3.349	3.449	3.534	3.620	FEM
				3.141	3.283	3.392	3.496	3.540	3.588	Present
1E5	1E5	0.1	0.1	3.173	3.276	3.373	3.466	3.554	3.638	FEM
				3.169	3.311	3.420	3.505	3.571	3.621	Present
1E5	1E5	1	1	3.399	3.479	3.557	3.634	3.708	3.780	FEM
				3.373	3.517	3.634	3.729	3.807	3.872	Present
1E5	1E5	10	10	4.156	4.201	4.247	4.294	4.341	4.388	FEM
				4.109	4.274	4.419	4.549	4.667	4.776	Present

4. Numerical results

Many attempts have been made, with different discretization schemes, to select the number of sampling points that yields stable and accurate solutions. It has been found that 15 sampling points achieve the assigned accuracy (0.1%) for the studied model (Essam, 2012).

Both the stability and the natural frequency of the beam-foundation system depend mainly on the integrated (overall) stiffness of the system. The integrated stiffness of the system is the magnitude of the force that produces a unit displacement. It is expected that the integrated stiffness of the system increases as the flexural rigidity of the beam (or α) increases, as the foundation stiffness increases, as the end restrains stiffness increases and as the applied axial load decreases.

The foundation stiffness parameters ($\overline{k}_1$ and $\overline{k}_2$) and the loading ratio γ are defined as

$$\overline{k}_1 = \frac{k_1 L^4}{EI} \qquad \overline{k}_2 = \frac{k_2 L^2}{\pi^2 EI} \qquad \gamma = \frac{P_o}{P_{cr}} \tag{4.1}$$

In the present model, the influences of 8-independent parameters (4 for elastic end restraints, 2 for foundation, tapering ratio α and external axial load P_o) on two parameters (stability parameter λ_b and frequency parameter λ) are to be studied.

A MATLAB code is used to study the variations of the stability and frequency parameters for the practical range of the beam and foundation parameters. Although the present solutions are obtained in terms of dimensionless parameter, the properties used to obtain the present numerical results are for a reinforced concrete beam with: $E = 2.1 \cdot 10^{10}$ Pa, $L = 5$ m, $b = 0.2$ m, $d_o = 0.5$ m, $\rho = 2500$ kg/m^3.

4.1. Investigations of the stability parameter (λ_b)

The variations of the stability parameter λ_b with different parameters of the beam-foundation system are shown in Figs. 2a-c. As expected, the stability parameter increases with the increase of the tapering ratio, the foundation stiffness, the stiffness of the elastic end restraints and with the decrease of the external compression load.

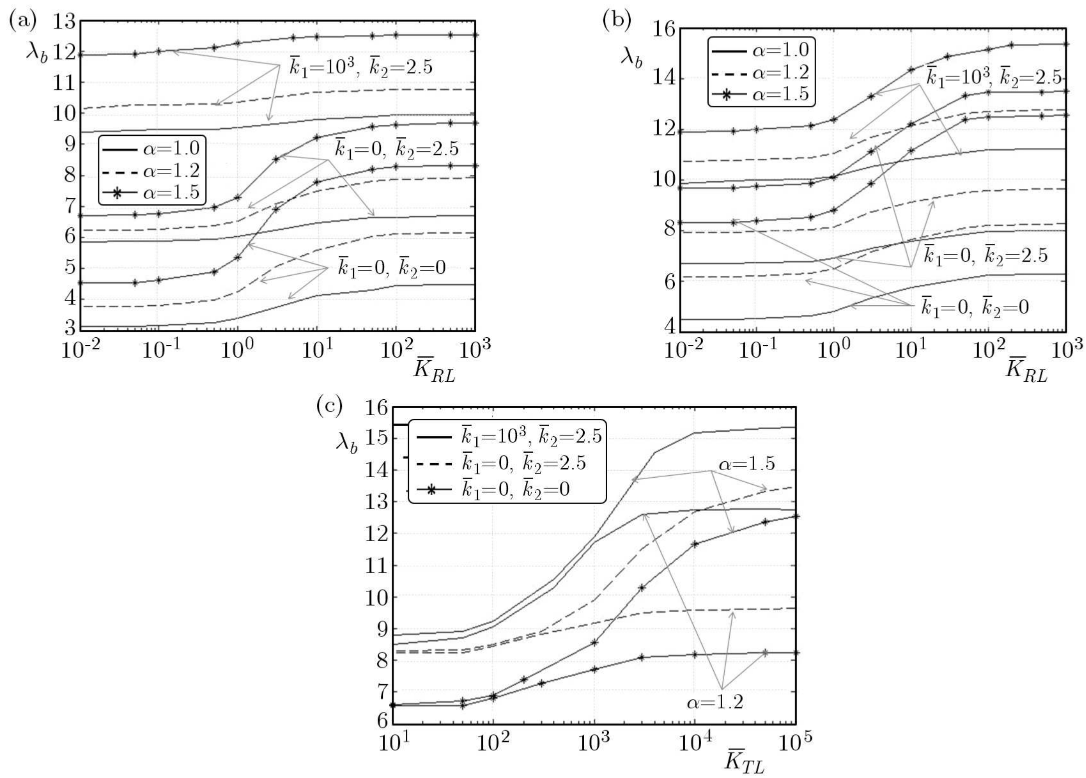

Fig. 2. Influence of the support stiffness on the stability parameter for: (a) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = 0$; (b) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = 10^3$; (c) $\overline{K}_{RO} = \overline{K}_{RL} = 10^3$, $\overline{K}_{TO} = 10^5$

The effects of the transition of end conditions from the pinned-pinned (P-P) case to pinned-clamped (P-C) case on the stability parameter λ_b are shown in Fig. 2a. It is clear that the value of the stability parameter increases as the stiffness of elastic end restraints increases. It is also observed that, to represent the conventional clamped end condition, a value of the rotational stiffness parameter $\overline{K}_{RO} = 100$ is adequate. In addition, in the case of weak foundations, the effect of variation in $\overline{K}_{RO}$ on the stability parameter is more significant for large values of the tapering ratio α.

The variations of the stability parameter due to the transition of the end conditions from the pinned-clamped (P-C) case to the clamped-clamped case (C-C) are shown in Fig. 2b for different values of the system parameters. It is observed that the effect of variation of $\overline{K}_{RO}$ is more significant for large values of the tapering ratio α for all types of foundations.

The variation of the stability parameter due to relaxation of the lateral translation at one end of the beam, while preventing the lateral translation at the other end and the rotations at both ends is shown in Fig. 2c. It is observed that to prevent the lateral movement at the beam end, a value of the translational stiffness parameter $\overline{K}_{TL} = 10^5$ is enough. However, it is found that the effect of variation in $\overline{K}_{TL}$ on the stability parameter λ_b is more significant than the variation of $\overline{K}_{RO}$, especially for large values of the tapering ratio α. Furthermore, it is noticed that the effect of the tapering ratio α is negligible for the case when the lateral translation is released at one end and other elastic restraints kept rigid (C-CF).

4.2. Investigations of the frequency parameter (λ)

The variations of the frequency parameter λ with different parameters of the beam and the underlying foundation are shown in Figs. 3-5. It is clear in all figures that the frequency parameter increases with the increase in the tapering ratio, the foundation stiffness, the end restraints stiffness and with the decrease of the axial compression load.

The effects of the transition from the case of (P-P) condition to the case of (C-C) condition on the frequency parameter λ are shown in Fig. 3a. It is found that the effect of the tapering ratio α on the frequency parameter λ decreases as the foundation stiffness increases.

Figure 3b shows the effect of the transition of the end restraints from the (C-P) case to the (C-C) case on the frequency parameter λ. It is observed that the effect of the axial compression load is more noticeable for the (C-P) case in the case of weak foundation. Also, it is found that the effect of α is more noticeable for the (C-C) case while the effect of foundation stiffness is noticeable for the (C-P) case.

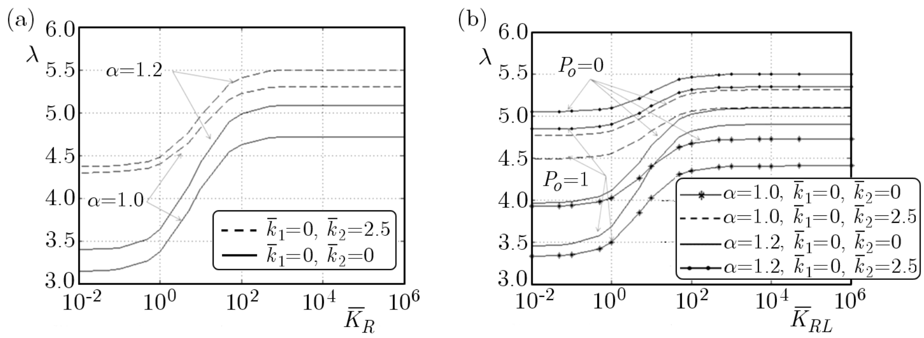

Fig. 3. Influence of the support rotational stiffness on the frequency parameter for: (a) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = \overline{K}_{RO} = \overline{K}_R$, $P_o = 0$; (b) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = 10^3$

The effects of the external axial compression and the variation of the rotational elastic stiffness at one end on the frequency parameter for prismatic beams are shown in Fig. 4a. It is clear that as the axial compression load increases, the natural frequency of the system decreases.

Indeed, in the deformed configuration, the lateral component of the external axial compression is in the opposite direction of the lateral restoring force resulting from the beam and foundation stiffness. Consequently, as the axial compression increases, the total restoring force of the system decreases, which leads to a flexible system with a low natural frequency. At a certain (critical) value of the external axial compression (P_{cr}), the system is transformed into an aperiodic system, and no free vibration occurs.

Figure 4b is another version of Fig. 4a for the case of the tapered beam with a high tapering ratio $(\alpha = 1.5)$. It is noticed that the influence of foundation stiffness on the natural frequency of the system is more significant for the (C-P) case than (C-C) case.

The influences of the variation of the translational stiffness at one end on the frequency parameter are shown in Fig. 4c for different values of the tapering ratio. It is clear that the natural frequency of the system increases as the end lateral stiffness increases.

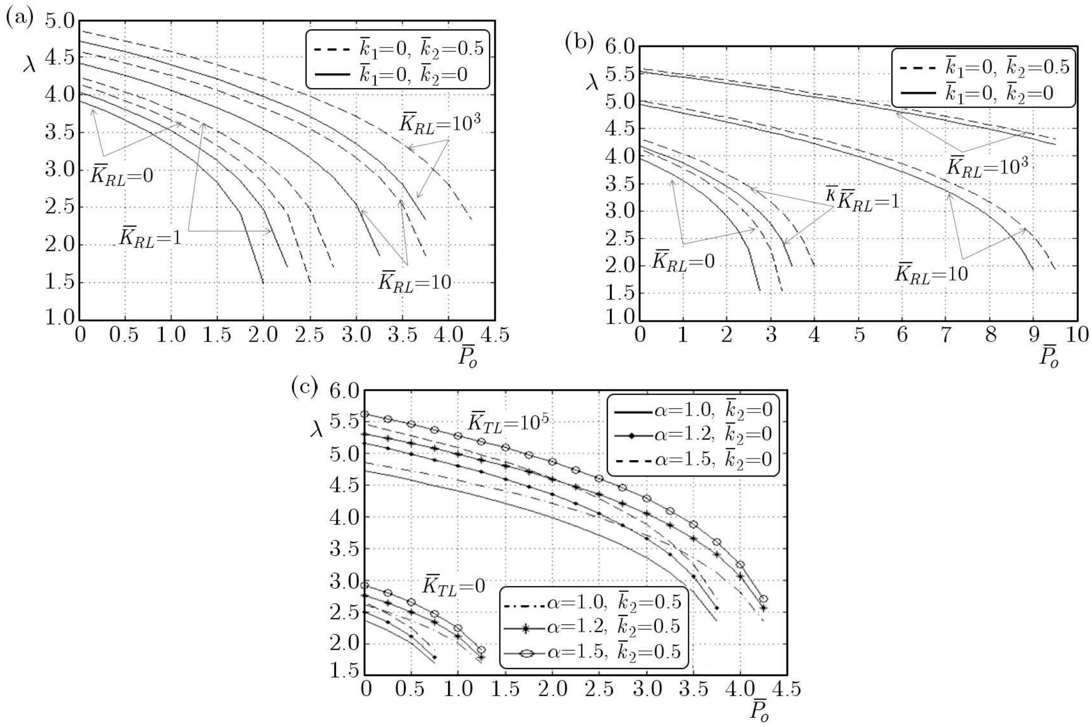

Fig. 4. Influence of the compression axial load on the frequency parameter for: (a) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = 10^3$, $\alpha = 1$; (b) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = 10^3$, $\alpha = 1.5$; (c) $\overline{K}_{TO} = 10^5$, $\overline{K}_{RO} = \overline{K}_{RL} = 10^3$, $\overline{k}_1 = 0$

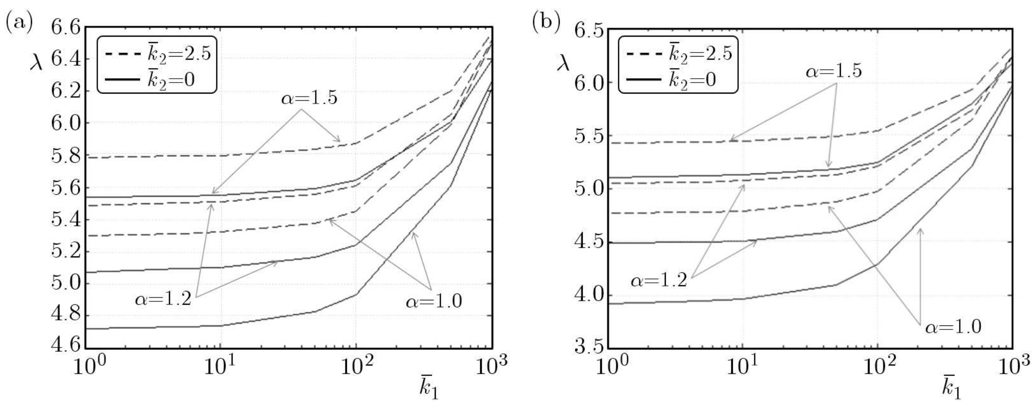

Fig. 5. Influence of the foundation parameters on the frequency parameter: (a) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = \overline{K}_{RL} = 10^3$, (b) $\overline{K}_{TO} = \overline{K}_{TL} = 10^5$, $\overline{K}_{RO} = 10^3$, $\overline{K}_{RL} = 0$

The effects of variations of both the foundation linear stiffness $\overline{k}_1$ (Winkler effect) and the foundation shear stiffness $\overline{k}_2$ (Pasternak effect) on the frequency parameter are shown in Figs. 5a and 5b.

In Fig. 5a, the influence of variation of the linear foundation stiffness $\overline{k}_1$ on the frequency parameter for the case of (C-C) end condition is shown. Further, in Fig. 5b, the influence of variation of the linear foundation stiffness $\overline{k}_1$ on the frequency parameter for the case of (C-P) condition is shown. It is clear that the influence of the linear foundation stiffness is stronger for the (C-P) case. However, the influence of the foundation shear stiffness is more significant for prismatic beams.

5. Sammary and conclusion

The stability and free vibrational behavior of axially-loaded centrally tapered beam-columns with elastic end restraints resting on two-parameter foundations are investigated using the DQM. The governing differential equation with variable coefficients is derived and discretized over the studied domain. The boundary conditions implantation at beam ends due to elastic end restraints are suggested and verified. Then, the governing differential equation is transformed into a system of $N-4$ homogeneous algebraic equations in $N-4$ unknown functional values of the lateral displacement in addition to the two parameters P_o and ω. Using the two parameter eigenvalue analysis, the values of the critical loads (P_{cr}) for the static case $(\omega = 0)$ and the natural frequencies ω_n for a prescribed value of the axial compression load $P_o < P_{cr}$ are obtained. It is found that the natural frequencies and the critical loads for the tapered beams increase as the integrated stiffness of the beam-foundation system increases. The integrated stiffness of the system is a qualitative combination of the beam flexural rigidity, the foundation stiffness and the elastic end restraints stiffness. Furthermore, it is found that the natural frequency of the beam-foundation system decreases as the axial compression load increases.

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