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Acoustic propagation in inhomogeneous fluids: regularization via the introduction of fine particles*)

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It is shown, using analytical methodologies, that the velocity field blow-up suffered by vertically ascending acoustic waves in an isothermal atmosphere can be eliminated via the introduction of fine particles. Assuming the inhomogeneous generalization of the particle-laden flow model known as the (linearized) Marble–Thompson model-1, it is established that bounded, exponentially decreasing, shock amplitudes can be obtained provided the mass fraction of particles exceeds a critical value, for which an exact expression is derived. Lastly, supporting numerical results are presented, special cases are discussed, and possible follow-on studies are noted.

Key words: inhomogeneous fluids, Laplace transform, particle-laden flow, singular surface theory.

1. Introduction

With the linearized (1D) Euler equations as his model system, Rayleigh [1], in 1890, showed that a time-harmonic acoustic wave propagating upwards in an isothermal atmosphere will always suffer amplitude blow-up as $z \to \infty$; see also Lamb [2, §309] and Whitham [3, §6.6]. In 1908, Lamb [4, §7] investigated the impact of Newtonian viscosity on Rayleigh's solution; his approach, however, was to simply add a viscosity term to Rayleigh's equation of motion (EoM), which in [4] had been re-derived in Lagrangian coordinates. In the context of the same problem that Rayleigh had considered some 18 years earlier, Lamb found that the main effect of including Newtonian viscosity was to *reduce* the rate at which amplitude blow-up occurs as $z \to \infty$: specifically, from exponential in z, which Rayleigh [1] encountered in his study of the lossless case, to *linear* in z when viscosity is included in the EoM [4, p. 131].

The subsequent studies of the lossless case by Carslaw and Jaeger [5, §82] and, more recently, Keiffer *et al.* [6, §3.3] show that the amplitudes of acoustic *singular surfaces* [7] also suffer blow-up as they propagate upwards in an isothermal atmosphere.

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In this paper, we revisit the problem of an upward propagating acoustic shock in a lossless, isothermal atmosphere. We show that the amplitude blow-up reported by earlier authors can be eliminated via the introduction of fine particles; i.e., by regarding the atmosphere as a dusty gas. Our approach, which is based on a classically-formulated dusty gas model and employs the Laplace transform, examines not only the solution for the fluid (i.e., gas) phase, but that of the solid (i.e., particle) phase as well, the latter being directly derivable from the former. We conclude our presentation by noting applicable special cases of our model system and possible extensions of the present study.

Before presenting our analysis and findings, however, we pause briefly to formulate (at a rather general level) the model system that we have adopted as the basis of this study.

2. The inhomogeneous Marble–Thompson model-1

When generalized to allow for the possibility that the ambient state of both phases might be inhomogeneous, the *linearized*, multi-D, version of the particle-laden flow model that JORDAN [8] has termed the *Marble–Thompson model-1* (MT-1) becomes

$$\partial\varrho/\partial t + \boldsymbol{\nabla}\cdot(\rho_{\rm a}\mathbf{u}) = 0, \tag{2.1a}$$

$$\partial(\rho_{\rm a}\mathbf{u})/\partial t + \boldsymbol{\nabla}p = -(6\pi\mu a)n_{\rm a}(\mathbf{u}-\boldsymbol{\mathfrak{v}}) - [\boldsymbol{\nabla}P_{\rm a} - (\varrho+\rho_{\rm a})\mathbf{b}], \tag{2.1b}$$

$$\partial(\varrho_{\rm p}\boldsymbol{\mathfrak{v}})/\partial t = \tfrac{9}{2}(\mu/a^2)(\mathbf{u}-\boldsymbol{\mathfrak{v}}) + \varrho_{\rm p}\mathbf{b}, \tag{2.1c}$$

$$\partial p/\partial t + \mathbf{u}\cdot(\boldsymbol{\nabla}P_{\rm a}) = c_{\rm a}^2\left[\partial\varrho/\partial t + \mathbf{u}\cdot(\boldsymbol{\nabla}\rho_{\rm a})\right], \tag{2.1d}$$

$$\partial\mathfrak{n}/\partial t + \boldsymbol{\nabla}\cdot(n_{\rm a}\boldsymbol{\mathfrak{v}}) = 0, \tag{2.1e}$$

where a subscript "a" denotes the ambient state value of the quantity to which it is attached. Here, $\mathbf{u} = (u, v, w)$ and $\boldsymbol{\mathfrak{v}} = (\mathfrak{u}, \mathfrak{v}, \mathfrak{w})$ are, respectively, the fluid and particle velocity vectors; $p(x, y, z, t) = P(x, y, z, t) - P_{\rm a}(x, y, z)$ is known as the *acoustic* (or over) pressure, where $P(> 0)$ is the thermodynamic pressure; $\varrho(x, y, z, t) = \rho(x, y, z, t) - \rho_{\rm a}(x, y, z)$, where $\rho(> 0)$ is the mass density of the clean fluid; $\mathfrak{n}(x, y, z, t) = n(x, y, z, t) - n_{\rm a}(x, y, z)$, where $n(> 0)$ is the number of particles per unit volume; and $\mathbf{b} = \mathbf{b}(x, y, z)$ is the external (per unit mass) body force vector. Moreover, $c_{\rm a} = c_{\rm a}(x, y, z)$, the ambient state value of the sound speed in the clean fluid, is given by

$$c_{\rm a} = \sqrt{A_{\rm a}/\rho_{\rm a}}, \tag{2.2}$$

where $A_{\rm a} = A_{\rm a}(x, y, z)$ denotes the ambient state value of the adiabatic bulk modulus [9] of the clean fluid; $\mu(> 0)$ is the (assumed constant) shear viscosity coefficient of the clean fluid; $\varrho_{\rm p}$, which we take to be constant, is the density of the material that constitutes the particles; and a, the particle radius, is also assumed constant.

An important assumption in the formulation of the MT-1 model is

$$c_{\mathrm{pp}} \simeq 0, \tag{2.3}$$

i.e., the particle specific heat (at constant pressure) is negligibly small [10, pp. 553–556]; an important consequence of this is

$$\vartheta_{\mathrm{p}} = \vartheta, \tag{2.4}$$

where $\vartheta_{\mathrm{p}} = \vartheta_{\mathrm{p}}(x, y, z, t)$ and $\vartheta = \vartheta(x, y, z, t)$ denote the particle absolute temperatures and the absolute temperature of the clean fluid, respectively. Also, implicit in Eq. (2.1d) is the assumption that the flow is *isentropic* [10]. In the context of general (i.e., nonlinear) compressible flow theory, this means

$$D\eta/Dt = 0, \tag{2.5}$$

where $\eta = \eta(x, y, z, t)$ is the specific entropy of the clean fluid and D/Dt denotes the material derivative operator. In the present study, which of course is carried out under the linear approximation, the isentropic nature of the flow follows from Eq. (2.4); see [8] and [10, §11.5].

3. Mathematical preliminaries

3.1. Governing system: reduction to 1D

In this study we consider the case of 1D propagation along the $+z$-axis in the atmosphere under the assumption that the "flat Earth" approximation [11, §5.3.2] holds and that the Earth's rotation can be neglected, where the $+z$-axis is taken to be directed vertically upwards. Under this propagation geometry, the following simplifications are realized: $\mathbf{u} = (0, 0, w(z, t))$, $\boldsymbol{\mathfrak{v}} = (0, 0, \mathfrak{w}(z, t))$, $P = P(z, t)$, $\rho = \rho(z, t)$, $n = n(z, t)$, and $\mathbf{b} = (0, 0, -g)$, where g denotes the acceleration due to gravity. On making these replacements, and taking note of the fact that all quantities that carry a subscript "a" are now functions of (at most) z, System (2.1) is reduced to

$$\varrho_t + \rho'_{\mathrm{a}}(z)w + \rho_{\mathrm{a}}(z)w_z = 0, \tag{3.1a}$$

$$\rho_{\mathrm{a}}(z)w_t + p_z = -(m/\tau)n_{\mathrm{a}}(z)(w - \mathfrak{w}) - \{P'_{\mathrm{a}}(z) + g[\varrho + \rho_{\mathrm{a}}(z)]\}, \tag{3.1b}$$

$$\mathfrak{w}_t = -g + (w - \mathfrak{w})/\tau, \tag{3.1c}$$

$$p_t + P'_{\mathrm{a}}(z)w = -\rho_{\mathrm{a}}(z)c^2_{\mathrm{a}}(z)w_z, \tag{3.1d}$$

$$\mathfrak{n}_t + n'_{\mathrm{a}}(z)\mathfrak{w} + n_{\mathrm{a}}(z)\mathfrak{w}_z = 0, \tag{3.1e}$$

where a prime denotes $\mathrm{d}/\mathrm{d}z$ and Eq. (3.1a) has been used to eliminate ϱ_t from Eq. (3.1d). Here, the momentum relaxation time, $\tau(>0)$, is given by

$$\tau = \frac{2\varrho_{\mathrm{p}}a^2}{9\mu} = \frac{m}{6\pi\mu a}, \tag{3.2}$$

where $m(>0)$, the mass of each individual particle, is assumed constant.

3.2. Additional assumptions

To simplify the forthcoming analysis, and our presentation thereof, we now introduce the notation

$$\begin{aligned} P_0 &:= \lim_{z\to 0} P_\mathrm{a}(z), \quad \rho_0 := \lim_{z\to 0} \rho_\mathrm{a}(z), \quad \vartheta_0 := \lim_{z\to 0} \vartheta_\mathrm{a}(z), \\ \eta_0 &:= \lim_{z\to 0} \eta_\mathrm{a}(z), \quad c_0 := \lim_{z\to 0} c_\mathrm{a}(z), \quad n_0 := \lim_{z\to 0} n_\mathrm{a}(z), \end{aligned} \tag{3.3}$$

and, more importantly, invoke the following assumptions:

(I) Following earlier authors, we hereafter assume that *air*, i.e., the mixture of gases which comprises the Earth's atmosphere, behaves like a (single species) *perfect gas* [10, p. 79]. In terms of our model system, this means that P_a, ρ_a, and ϑ_a also satisfy the following special case of the *ideal gas law* [10, §2.5]:

$$P_\mathrm{a} = (c_p - c_v)\rho_\mathrm{a}\vartheta_\mathrm{a} \qquad (c_p, c_v := \text{const}), \tag{3.4}$$

and that the ambient state value of the adiabatic bulk modulus is given by (see [9, p. 30])

$$A_\mathrm{a}(z) = \gamma P_\mathrm{a}(z) \qquad \text{(perfect gases)}. \tag{3.5}$$

Here, $c_p > c_v > 0$ are the specific heats at constant pressure and volume, respectively, of the clean gas; $\gamma = c_p/c_v$, where $\gamma = 1.4$ in the case of air [9, p. 28]; and we observe that Eq. $(3.3)_5$ evaluates to

$$c_0 = \sqrt{\gamma P_0/\rho_0} \qquad \text{(perfect gases)}. \tag{3.6}$$

(II) When Eq. (3.1c) is recast in a suitable dimensionless form, the gravitational body force term it contains, becomes $-g\tau|W_0|^{-1}$, where $|W_0|(\neq 0)$ denotes the magnitude of the signal that is driving the gas phase. In many situations of practical interest, τ is sufficiently small so that the following inequality is easily satisfied[1]:

$$\frac{g\tau}{c_0} < \epsilon(\ll 1), \tag{3.7}$$

where $\epsilon = c_0^{-1}|W_0|$ is the acoustic Mach number. [It is noteworthy that $g\tau$ is the *magnitude* of the (falling body) terminal velocity of a single particle, as is easily established on setting $w = 0$ in Eq. (3.1c) and recalling Eq. (3.2).]

[1] In the case of cigarette smoke in air, e.g., $\tau \sim 1$ µs [10, p. 552]; therefore, using $c_0 = 331\,\mathrm{m/s}$ [9, p. 29], which corresponds to dry air at $\vartheta_0 = 273.15\,\mathrm{K}$, and $g = 9.81\,\mathrm{m/s^2}$, the value near Earth's surface, $c_0^{-1}g\tau \approx 2.96 \times 10^{-8}$.

According to the linearization scheme under which System (2.1) was derived, ϵ is not only small with respect to unity, it is regarded as an *infinitesimal*, and therefore negligibly small, parameter. Seeking both practicality and analytical simplicity, we hereafter assume that τ is such that the above inequality is always satisfied and, as this implies that the impact of gravity on the motion of the particles is always negligibly small, replace g with zero in the particle momentum equation.

(III) Since g no longer appears in the particle momentum equation, we now take the ambient state of both phases to be not only inhomogeneous, i.e., P_{a}, ρ_{a}, ϑ_{a}, and n_{a} may vary with (at most) z, but also *quiescent* [9, p. 14], meaning that $\mathbf{u}_{\mathrm{a}} = \boldsymbol{\mathfrak{v}}_{\mathrm{a}} = (0,0,0)$. Achieving the latter while simultaneously satisfying System (3.1), in particular, Eq. (3.1b), requires that

$$P'_{\mathrm{a}}(z) = -g\rho_{\mathrm{a}}(z), \tag{3.8}$$

which is the condition of hydrostatic equilibrium for our 1D atmosphere; see [10, Example 2.4].

3.3. Equation of motion

On carrying out the simplifications contained in Assumptions (I)–(III), System (3.1) reduces to

$$\varrho_t + \rho'_{\mathrm{a}}(z)w + \rho_{\mathrm{a}}(z)w_z = 0, \tag{3.9a}$$

$$\rho_{\mathrm{a}}(z)w_t + p_z = -(m/\tau)n_{\mathrm{a}}(z)(w - \mathfrak{w}) - g\varrho, \tag{3.9b}$$

$$(1 + \tau\partial_t)\mathfrak{w} = w, \tag{3.9c}$$

$$p_t + \gamma P_{\mathrm{a}}(z)w_z = g\rho_{\mathrm{a}}(z)w, \tag{3.9d}$$

$$\mathfrak{n}_t + n'_{\mathrm{a}}(z)\mathfrak{w} + n_{\mathrm{a}}(z)\mathfrak{w}_z = 0, \tag{3.9e}$$

Now eliminating p, ϱ, and $\mathfrak{w}$ between the first four equations of System (3.9), we obtain the equation of motion (EoM) for the gas phase:

$$\left[1 + m\left(\frac{n_{\mathrm{a}}(z)}{\rho_{\mathrm{a}}(z)}\right)\right]w_{tt} - c_{\mathrm{a}}^2(z)w_{zz} + \gamma g w_z = \tau(c_{\mathrm{a}}^2(z)w_{tzz} - \gamma g w_{tz} - w_{ttt}). \tag{3.10}$$

In all that follows relating to the 1D case, however, we make the simplifying assumption $n_{\mathrm{a}}(z) \propto \rho_{\mathrm{a}}(z)$, specifically, that

$$n_{\mathrm{a}}(z) = (n_0/\rho_0)\rho_{\mathrm{a}}(z); \tag{3.11}$$

consequently, our EoM is reduced to

$$(1 + \kappa_0)w_{tt} - c_{\mathrm{a}}^2(z)w_{zz} + \gamma g w_z = \tau(c_{\mathrm{a}}^2(z)w_{tzz} - \gamma g w_{tz} - w_{ttt}), \tag{3.12}$$

where $\kappa_0 = n_0 m/\rho_0$ denotes the mass fraction of particles [10, p. 554].

In the next section we shall investigate Eq. (3.12) under the simplest, and rather limited (see Section 6.1), model of Earth's atmosphere; specifically, in Section 4 we present both analytical and numerical results for the case in which the ambient state of the *entire* atmosphere is assumed to behave *isothermally* with respect to acoustic propagation.

4. Isothermal ambient state: ormulation and analysis

4.1. Gas phase

The distinguishing assumption of this model, as Rayleigh [1] and others have discussed, is

$$\vartheta_{\mathrm{a}}(z) = \vartheta_0 \Rightarrow \rho_{\mathrm{a}}(z) = \rho_0 \exp(-z/H),\ P_{\mathrm{a}}(z) = P_0 \exp(-z/H),\ c_{\mathrm{a}}^2(z) = c_0^2, \tag{4.1}$$

where $H = c_0^2/(\gamma g)$ is known as the "scale height of the atmosphere" [10, p. 69]. (In the case of Earth, taking $\vartheta_0 = 283\,\mathrm{K}$ yields the value $H \approx 8284\,\mathrm{m}$.)

In the remainder of this section, we investigate the following initial-boundary value problem (IBVP):

$$(1+\kappa_0)w_{tt} - c_0^2 w_{zz} + \gamma g w_z = \tau(c_0^2 w_{tzz} - \gamma g w_{tz} - w_{ttt}), \qquad (z,t) \in (0,\infty)\times(0,\infty), \tag{4.2a}$$

$$w(0,t) = W_0\Theta(t), \quad \lim_{z\to\infty} |w(z,t)| < \infty, \quad t > 0, \tag{4.2b}$$

$$w(z,0) = 0, \quad w_t(z,0) = 0, \quad w_{tt}(z,0) = 0, \quad z > 0, \tag{4.2c}$$

which might be regarded as the acoustic version of Stokes' first problem[2]. Here, $\Theta(\zeta)$ denotes the Heaviside unit step function, we recall that $W_0(\neq 0)$ is the (constant) amplitude of the input signal that is driving the gas phase, and we observe that once w is determined, $\mathfrak{w}$ can be obtained with the aid of Eq. (3.9c).

Turning to the Laplace transform, we apply this well known (linear) operator to Eq. (4.2a) and the boundary conditions (BC)s. After employing the initial conditions and simplifying, we are led to consider the subsidiary equation

$$c_0^2(1+\tau s)\overline{w}'' - \gamma g(1+\tau s)\overline{w}' - s^2[(1+\kappa_0)+\tau s]\overline{w} = 0, \tag{4.3}$$

where s is the Laplace transform parameter and a bar over a quantity denotes the image of that quantity in the Laplace transform domain. Solving this ODE subject to the transformed BCs, which take the form

$$\overline{w}(0,s) = W_0/s, \qquad \lim_{z\to\infty} |\overline{w}(z,s)| < \infty, \tag{4.4}$$

yields

[2]See, e.g., SCHLICHTING [12]. From the physical acoustics standpoint, however, this IBVP is considered a *signaling problem*.

$$\overline{w}(z,s) = (W_0/s)\exp[z/(2H)] \times \exp\left[-\left(\frac{z}{2H}\right)\sqrt{\frac{1+\tau s+4c_0^{-2}H^2s^2(1+\kappa_0+\tau s)}{1+\tau s}}\,\right], \tag{4.5}$$

which is the exact solution of IBVP (4.2) in the transform domain.

In principle, the exact time-domain solution, $w(z,t)$, can be determined by applying the Laplace Inversion Theorem [5, 13] to Eq. (4.5). This procedure, however, would not only be extremely laborious to perform analytically, but the resulting sum of integrals is unlikely to elucidate the physics we seek to understand. Instead, we employ a simple, transform-based, methodology that Boley developed to determine the time-domain amplitude-value of a discontinuity that a given solution, or any of its time derivatives, might exhibit; see, e.g., [14, §4]. The main advantage of what some authors refer to as "Boley's criterion" is that it is applied in the transform domain, specifically, to the large-s expansion of one's transform domain solution.

To this end, we expand Eq. (4.5) for large-s, which yields

$$\overline{w}(z,s) \sim W_0 s^{-1}\exp(-sz/c_0)\exp\left[\left(1-\frac{c_0\kappa_0}{\gamma g\tau}\right)\frac{z}{2H}\right] \times\left[1+\left(\frac{\kappa_0(4+\kappa_0)}{8c_0\tau^2}-\frac{c_0}{8H^2}\right)\frac{z}{s}+\mathcal{O}(s^{-2})\right] \qquad (s\to\infty). \tag{4.6}$$

On applying the theorem given in [14, §4] (i.e., Boley's criterion) to Eq. (4.6), it is readily established that

$$[\![w]\!](t) = W_0\exp\left[-\left(\frac{\kappa_0}{\kappa_0^*}-1\right)\frac{c_0 t}{2H}\right], \tag{4.7}$$

which is the amplitude of the *jump* discontinuity exhibited by w across $\Sigma(t) = c_0 t$. Here, we introduce

$$\kappa_0^* := \gamma g\tau/c_0, \tag{4.8}$$

which is a critical value of κ_0. (In the case of Earth, taking $\tau = 1\,\mu\text{s}$ [10, p. 552] and $\vartheta_0 = 283\,\text{K}$ yields the value $\kappa_0^* \approx 4.07\times10^{-8}$.) In this presentation, we follow Morro [15, §3] and Straughan [16] and define the amplitude of the jump in a function $\mathfrak{F} = \mathfrak{F}(z,t)$ across a singular surface $z = \Sigma(t)$ as

$$[\![\mathfrak{F}]\!](t) := \mathfrak{F}^- - \mathfrak{F}^+, \tag{4.9}$$

where $\mathfrak{F}^\mp := \lim_{z\to\Sigma(t)^\mp}\mathfrak{F}(z,t)$ are assumed to exist, and where a "+" superscript corresponds to the region into which Σ is advancing while a "$-$" superscript corresponds to the region behind Σ.

With respect to the gas phase, then, the physical interpretation of the plane $z = \Sigma(t) = c_0 t$ is that of a *shock wave* [7] propagating in the $+z$-direction with speed c_0. Here, we observe that $w^+ = 0$ in the case of IBVP (4.2).

Hence, rejecting the case $\kappa_0 < \kappa_0^*$ for obvious reasons, and also rejecting the case $\kappa_0 = \kappa_0^*$ because of its unstable character, we hereafter limit our focus to

$$\kappa_0 > \kappa_0^*. \tag{4.10}$$

This case, as illustrated in Section 4.3, yields solutions that satisfy IBVP (4.2) in its entirety, in particular, the second BC therein.

4.2. Particle phase

In the case of the particle phase we have, using the Laplace transformed version of Eq. (3.9c) and Eq. (4.5),

$$\begin{aligned}\overline{\mathfrak{w}}(z,s) = \left(\frac{\tau^{-1}W_0}{s^2 + s/\tau}\right)\exp[z/(2H)]\\ \times \exp\left[-\left(\frac{z}{2H}\right)\sqrt{\frac{1+\tau s + 4c_0^{-2}H^2 s^2(1+\kappa_0+\tau s)}{1+\tau s}}\right],\end{aligned} \tag{4.11}$$

from which we easily obtain

$$\begin{aligned}&\overline{\mathfrak{w}}(z,s) \sim (W_0/\tau)s^{-2}\exp(-sz/c_0)\\ &\times \exp\left[\left(1 - \frac{c_0\kappa_0}{\gamma g \tau}\right)\frac{z}{2H}\right]\left[1 + \left(\frac{\kappa_0(4+\kappa_0)}{8c_0\tau^2} - \frac{c_0}{8H^2}\right)\frac{z}{s} + \mathcal{O}(s^{-2})\right] \quad (s \to \infty).\end{aligned} \tag{4.12}$$

Applying Boley's criterion to this expansion reveals that while $[\![\mathfrak{w}]\!](t) = 0$,

$$[\![\mathfrak{w}_t]\!](t) = \tau^{-1}[\![w]\!](t) = (W_0/\tau)\exp\left[-\left(\frac{\kappa_0}{\kappa_0^*} - 1\right)\frac{c_0 t}{2H}\right] \tag{4.13}$$

and

$$[\![\mathfrak{w}_z]\!](t) = -\left(\frac{W_0}{c_0\tau}\right)\exp\left[-\left(\frac{\kappa_0}{\kappa_0^*} - 1\right)\frac{c_0 t}{2H}\right], \tag{4.14}$$

where we observe that both of these jumps *also* occur across $\Sigma(t) = c_0 t$. Here, $[\![\mathfrak{w}_z]\!](t)$ was determined using the expression for $[\![\mathfrak{w}_t]\!](t)$ and the $[\![\mathfrak{F}]\!](t) = 0$ special case of

$$\frac{\mathfrak{d}[\![\mathfrak{F}]\!](t)}{\mathfrak{d}t} = [\![\mathfrak{F}_t]\!](t) + c_0[\![\mathfrak{F}_z]\!](t), \tag{4.15}$$

which is often referred to as the *kinematic condition of compatibility*[3], and it should be noted that $\mathfrak{w}_z^+ = 0$ in Eq. (4.14).

Thus, with respect to the particle phase, the physical interpretation of the plane $z = c_0 t$ is that of an *acceleration wave* [7], which is the weak discontinuity of lowest possible order [17, p. 182], that is propagating in the $+z$-direction with speed c_0. Thus, like the case in which the ambient state of both phases is assumed to be homogeneous [8], the velocity field of the particle phase associated with the present IBVP is found to admit a singular surface one order *higher* than that of its gas phase counterpart.

4.3. Numerical results

While clearly important, our singular surface results do not give us any information on the behavior of the solution profiles *behind* $\Sigma(t)$, i.e., on the interval $0 < z < c_0 t$. Moreover, it is of interest to compare the evolution of the velocity profile of the gas phase with that of its particle phase counterpart. As such, in this subsection we compute and plot the solution of IBVP (4.2) by numerically inverting Eq. (4.5), as well as Eq. (4.11), using the modified Tzou series[4]

$$\begin{aligned}(4.16)\qquad \mathfrak{F}(z,t) \approx{}& \frac{\exp(4.7)}{t}\\ &\times \left\{ \frac{1}{2}\overline{\mathfrak{F}}\left(z, \frac{4.7}{t}\right) + \mathrm{Re}\left[\sum_{j=1}^{M}(-1)^j\, \overline{\mathfrak{F}}\left(z, \frac{4.7 + \mathrm{i}j\pi}{t}\right) \mathrm{sinc}\left(\frac{j\pi}{M}\right)\right]\right\} \quad (t > 0),\end{aligned}$$

where $M \gg 1$ is an integer and

$$(4.17)\qquad \mathrm{sinc}(\zeta) := \begin{cases} \zeta^{-1}\sin(\zeta), & \zeta \neq 0, \\ 1, & \zeta = 0. \end{cases}$$

The plots shown in Fig. 1, wherein we have set $\mathsf{W} := w/W_0$ and $\mathcal{W} := \mathfrak{w}/W_0$ to reduce the number of parameters, clearly illustrate the main analytical results presented in the previous subsection; e.g., the fact that $\Sigma(t) = c_0 t$ for *both* phases. Also, while we employ a relatively large value of $\tau(= 0.01\,\mathrm{s})$, so as to produce clear, easy to view plots, the left-hand side of the inequality in Eq. (3.7) remains quite small ($\sim \mathcal{O}(10^{-4})$).

In the upper panel of Fig. 1 we see that, as predicted by Eq. (4.7), the amplitude of the shock which the gas phase exhibits is a (exponentially) decreasing function of t when $\kappa_0 > \kappa_0^*$. What is also clear from this panel is that

$$(4.18)\qquad W_0^{-1}[\![w]\!](t) = [\![\mathsf{W}]\!](t) < \mathsf{W} < 1 \qquad (z,t) \in (0, c_0 t) \times (0, \infty);$$

[3]See, e.g., [16] and the references cited therein; see also BLAND [17, §6.9], who refers to this relation as "Hadamard's lemma".

[4]See [6] and the references cited therein.

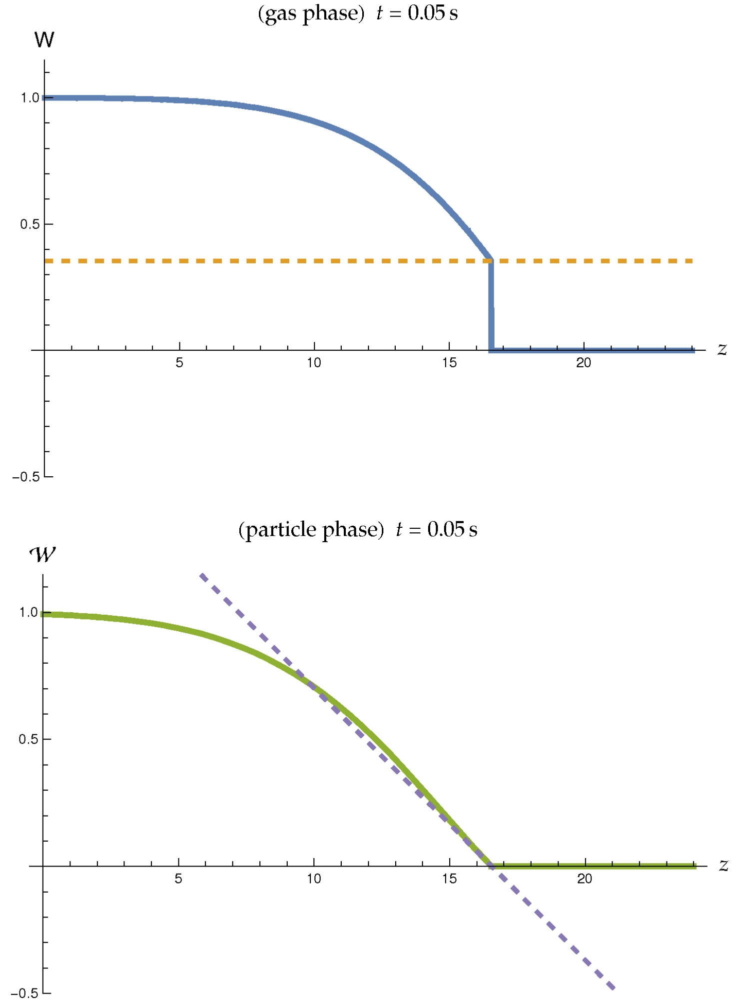

FIG. 1. Blue curve: W vs. z based on Eq. (4.5). Green curve: $\mathcal{W}$ vs. z based on Eq. (4.11). Orange-dashed line: $[\![\mathsf{W}]\!](t)$ vs. z (see Eq. (4.7)). Purple-dashed line: $(z - c_0 t)\,[\![\mathcal{W}_z]\!](t)$ vs. z (see Eq. (4.14)). Herein, we have taken $\tau = 0.01\,\mathrm{s}$, $\gamma = 1.4$, $g = 9.81\,\mathrm{m/s^2}$, $c_0 = 331\,\mathrm{m/s}$, and $\kappa_0 = 1000\kappa_0^*$, where for these values of τ, γ, g, and c_0 Eq. (4.8) yields $\kappa_0^* \approx 0.000415$. Both plots were generated using Eq. (4.16) with $M = 25000$.

i.e., behind $z = \Sigma(t)$, the W vs. z profile is bounded and strictly decreasing when $\kappa_0 > \kappa_0^*$. The lower panel of Fig. 1, on the other hand, illustrates the fact that the acceleration wave amplitude given in Eq. (4.14) is simply the slope of the tangent to the bounded, *and* continuous, $\mathcal{W}$ vs. z (i.e., particle phase) profile at the wavefront, which is located at the point $(z, \mathcal{W}) = (\Sigma(t), 0)$. We also see in the lower panel of Fig. 1 that, like the absolute value of its slope at the wavefront, the $\mathcal{W}$ vs. z profile is strictly decreasing, for $z \in (0, c_0 t)$.

5. Discussion

In the present investigation we have shown that by introducing more and more fine particles, i.e., increasing the mass fraction of particles κ_0 (> 0), we are able to first reduce the rate ($\Rightarrow \kappa_0 \in (0, \kappa_0^*)$), then "zero-out" the rate ($\Rightarrow \kappa_0 = \kappa_0^*$), and finally *reverse* the rate ($\Rightarrow \kappa_0 > \kappa_0^*$) at which amplitude blow-up occurs in the case of IBVP (4.2).

The reason for this is clear: the fact that $\rho_{\mathrm{a}}(z) \to 0$ as $z \to \infty$ means that, under the isothermal model, our upward propagating acoustic signal begins to encounter a medium that is tending to the vacuum state; as such, the continuum assumption, on which our model system is based, begins to breakdown. By introducing fine particles, however, i.e., increasing $\kappa_0(> 0)$ sufficiently, we are able to *compensate* for the decreasing gas density ahead of our signal. In fact, for $\kappa_0 > \kappa_0^*$ the behavior of our acoustic signal is, qualitatively, the same as if it were propagating in a dusty gas—one also described by the linearized MT-1 model—but wherein the gas phase is a *homogeneous* perfect gas; compare the plots in Fig. 1 with their counterparts in [8, §3.2].

And lastly, it is noteworthy that, as in the case of $[\![w]\!](t)$, the behavior exhibited by the shock amplitude in [15], wherein a "hidden variables" based fluid model was assumed, depends on a critical value; see [15, p. 198], and note that Γ is a critical value of the volume gradient.

6. Possible follow-on studies

6.1. Taylor's two-layer atmosphere model

While the isothermal model provides a suitable approximation of the actual ambient temperature profile in the stratosphere, it fails to do the same at lower altitudes, specifically, in the troposphere, where the ambient temperature profile is known to be a linearly decreasing function of z; see, e.g., [10, Fig. 2.3]. In 1929, TAYLOR [18], seeking to formulate a more accurate description of propagation in Earth's atmosphere, combined these two models, which until then had been treated separately, into a single formulation based on a "two-layer" atmosphere. In its most general form, Taylor's model reads

$$(6.1)\quad \vartheta_{\mathrm{a}}(z) = \begin{cases} \vartheta_0 - \beta z, & z \in (0, z_{\mathrm{i}}), \\ \vartheta_{\mathrm{s}}, & z \geq z_{\mathrm{i}}, \end{cases}$$

$$\Rightarrow \quad \varrho_{\mathrm{a}}(z) = \varrho_0 \begin{cases} (1 - \beta z/\vartheta_0)^{\varpi}, & z \in (0, z_{\mathrm{i}}), \\ \exp[-(z - z_{\mathrm{i}})/\widehat{H}](1 - \beta z_{\mathrm{i}}/\vartheta_0)^{\varpi}, & z \geq z_{\mathrm{i}}, \end{cases}$$

$$P_{\mathrm{a}}(z) = P_0 \begin{cases} (1-\beta z/\vartheta_0)^{1+\varpi}, & z \in (0, z_{\mathrm{i}}), \\ \exp[-(z-z_{\mathrm{i}})/\widehat{H}](1-\beta z_{\mathrm{i}}/\vartheta_0)^{1+\varpi}, & z \geq z_{\mathrm{i}}, \end{cases}$$

$$c_{\mathrm{a}}^2(z) = \begin{cases} c_0^2(1-\beta z/\vartheta_0), & z \in (0, z_{\mathrm{i}}), \\ c_{\mathrm{s}}^2, & z \geq z_{\mathrm{i}}. \end{cases}$$

Here, the parameter β (> 0) carries (SI) units of K/m, i.e., β is the magnitude of the temperature gradient in the troposphere; $\widehat{H} := \vartheta_{\mathrm{s}}(1-1/\gamma)/\beta_1$, where, following LAMB [2, p. 546], we have set $\beta_1 := g/c_p$; the *tropopause* [19, §4], i.e., the interface between the two layers, lies at $z = z_{\mathrm{i}}$; and we have also set $\vartheta_{\mathrm{s}} := \vartheta_0 - \beta z_{\mathrm{i}}$, $c_{\mathrm{s}}^2 := c_p(\gamma-1)\vartheta_{\mathrm{s}}$, and

$$\varpi := -1 + \frac{g}{\beta(c_p - c_v)} = -1 + \frac{\gamma\beta_1}{\beta(\gamma-1)}, \tag{6.2}$$

where we note that ϖ is denoted by n in [18]. It is noteworthy that in his analysis, Taylor [18] took $z_{\mathrm{i}} = 13000\,\mathrm{m}$, $\vartheta_0 = 283\,\mathrm{K}$, and $\beta = \frac{1}{2}\beta_1$, from which he obtained $\vartheta_{\mathrm{s}} \approx 220\,\mathrm{K}$; Pekeris [19, §4], in contrast, took $z_{\mathrm{i}} = 10300\,\mathrm{m}$, $\vartheta_{\mathrm{s}} = 220\,\mathrm{K}$, and $\beta = \frac{7}{11}\beta_1$, which gave him $\vartheta_0 \approx 284\,\mathrm{K}$.

Of course, IBVPs formulated around Eq. (3.12) and Taylor's two-layer model lend themselves to treatment by the Laplace transform, and the use of Boley's criterion, after suitable interface condition(s) are specified. An obvious application of such IBVPs is the modeling of vertical propagation in regions of the atmosphere containing particulates, e.g., volcanic dust/ash, in which case the values of τ and κ_0 may *not* be the same in both layers.

6.2. The special case $P_{\mathrm{a}}(x,y,z) :=$ const.

Invoking this assumption, along with taking $\mathbf{b} = (0,0,0)$, allows us to eliminate ϱ, p, and $\mathfrak{v}$ between the second, third, and fourth equations of System (2.1). The resulting *multi-D* EoM reads

$$\begin{aligned}\left[1 + m\left(\frac{n_{\mathrm{a}}(x,y,z)}{\rho_{\mathrm{a}}(x,y,z)}\right)\right]\mathbf{u}_{tt} + \tau\mathbf{u}_{ttt} \\ = \rho_{\mathrm{a}}^{-1}(x,y,z)(1+\tau\partial_t)\boldsymbol{\nabla}[\rho_{\mathrm{a}}(x,y,z)c_{\mathrm{a}}^2(x,y,z)\boldsymbol{\nabla}\cdot\mathbf{u}].\end{aligned} \tag{6.3}$$

If the product $\rho_{\mathrm{a}}(x,y,z)c_{\mathrm{a}}^2(x,y,z) = \text{const}$ (i.e., if $A_{\mathrm{a}}(x,y,z) = \text{const}$) as well, then Eq. (6.3) simplifies to

$$\left[1 + m\left(\frac{n_{\mathrm{a}}(x,y,z)}{\rho_{\mathrm{a}}(x,y,z)}\right)\right]\mathbf{u}_{tt} + \tau\mathbf{u}_{ttt} = c_{\mathrm{a}}^2(x,y,z)[\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \tau\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}_t)]. \tag{6.4}$$

6.3. Poroacoustic special case

System (2.1), we observe, is a special case of the linearized, inhomogeneous fluid, version of the dual-phase continuum model discussed by Ciarletta and Straughan; see Eqs. (2.1)–(2.4) of [20], as well as [21, §1.9.1] and the references cited therein. As such, it becomes a model of poroacoustic propagation—one based on Darcy's law—on setting $\mathfrak{v} = (0, 0, 0)$, omitting Eqs. (2.1c) and (2.1e), and replacing the product $(6\pi a)n_{\rm a}(x, y, z)$ in Eq. (2.1b) with the (*not* necessarily constant) ratio χ/K, where $\chi \in (0, 1)$ and K denote, respectively, the porosity and permeability of the rigid, stationary, porous solid under consideration; see [20, §2], as well as [21, §8.1] and the reference cited therein.

6.4. Analytically intractable problems

Often, problems of practical interest are, from the analytical standpoint, intractable. With regard to System (2.1) and its aforementioned special cases, it appears that many (most?) such problems can be treated numerically using the elaborate, and in some cases elegant, homogenization methodologies one can find in the literature; see, e.g., [22, 23, 24], and the references cited therein. Of those which we have studied, the "two-pronged" approach described by E [25] appears to be especially promising. Briefly, E prescribes the following: Perform homogenization via what has been termed the *Heterogeneous multi-scale method* (HMM)[5], followed by use of the appropriate finite-volume scheme [22] to solve the resulting "macro-scale" model. In future works we hope to apply E's approach to not only System (2.1) and its special cases, but to their finite-amplitude (i.e., weakly-nonlinear) extensions as well.

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