THE GEOMETRICAL FACTORS OF A NAVIGATIONAL SYSTEMS

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ABSTRACT

Vessels engaged in coastal navigation must have their position determined with high accuracy. This can be ensured only by satellite systems such as GPS and GLONASS. The paper presents generalized concepts of geometrical factors of a navigational system. Such factors are used in the analysis of the accuracy of various radionavigational systems aimed at selecting the best system for a given area.
The modern process of navigation is described in a four-dimensional space – three geometric dimensions and time. For this reason both the description and analysis of navigational systems should be performed in the same space. The traditional geometric factor of the land-based radionavigational system was generalised to include the factors GDOP, PDOP, HDOP, VDOP and TDOP for the needs of the accuracy analysis of a GPS system. These terms are related to the so-called geometry of navigational system – through mutually related positions of gradients of navigational functions determining position lines (hyperplanes). They are connected with non-linear regression through a probabilistic relation between the measured navigational parameters. Consequently, the concept of geometric factors in the process of navigational parameters estimation can be also extended to include a larger number of dimensions appropriate for the state vector.

INTRODUCTION

The modern process of navigation is described in four-dimensional space in terms of the physical space – three geometrical dimensions plus time. Consequently, a description and analysis of navigational positionig system accuracy should be performed in the space of the same dimensions. It is particularly important when navigational pseudorange satellite systems are used. These systems, from the point of view of measured navigational parameters analysis, have spatial-time structure. The classical concepts of navigational geometry in two dimensions need to be extended to at least four dimensions.
Both in theoretical considerations as well as in practice it is necessary to compare two navigational systems. From navigational perspective, it is important which system makes it possible in a given area to determine a position with greater accuracy. For comparison, we make use of accuracy zones of these systems. Within these zones, lines of equal accuracy of position are drawn. These lines correspond to constant values of

distance root mean square errors. The lines of equal accuracies are sets of points on the surface of the earth's ellipsoid (or reference plane), satisfying this condition:

$$drms = \text{const}, \qquad (1)$$

where *drms* - distance root mean square.

In general, we refer to hyperplanes of equal accuracies as to sets of points in the navigational space V_N satisfying the above condition. In the case of equally accurate measurements of navigational parameters, we can determine the distance root mean square error from this relationship:

$$drms = \sigma \text{DOP}, \qquad (2)$$

where:
σ - mean measuring error,
DOP - Dilution of Position.

The geometric factor of the navigational system is a single-parameter (scalar) estimation of the system accuracy in the case when measurements are equally accurate. The factor accounts for the location of aids to navigation (radio-navigational system station, navigational satellites, celestial bodies) relative to the observer, i.e. the so called system geometry. The navigational system geometry is defined by the angles at which position lines (hyperplanes) intersect and the observer's distance to individual aids to navigation.
The above classical geometric factor of the navigational system, DOP, features the accuracy of positioning in the horizontal plane. However, such a concept of the system geometry is now insufficient. The development of satellite technology and practical use of navigational satellite systems called for a more generalised definition of the navigational system geometric factor to include four dimensions.

GEOMETRY OF THE NAVIGATIONAL POSITIONING SYSTEM

Many factors affect the accuracy of position coordinates determination. The most important of these factors are as follows:
- **accuracy of the mathematical model chosen for the calculations of position coordinates in a given navigational positioning system (calculations on the ellipsoid surface, on the sphere, on a reference plane, analytical method, numerical method etc.),**
- **accuracy of navigational measurements, expressed by means of measuring errors covariance matrix R,**
- **number of measurements (position lines, areas or hyperplanes), expressed as the dimension of measurement space – *n*,**
- **geometry of navigational positioning system (geometry of position lines, areas or hyperplanes).**

Let us consider the latter factor, namely the geometry of navigational positioning system. It should be mentioned that it also comprises information on the number of measurements. Besides, the geometry is related with the accepted model of coordinates calculations. In a general case, we can accept the following model for the calculation of

coordinates [2]:

$$\mathbf{z} = \mathbf{G}\mathbf{x}, \qquad (3)$$

where:

z – n-dimensional vector of measurements,

x – m-dimensional state vector (of position coordinates),

G – *(n* x *m*)-dimensional matrix functionally combining the position coordinates with navigational measurements.

The matrix G is usually the Jacobian matrix of navigational vector function f (non-linear). In the geometrical interpretation, the matrix rows are the gradients of navigational lines (areas, hyperplanes). Its general form is as follows:

$$\mathbf{G} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}. \qquad (4)$$

In the deterministic case, i.e. when *n = m*, we can obtain the solution of equation (3) by the Newton's method of solving non-linear equations [2], [8]. When we have an excessive number of measurements, the solution will be obtained by the least squares method. In both cases, we evaluate the accuracy by position covariance matrix [2], [3], [4], [10]:

$$\mathbf{P} = \sigma^2 \left(\mathbf{G}^T\mathbf{G}\right)^{-1} \qquad (5)$$

in the case of homogeneous and equally accurate navigational measurements,

$$\mathbf{P} = \left(\mathbf{G}^T\mathbf{R}^{-1}\mathbf{G}\right)^{-1} \qquad (6)$$

in any case (also in the case of correlated measurements).

As it can be seen in the equations above, the matrix G plays a very important part in the accuracy assessment of position co-ordinates. The geometry of the navigational positioning system can be expressed in quantitative terms by defining the system geometry matrix Γ in the form of the following equation [2]:

$$\Gamma = \mathbf{G}^T\mathbf{G}. \qquad (7)$$

This matrix, being Gram matrix of estimation space base (columns of the matrix G), is connected with a natural generalisation of the classical geometric factor of the navigational positioning system. The matrix is also related with Fisher's information matrix [2]. In addition, as we can see from the relationship (5) it is connected with the covariance matrix of position coordinates since

$$\mathbf{P} = \sigma^2\,\Gamma^{-1}. \qquad (5a)$$

In the general case the matrix Γ has this form:

$$\Gamma = \begin{bmatrix} \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_1}\right)^2 & \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_2} & \cdots & \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_m} \\ \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_2} & \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_2}\right)^2 & \cdots & \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_2}\frac{\partial f_i}{\partial x_m} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_m} & \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_1}\frac{\partial f_i}{\partial x_3} & \cdots & \sum_{i=1}^{n}\left(\frac{\partial f_i}{\partial x_m}\right)^2 \end{bmatrix}. \qquad \textbf{(8)}$$

In special cases, e.g. for navigational satellite systems GPS and GLONASS we obtain

$$\Gamma_{\textbf{GPS (GLONASS)}} = \begin{bmatrix} \sum_{i=1}^{n}\left(\frac{\partial d_i}{\partial \varphi}\right)^2 & \sum_{i=1}^{n}\frac{\partial d_i}{\partial \varphi}\frac{\partial d_i}{\partial \lambda} & \sum_{i=1}^{n}\frac{\partial d_i}{\partial \varphi}\frac{\partial d_i}{\partial h} & c\sum_{i=1}^{n}\frac{\partial d_i}{\partial \varphi} \\ \sum_{i=1}^{n}\frac{\partial d_i}{\partial \varphi}\frac{\partial d_i}{\partial \lambda} & \sum_{i=1}^{n}\left(\frac{\partial d_i}{\partial \lambda}\right)^2 & \sum_{i=1}^{n}\frac{\partial d_i}{\partial \lambda}\frac{\partial d_i}{\partial h} & c\sum_{i=1}^{n}\frac{\partial d_i}{\partial \lambda} \\ \sum_{i=1}^{n}\frac{\partial d_i}{\partial \varphi}\frac{\partial d_i}{\partial h} & \sum_{i=1}^{n}\frac{\partial d_i}{\partial \lambda}\frac{\partial d_i}{\partial h} & \sum_{i=1}^{n}\left(\frac{\partial d_i}{\partial h}\right)^2 & c\sum_{i=1}^{n}\frac{\partial d_i}{\partial h} \\ c\sum_{i=1}^{n}\frac{\partial d_i}{\partial \varphi} & \sum_{i=1}^{n}\frac{\partial d_i}{\partial x_1}\frac{\partial d_i}{\partial x_3} & c\sum_{i=1}^{n}\frac{\partial d_i}{\partial h} & nc^2 \end{bmatrix}, \qquad \textbf{(9)}$$

where:

c – **velocity of light,**
d – **measured pseudorange,**
h – **geodetic height (called by some authors by ellipsoidal height),**
φ – **latitude,**
λ – **longitude.**

For the land – based hyperbolic radionavigational system the geometry matrix will have this form:

$$\Gamma = 4\cdot\begin{bmatrix} \sin^2 A_{12}\sin^2\frac{\omega_{12}}{2} + \sin^2 A_{12}\sin^2\frac{\omega_{23}}{2} & -\sin A_{12}\cos A_{12}\sin^2\frac{\omega_{12}}{2} - \sin A_{12}\cos A_{23}\sin^2\frac{\omega_{23}}{2} \\ -\sin A_{12}\cos A_{12}\sin^2\frac{\omega_{12}}{2} - \sin A_{12}\cos A_{23}\sin^2\frac{\omega_{23}}{2} & \cos^2 A_{12}\sin^2\frac{\omega_{12}}{2} + \cos^2 A_{12}\sin^2\frac{\omega_{23}}{2} \end{bmatrix}, \qquad \textbf{(10)}$$

where:

A_{ij} – **average azimuth between the *i*-th and *j*-th station,**
ω_{ij} – **base angle between the *i*-th and *j*-th station.**

GEOMETRIC FACTORS OF THE NAVIGATIONAL POSITIONING SYSTEM

The generalisation of the geometric factors of navigational positioning system appeared along with the needs of spatial accuracy interpretation of position coordinates. At first, it referred to position determination by LORAN-C in aviation, then the concept was extended to cover GPS and other satellite systems. Nowadays geometric factors are defined for various coordinate systems and for various systems of navigation.
The most common geometric factors of the navigational position system are as follows:

- **GDOP – Geometric Dilution of Precision; it refers to the accuracy in a four-dimensional space (*φ, λ, H, Δt*);**
- **PDOP – Position Dilution of Precision; it refers to the accuracy in a three-dimensional space (*φ, λ, H*);**
- **HDOP – Horizontal Dilution of Precision; it refres to the accuracy in a two-dimensional space (*φ, λ*); this factor corresponds to the classical factor of the navigational system geometry;**
- **VDOP – Vertical Dilution of Precision; it refers to the accuracy in a one-dimensional space (*H*);**
- **TDOP – Time Dilution of Precision; it refers to the accuracy in a one-dimensional space (*Δt*).**

Other factors were also introduced. These characterise the accuracy along a meridian and a parallel:

- **NDOP – North Dilution of Precision;**
- **EDOP – East Dilution of Precision.**

Both factors started to be used after the DGPS system was introduced. They are equivalent to VDOP and TDOP for the geographical coordinates. The above factors are computed using the system geometry matrix. Let us denote its elements with γ_{ij}, and the elements of the inverse matrix with γ'_{ij}. With this notation, we can compute particular values of DOP from these formulas:

$$\mathrm{GDOP} = \sqrt{\mathrm{tr}\,\Gamma^{-1}} = \sqrt{\gamma'_{11}+\gamma'_{22}+\gamma'_{33}+\gamma'_{44}}\ , \quad (11)$$

$$\mathrm{PDOP} = \sqrt{\gamma'_{11}+\gamma'_{22}+\gamma'_{33}}\ , \quad (12)$$

$$\mathrm{HDOP} = \sqrt{\gamma'_{11}+\gamma'_{22}}\ , \quad (13)$$

$$\mathrm{VDOP} = \sqrt{\gamma'_{33}}\ , \quad (14)$$

$$\mathrm{TDOP} = \sqrt{\gamma'_{44}}\ , \quad (15)$$

$$\mathrm{NDOP} = \sqrt{\gamma'_{11}}\ , \quad (16)$$

$$\mathrm{EDOP} = \sqrt{\gamma'_{22}}\ . \quad (17)$$

Due to a complex form of the matrix determinant (9), for pseudorange navigational satellite systems the factors (11) – (17) do not have a simple geometric interpretation, whereas in the case of the hyperbolic radionavigational system we obtain

$$\mathrm{HDOP}=0{,}5\,\mathrm{cosec}\,\theta\,[\mathrm{cosec}^2(0{,}5\omega_{12}) + \mathrm{cosec}^2(0{,}5\omega_{23})]^{1/2}, \quad (18)$$

where θ - the angle of intersection of the position lines.

There are extensions of these factors covering the cartesian coordinate system (*X*, *Y*), which convert the respective geographical (ellipsoid) coordinates. This is used in hydrographic survey with the use of plotting boards (in UTM mapping). These factors, however, XDOP and YDOP, do not change the essence of the matter; all that is needed is the conversion of the coordinates.
Another generalisation consists in the computation of geometric factors along the track and across the track. These factors are denoted, respectively, ADOP *(along-track)* and XDOP *(cross-track)*. The latter, however, has a notation that may be confused with a factor used in the UTM mapping.
The notion of geometric factor can also be applied to dead reckoning navigation. In this case the following relationship will be equivalent to the equation (3)

$$\mathbf{v} = \mathbf{Ax}, \tag{19}$$

where:
v – velocity vector,
A – transition matrix.
With this notation, the geometric factor of dead reckoning navigation is written as:

$$\mathrm{DOP_{DR}} = \sqrt{\mathrm{tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}} . \tag{20}$$

For frequent cases when

$$\mathbf{A} = \frac{1}{\Delta t}\mathbf{I}, \qquad \mathbf{I} \text{ – unit matrix,} \tag{21}$$

formula (20) will have this form

$$\mathrm{DOP_{DR}} = \Delta t\sqrt{2} , \tag{20a}$$

which obviously means that the accuracy of reckoned position decreases in proportion to time.

P.J.G. Teunissen [11], [12], in turn, has proposed a differently defined factor Ambiguity Dilution of Precision (ADOP):

$$\mathrm{ADOP} = \sqrt{\det \mathbf{Q}_{\mathrm{a}}} , \tag{22}$$

where $\mathbf{Q_a}$ – ambiguity covariance matrix. Although different, this definition of a system geometry factor is equivalent in view of optimization. As it was shown in [2], the optimization, i.e. the minimization of the geometric factor can be changed into the minimization of the confidence area (area of surface or volume), which, in turn, corresponds to the minimization of covariance matrix P determinant or, respectively, maximization of the system geometry matrix Γ determinant.

CONCLUSION

The concept of the geometry factor of the navigational positioning system is very useful in the accuracy analysis of navigational systems [2], [4] – particularly in the designing of aids to navigation, assessment of their integrity [9] and the ambiguity of phase measurements [11], [12]. More applications can be pointed out, such as the comparison of accuracy of various navigational systems, optimization of the choice of land-based

radio-navigational system chain, or the already implemented optimization of satellite configuration etc. One should bear in mind that this factor contains "pure" geometry, so it does not take into account the accuracy of measurement in the case when measurements are not equally accurate and/or are correlated. In these cases a full analysis should be performed with the use of the covariance matrix of position coordinates described by the formula (6) and appropriate areas of confidence – ellipses or ellipsoids of errors [2].

In a general case a similar analysis can be applied to any problem described by a formula analogous to the relation (3), including the assessment of estimators of generalised state vector in navigation, when the matrix G is obtained from regression etc. However, in these cases the geometric interpretation will not be connected with geometry in terms of geographic coordinates.

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