ALGORITHM OF DETERMINING THE COEXISTENCE LEVELS OF UNKNOWNS AND THEIR BINDING FUNCTIONS IN LINEAR EQUATION SYSTEMS

Mieczysław Kwaśniak

Department of Engineering Surveying
Warsaw University of Technology
Pl. Politechniki 1, PL-00 661 Warsaw, Poland

ABSTRACT

The study presents a unified algorithm of determining the coexistence levels in any system of linear equations. The coexistence levels can be determined both for the system unknowns (or groups of) as well as for functions binding those unknowns (equations or their groups). Because we deal with systems of linear equations also in geodetic networks, the presented algorithm allows the determination of coexistence levels of network points or observations made within that network. Due to the possibility of grouping of unknowns and equations in the algorithm, there are no limitations of space for the geodetic network. The functioning of the presented algorithm has been illustrated by the example of a linear, horizontal geodetic network. Exemplary tasks have also been shown, in which the coexistence levels can be helpful.

1. INTRODUCTION

If we examine any combination of connections of elements through other elements, so for such network structure there exists a notion of distance between elements in topological sense. Two elements of a given network structure can be connected directly, but they can also be connected through other elements. A good measure of such distance between elements is the so called coexistence level. It is a distance measure of two elements of network structure along the shortest way, expressed in number of indirect elements. In geodetic literature, this notion has been introduced in relation to geodetic network by Prof. Adamczewski (Adamczewski 1971), who defined the coexistence level of points. In the work (Prószyński, Kwaśniak 2002), the notion of coexistence level of observations in geodetic network has been introduced along with the algorithm for the determination such levels basing on the codes assigned to observations.

In this paper, the notion of coexistence level has been related to any system of linear equations, in which in subsequent equations the relations between unknowns are written. Presented is a new unified algorithm for the determination of coexistence levels of unknowns and coexistence levels of equations. Because the system of observation equations in geodetic networks is also a system of linear equations, the presented algorithm can also be used for the determination of coexistence levels of points or observations within the network. The proposed algorithm is illustrated on numerical example.

2. UNIFIED ALGORITHM FOR DETERMINATION OF COEXISTENCE LEVELS

Let's assume that we have given a system of linear equations:

$$
\begin{aligned}
&a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = l_1 \\
&a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = l_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = l_m
\end{aligned}
\tag{1}
$$

Each of those equations constitutes information about connections of the unknowns x_1, x_2, ... , x_n. We can say that trough the equations above, the space of unknowns is connected with the space of information. We are interested in structure topology of connections of those two spaces. To generalize the proposed algorithm, let's assume the possibility of grouping both of the unknowns, as well as equations and search for connections among such groups.
The input data for the proposed algorithm are the coefficients of equation system, presented as matrix A($m\times n$). Depending on our interests – coexistence levels of unknowns (or their groups) or equations (or their groups), we submit matrix A for processing according to the formulas below:

<u>For coexistence levels of unknowns:</u>

$$
\{\mathbf{A}_c\}_{i,s} = \begin{cases} 0 & \text{for } \sum_{j=1}^{n_s} \left|\{\mathbf{A}\}_{i,j}\right| = 0 \\ 1 & \text{for } \sum_{j=1}^{n_s} \left|\{\mathbf{A}\}_{i,j}\right| > 0 \end{cases} \qquad i = 1, 2, \ldots, m \qquad s = 1, \ldots, p \tag{2}
$$

where s is the number of unknown group, n_s is the number of unknowns in the given group and p is the number of groups.

<u>For coexistence levels of equations:</u>

$$
\{\mathbf{A}_c\}_{z,i} = \begin{cases} 0 & \text{for } \sum_{j=1}^{n_z} \left|\{\mathbf{A}\}_{j,i}\right| = 0 \\ 1 & \text{for } \sum_{j=1}^{n_z} \left|\{\mathbf{A}\}_{j,i}\right| > 0 \end{cases} \qquad i = 1, 2, \ldots, n \qquad z = 1, \ldots, q \tag{3}
$$

where z is the number of equation group, n_z is the number of equations in the given group and q is the number of groups.
Next we create matrix $\mathbf{A}_*$:

$$
\mathbf{A}_* = \begin{cases} \mathbf{A}_c^{\mathrm{T}}\mathbf{A}_c & \text{for the determination of coexistence levels of unknowns} \\ \mathbf{A}_c\mathbf{A}_c^{\mathrm{T}} & \text{for the determination of coexistence levels of equations} \end{cases} \tag{4}
$$

Matrix $\mathbf{A}_*$ we transform into a binary form B:

$$\{\mathbf{B}\}_{i,j} = \begin{cases} 0 & \text{for } \{\mathbf{A}_*\}_{i,j} = 0 \\ 1 & \text{for } \{\mathbf{A}_*\}_{i,j} > 0 \end{cases} \qquad i, j = 1, 2, \dots, w \qquad (5)$$

where $w = p$ if we determine the coexistence levels of unknowns or $w = q$ when we determine coexistence levels of equations.
Non-diagonal elements of matrix B with values equal to 1 show us pairs of unknowns (or their groups) or pairs of equations (or their groups) remaining in the so called direct coexistence, which means coexistence level $r_{i,j} = 1$. For $i = j$ we deal with a special case, where the given quantity remains in direct coexistence with itself ($r_{i,j} = 0$). This fact we note in a table of coexistence levels when closing the determination of those levels. To shorten the text, within the course of this paper we will use the notation „unknown/equation" to point out the alternative of determining coexistence levels of unknowns or equations.
To determine higher coexistence levels, we will use the property of the von Neumann matrix series

$$(\mathbf{I} - \mathbf{P})^{-1} = \mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \mathbf{P}^3 + \dots = \sum_{k=0}^{\infty} \mathbf{P}^k \qquad (6)$$

which is convergent with $\|\mathbf{P}\| < 1$.
Using the property mentioned above, the paper (Nowak, Nowak 2005) contains the way of calculation of reliability matrix R. It has been also found that subsequent matrix powers $\mathbf{I} - \mathbf{A}\mathbf{A}^{\mathrm{T}}$ characterize the influence of the given observation on another observations within the geodetic network. This statement is confirmed by studies on networks of different types and sizes, presented in (Kwaśniak 2008), out of which it results that the elements of matrix R can be approximated by use of an equipotential function, of which the independent variable constitutes the coexistence level of observations. A similar conclusion, but related to the dependence between points within the geodetic network, as well as their coexistence levels has been presented in (Adamczewski 1971).
The conclusions from above studies have been an inspiration for the elaboration of an algorithm for the determination of higher coexistence levels than 1. Searching pairs of unknowns/equations remaining in the second and higher coexistence levels, we calculate the second term (and next) of the power series $\mathbf{B}^r$ (r = 2, 3, ...). New non-zero elements of matrix $\mathbf{B}^r$ (in comparison with elements of matrix $\mathbf{B}^{r-1}$) respond to the pairs of unknowns/equations remaining in r coexistence level. We take this fact into account in matrix $\mathbf{B}^r$, simultaneously realizing modifications of elements according to the formula below:

$$\{\mathbf{B}^r\}_{i,j} = \begin{cases} r & \text{for } \{\mathbf{B}^{r-1}\}_{i,j} = 0 \text{ and } \{\mathbf{B}^{r-1}\mathbf{B}\}_{i,j} > 0 \\ \{\mathbf{B}^{r-1}\}_{i,j} & \text{for } \{\mathbf{B}^{r-1}\}_{i,j} > 0 \end{cases} \qquad i, j = 1, 2, \dots, w \qquad (7)$$

We finish the determination of coexistence levels, if in matrix $\mathbf{B}^r$ there are no more elements with the value 0. The block diagram of algorithm for the determination of coexistence levels is presented in Fig. 1.

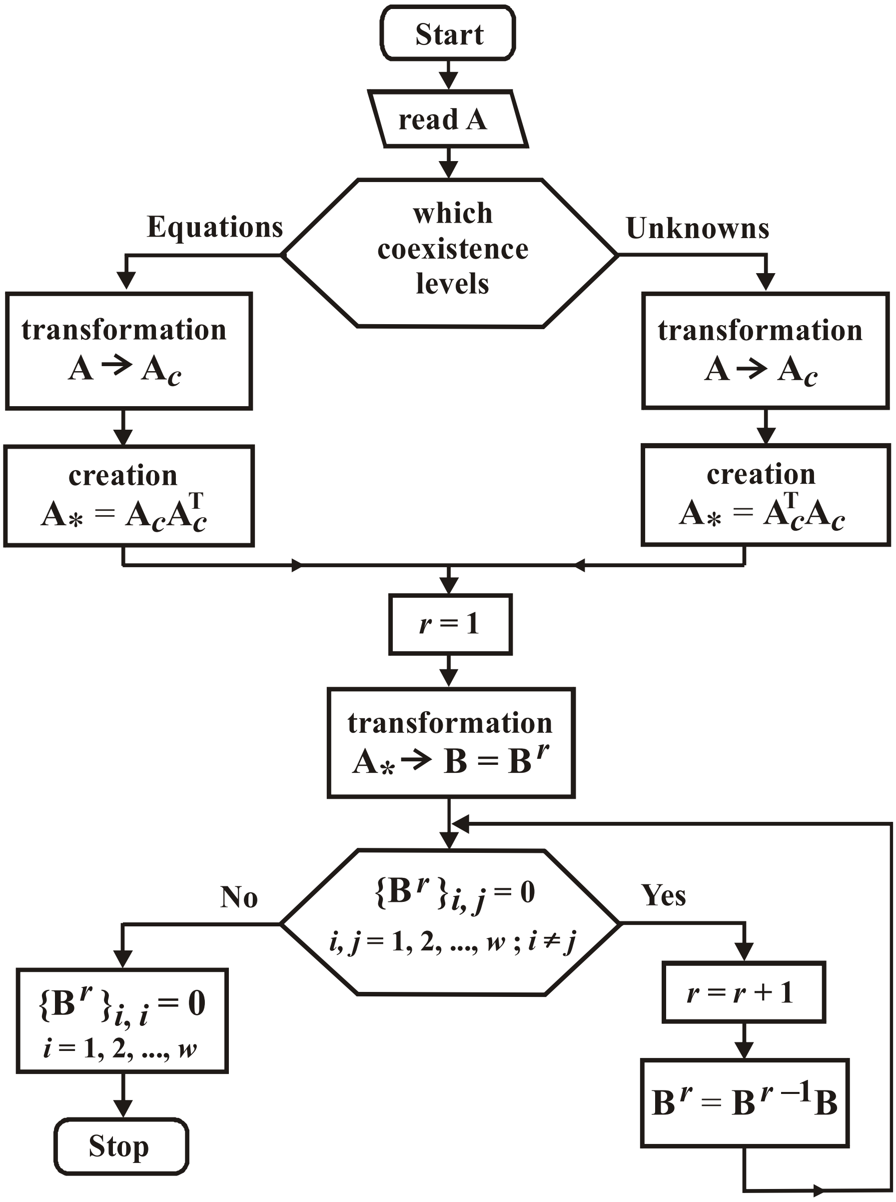


Fig. 1. Block diagram of the algorithm for determination of coexistence levels.

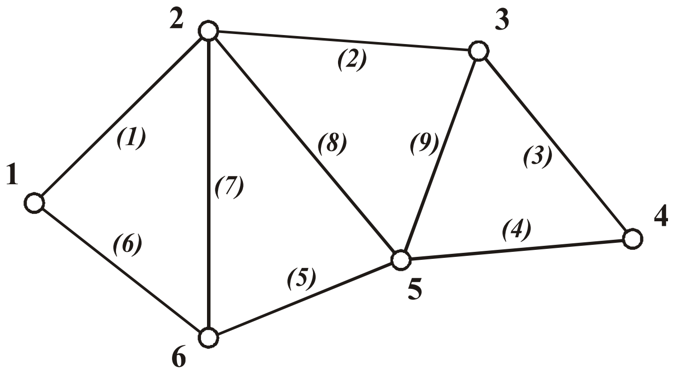
In practice there is often a need to determine the coexistence level for a specific pair of unknowns/equations, though not necessarily for all. The algorithm above allows the realization of such task as well. Let's assume that we want to determine the coexistence level between unknowns/equations having numbers i and j. After creation of matrix B, we choose its line $B_{i\bullet}$ (for $r = 1$ it's $B = B^r$). The subsequent procedure is realized in a loop and consists in verifying of condition $\{B^r_{i\bullet}\}_j = 0$. If this condition is fulfilled, than the coexistence level of the chosen pair of unknowns/equations $r_{i,j} = r$. Otherwise we augment the currently determined coexistence level $r = r + 1$ and calculate as follows:

$$\{B^r_{i\bullet}\}_k = \begin{cases} r & \text{for } \{B^{r-1}_{i\bullet}\}_k = 0 \text{ and } \{B^{r-1}_{i\bullet}B\}_k > 0 \\ \{B^{r-1}_{i\bullet}\}_k & \text{for } \{B^{r-1}_{i\bullet}\}_k > 0 \end{cases} \qquad k = 1, 2, \dots, w \qquad (8)$$

A special case of a system of linear equations is a system of observation equations in geodetic network, where the unknowns are the coordinates of points and the equations respond to the observations. So the algorithm presented above can be used for determination of coexistence points and observations in geodetic networks of any space dimension (leveling, two-dimensional and three-dimensional network). That's why the functioning of the proposed algorithm has been illustrated below on an example of the linear geodetic network.

3. NUMERICAL EXAMPLE

In a linear network, as shown in Fig. 2, determine the coexistence levels for the points and observations.



Point No.	X [m]	Y [m]
1	55	15
2	100	60
3	95	130
4	45	170
5	40	110
6	20	60

Fig. 2. Two-dimensional network with linear observations and approximate coordinates of points (*(x)* means number of observation).

<u>Realization of the algorithm with the goal to determine coexistence levels of points:</u>

1. **Creation of a coefficient matrix A of observation equations (sequence of unknowns: $X_1, Y_1, X_2, Y_2, \ldots, X_6, Y_6$):**

$$\mathbf{A} = \begin{bmatrix} -0.71 & -0.71 & 0.71 & 0.71 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.07 & -0.99 & -0.07 & 0.99 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.78 & -0.62 & -0.78 & 0.62 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.08 & 0.99 & -0.08 & -0.99 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.37 & 0.93 & -0.37 & -0.93 \\ 0.61 & -0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.61 & 0.79 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0.77 & -0.64 & 0 & 0 & 0 & 0 & -0.77 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.94 & 0.34 & 0 & 0 & -0.94 & -0.34 & 0 & 0 \end{bmatrix}$$

2. **Grouping of unknowns X_i and Y_i with transformation according to (2) into matrix A_c**

$$\mathbf{A}_c = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

3. **Establishing according to (4) of matrix $\mathbf{A}_*$ and transforming it according to (5) into a binary form B:**

$$\mathbf{B}=\begin{bmatrix}1&1&0&0&0&1\\1&1&1&0&1&1\\0&1&1&1&1&0\\0&0&1&1&1&0\\0&1&1&1&1&1\\1&1&0&0&1&1\end{bmatrix}$$

4. **Because B contains non-diagonal elements with value 0, we assume r = 2 an calculate according to (7) matrix $\mathbf{B}^2$:**

$$\mathbf{B}^2=\begin{bmatrix}1&1&2&0&2&1\\1&1&1&2&1&1\\2&1&1&1&1&2\\0&2&1&1&1&2\\2&1&1&1&1&1\\1&1&2&2&1&1\end{bmatrix}$$

5. **Because $\mathbf{B}^2$ contains non-diagonal elements with value 0, we assume r = 3 and calculate according to (7) matrix $\mathbf{B}^3$:**

$$\mathbf{B}^3=\begin{bmatrix}1&1&2&3&2&1\\1&1&1&2&1&1\\2&1&1&1&1&2\\3&2&1&1&1&2\\2&1&1&1&1&1\\1&1&2&2&1&1\end{bmatrix}$$

6. **Because $\mathbf{B}^3$ does not contain any elements having value 0 anymore, we neutralize diagonal elements of matrix $\mathbf{B}^3$ and obtain finally a matrix of coexistence levels of points $\mathbf{K}_p$:**

$$\mathbf{K}_p=\begin{bmatrix}0&1&2&3&2&1\\1&0&1&2&1&1\\2&1&0&1&1&2\\3&2&1&0&1&2\\2&1&1&1&0&1\\1&1&2&2&1&0\end{bmatrix}$$

Analogically we can determine the matrix of coexistence levels of observations $\mathbf{K}_o$.
In case if the goal of determination would be a coexistence level for a specific pair of points, i.e. i =1, j = 4, than from matrix B we chose line i (r =1):

$$\mathbf{B}_{i\bullet}=\begin{bmatrix}1&1&0&0&0&1\end{bmatrix}$$

If the j-element $\left\{\mathbf{B}_{i\bullet}\right\}_j$ is zero, than r = 2 and this line we multiply according to (8) by matrix B:

$$\mathbf{B}_{i\bullet}^2=\begin{bmatrix}1&1&2&0&2&1\end{bmatrix}$$

Because $\left\{ \mathbf{B}_{i\bullet}^{2} \right\}_j = 0$, than $r = 3$ and line $\mathbf{B}_{i\bullet}^{2}$ we multiply by matrix B:

$$\mathbf{B}_{i\bullet}^{3} = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

This time $\left\{ \mathbf{B}_{i\bullet}^{3} \right\}_j = 3$, which means that the coexistence level of points 1 and 4 amounts to $r_{1,4} = 3$.

Additionally, presented below is matrix $\mathbf{A}_c$ as binary form of matrix A and matrix $\mathbf{K}_o$ of coexistence levels of observations for the analyzed network.

$$\mathbf{A}_c = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{K}_o = \begin{bmatrix} 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 2 & 3 & 2 & 2 & 1 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

4. SUMMARY

The algorithm presented in this paper for the determination of coexistence levels between elements of one type in linear equation systems can be useful in topological analysis of different type of network structures. The matrix $\mathbf{K}_b$ and $\mathbf{K}_o$ jointly unequivocally describe the topology of analyzed network. Due of the possibility of grouping of some unknowns, the analyzed network can be multidimensional. As shown in the numerical example, this algorithm can be used for the determination of coexistence levels both of points, as well as observations in geodetic networks. The algorithm tolerates observation repetitions within a network and ignores the influence of standardization of observation equations.

The coexistence levels of points determined by use of the proposed algorithm can be useful during designing or accuracy analysis of large geodetic networks. While the coexistence levels of observations can be used to accelerate the diagnostic process in case, if the geodetic network contains numerous gross errors.

REFERENCES

Adamczewski Z., (1971) Nieliniowa analiza dokładności sieci geodezyjnej (Non-linear accuracy analysis of geodetic network), Kwartalnik Naukowy PAN Geodezja I Kartografia, Nr 3, t. XX; str. 209-223.

Kwaśniak M., (2008) Estimation of post-adjustment correlations between observations on the basis of their topological coexistence in the network, Kwartalnik Naukowy PAN Geodezja i Kartografia, Vol. 57, No. 2, str. 45-60.

Nowak E., Nowak J., (2005): Boundary properties of the reliability matrix, Reports on Geodesy No. 3 (74), pp. 65-75.

Prószyński W., Kwaśniak M., (2002): Niezawodność sieci geodezyjnych (Reliability of geodetic networks), Oficyna Wydawnicza Politechniki Warszawskiej, Warszawa.