DIFFERENCES IN MATHEMATICAL MODELS OF GPS AND PSEUDOLITE OBSERVATION EQUATIONS

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ABSTRACT

Some error sources in GPS surveys, such as linearization error and orbital bias, are well known and do not require further investigation. The impact of these errors in GPS + pseudolite observations can be different and must be considered separately. This paper presents influence of orbital bias and linearization error on single differenced pseudolite observations.

1. INTRODUCTION

Pseudolite is basically a GPS satellite transmitter placed on the ground (Dai, 2003), (Rzepecka, 2005), (Wang, 2001). Overall concept of observation equation is the same as for GPS satellites. In details these equations are slightly different. Since many of error sources in satellite observations are eliminated in differencing process thanks to large distance between transmitter and receiver, in pseudolite observation, equations must be corrected to eliminate these differences.

First and obvious difference is lack of ionosphere term in pseudolite equation since the signal is travelling only through the troposphere. Troposphere must also be taken into account differently than for GPS satellites observations (Rapinski, 2005).

Two error sources require further investigation: pseudolite antenna location error and linearization error.

2. IMPACT OF LINEARIZATION ON PSEUDORANGE AND CARRIER PHASE OBSERVATION EQUATION

The simple mathematical models for pseudorange and carrier phase observations are:

$$R=\rho+c\Delta\delta \tag{1}$$

$$\Phi = \frac{1}{\lambda}\rho + N + f\Delta\delta,$$

where:

R – measured code pseudorange
ρ – geometric satellite (pseudolite)-reciever distance determined on the basis of approximate coordinates
c – speed of light
$\Delta\delta$ – clock bias
Φ – carrier phase in cycles
λ – wavelengh

N – integer ambiguity
f – frequency of satellite signal

Multiplying the second equation by λ we obtain the measurement's equation of carrier phase in metres:

$$\lambda\Phi = \rho + \lambda N + c\Delta\delta \qquad (2)$$

Usually the equations (1) are solved after linearization. In linearization process the second and higher terms of expansion in Taylor's series are neglected. It can be useful to check if the neglected terms are always not significant or there can be some conditions in which they are significant. In order to solve that problem the formulas of the second derivatives of nonlinear term from equations (1) and (2) are derived:

$$\frac{\delta^2\rho}{\delta x^2} = \frac{\rho^2 - \Delta x^2}{\rho^3}, \quad \frac{\delta^2\rho}{\delta y^2} = \frac{\rho^2 - \Delta y^2}{\rho^3}, \quad \frac{\delta^2\rho}{\delta z^2} = \frac{\rho^2 - \Delta z^2}{\rho^3} \qquad (3)$$

where:

$$\rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

The common influence Ψ of second terms of expansion in Taylor's series on models (1) and (2) can be estimated after assuming unit values of increments dx=dy=dz=1:

$$\Psi = \frac{1}{2}\frac{\delta^2\rho}{\delta^2 x}dx^2 + \frac{1}{2}\frac{\delta^2\rho}{\delta^2 y}dy^2 + \frac{1}{2}\frac{\delta^2\rho}{\delta^2 z}dz^2 = -\frac{1}{\rho}. \qquad (4)$$

The graph of Ψ vs ρ is presented in fig 1.

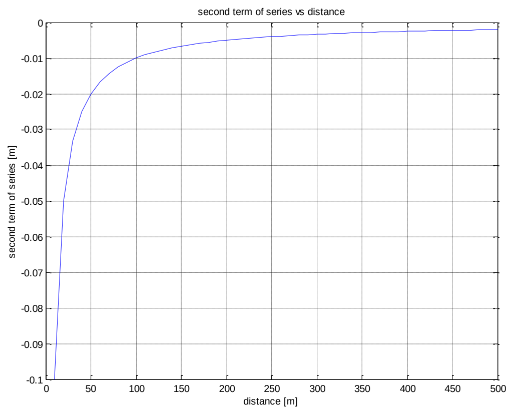

Fig. 1. Second term of series vs. distance.

For GPS satellites that are 20 000 km from the receiver the influence of second term of Taylor series is negligibly small (10^{-7} m). For the pseudolite observations that influence is more significant.

In data processing algorithms single and double differences of observations (1) are used (Fig. 2):

[illegible] **(single differences) (5)**

or [illegible] **(6)**

[illegible] **(double differences). (7)**

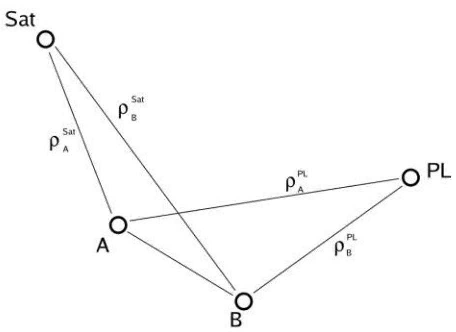

Fig. 2. Double differences with satellite and pseudolite

In the case showed in fig 2 the distance between unknown point and pseudolite is usually short. Thus the influence of the second term of Taylor series is significant. In the equation (6) the term: $\rho_B^{PL} - \rho_A^{PL}$ denotes nonlinear function of unknown point

coordinates. On the basis of (4) and (6) the common influence of the second terms of Taylor series for single differenced carrier phases amounts to

$$\frac{1}{\rho_A^{PL}} - \frac{1}{\rho_B^{PL}}. \tag{8}$$

The values of Ψ vs. ρ_A^{PL} and ρ_B^{PL} are showed in Fig. 3. The axes x and y correspond to distances ρ_A^{PL} and ρ_B^{PL} and axis z corresponds to value of Ψ.

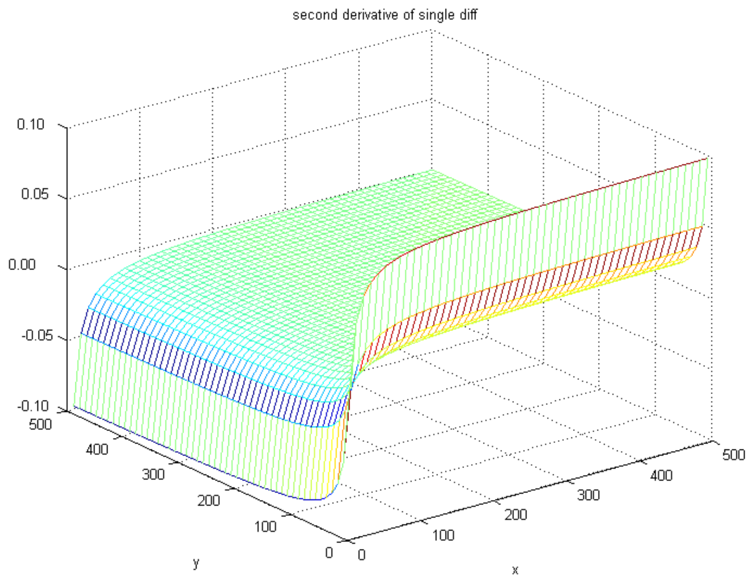

Fig 3. The second derivatives of single differences.

The value of Ψ is highest if the distances ρ_A^{PL} and ρ_B^{PL} differ significantly. In the case of equal values of ρ_A^{PL} and ρ_B^{PL} the quantity Ψ is very small.

Fig. 4. shows some cross sections of graph from Fig. 3. The values of the second derivatives differ most for distances in the range 0 – 50 m.

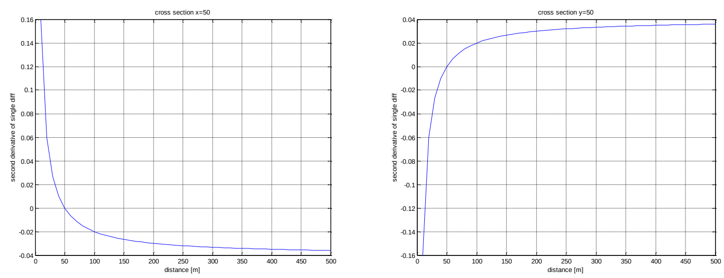

Fig. 4. The cross sections of the second derivatives of the single differences graph.

3. IMPACT OF PSEUDOLITE ANTENNA LOCATION ERROR ON PSEUDORANGE AND CARRIER PHASE OBSERVATION EQUATION

Between many error sources in GPS surveys orbital error is one of minor importance since it is reduced during single differencing of observations. Because of the geometry of GPS surveys (the distance between satellite and receiver is many times larger then distance between receivers) the impact of satellite orbit bias on both points of vector is nearly parallell. This allows to reduce the orbit bias in single differencing of observation data.

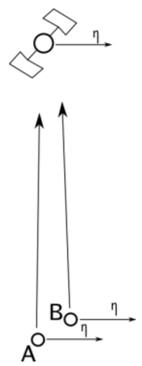

Fig. 5. Orbital bias for satellite surveys.

While dealing with orbital bias in GPS surveys is simple, pseudolite observations require more attention. Since distance between measurement points and pseudolite location is similar (from several to several hundret meters) the angle between directions from pseudolite to both points can vary from 0 to 180 degrees.

Assuming that pseudolite antenna and antennas of receivers are in one plane O, vector of orbital bias can be written as the sum of vector η - in plane O and ε - perpendicular to plane O.

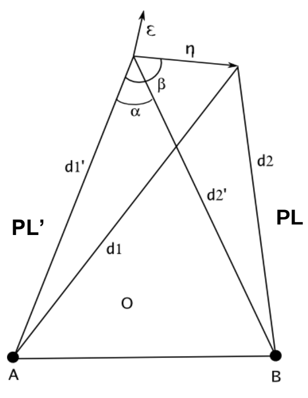

Fig. 6. Pseudolite antenna displacement and single differencing.

Naming measured distances as d_1 and d_2 and distances calculated from coordinates as d_1' and d_2' and taking into consideration small angles (PL' A PL) and (PL' B PL):

$$\Delta d_1 = d_1' - d_1 = \eta \cos\beta \qquad (9)$$
$$\Delta d_2 = d_2' - d_2 = \eta \cos(\beta - \alpha)$$

'Theoretical' and 'practical' single differences can be presented as follows:

$$sd_t = d_2' - d_1' \qquad (10)$$
$$sd_p = d_2 - d_1$$

The influence of orbital bias on single differences can be presented as difference of 'theoretical' and 'practical' single differences:

$$\Delta\delta_{orb} = sd_t - sd_p \qquad (11)$$

or on the basis of (9), (10) and (11):

$$\Delta\delta_{orb} = d_1' - d_1 - d_2' - d_2 = \eta\cos(\beta - \alpha) - \eta\cos(\beta) = 2\eta\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{2\beta - \alpha}{2}\right) \qquad (12)$$

where:

ε - bias perpendicular to plane O

η - bias in plane O

Element ε has no influence on further data processing since it gets reduced in single differencing process.

The worst case of pseudolite to point direction and bias direction angle is when:

$$\sin\left(\frac{2\beta - \alpha}{2}\right) = 1 \qquad (13)$$

So:

$$\Delta\delta_{orb} \le 2\eta\sin\left(\frac{\alpha}{2}\right) \qquad (14)$$

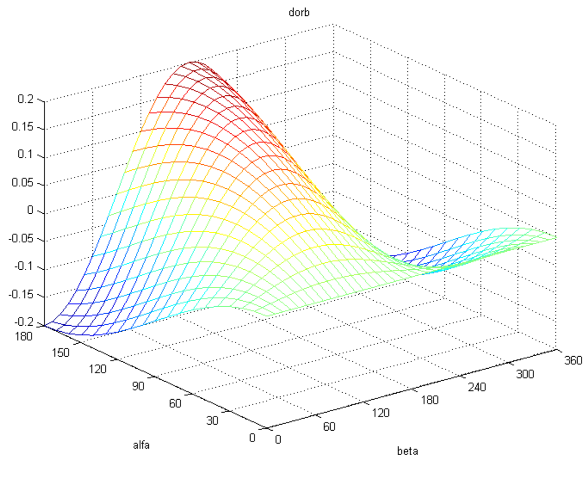

Fig. 7. Influence of orbital bias vs angle between PL and receivers (in the worst case).

As we can see in case of GPS satellites: if $\alpha \rightarrow 0$ than $\Delta\delta_{orb} \rightarrow 0$. In the case of pseudolite the influence of orbital bias can vary from 0 if $\alpha=0$ to 2η if $\alpha=180$ degrees.

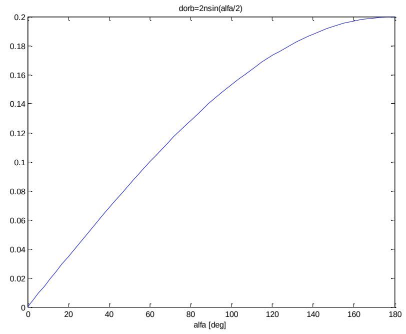

Fig. 8. Influence of orbital bias vs angle between PL and receivers (in the worst case).

4. CONCLUSIONS

1. **The impact of orbital error and linearization error for GPS satellite observation and pseudolite observation varies and needs to be considered separately.**
2. **The impact of linearization in pseudolite observations is inversely proportional to the distance between transmitter and receiver antennas.**
3. **The impact of linearization in single differenced pseudolite observations grows with difference between distances from pseudolite antenna to receivers.**
4. **Pseudolite location error is not negligible during single differencing of observations.**
5. **Pseudolite location error impacts single differenced observation proportionally to the angle between pseudolite and the receivers.**

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