compression, Hurwitz-Radon matrix, contour, curve reconstruction, tomography

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HURWITZ-RADON OPERATOR IN MONOCHROMATIC MEDICAL IMAGE RECONSTRUCTION

In this paper the method, dedicated for medical images reconstruction, will be presented. One of them called the method of the Hurwitz-monochromatic (e.g. black and white) images. The method is based on a family of Hurwitz-Radon matrices. The matrices possess columns composed of orthonormal vectors. The operator of Hurwitz-Radon (OHR), built from that matrices, is described. It is shown how to create the orthogonal and discrete OHR and how to use it in a process of curve interpolation. The method needs suitable choice of nodes, i.e. points of the curve to be compressed: they should be equidistance in one of coordinates. Application of MHR gives a high level of compression (up to 99 %) and a very good interpolation accuracy in the process of reconstruction of contours. Its use in the computer tomography is also effective. Orthogonal OHR can be regarded as a linear and discrete model in the supervised (machine) learning [5]. It is shown how to use it in approximation of data. Created from the family of N-1 HR matrices and completed with the identical matrix, system of matrices is orthogonal only for vector spaces of dimensions N=2,4,8. Orthogonality of columns and rows is very important and significant for stability and high precision of calculations.

1. INTRODUCTION

One can form the following question: is it possible to find a method of compression and decompression of monochromatic images' contours with better accuracy, non-worse complexity of calculation and similar or better level of compression comparing with nowadays existing methods? Our paper aims at giving the positive answer to this question. Current methods of contour reconstruction are: polynomial interpolation or approximation, Bézier curves, Hermite curves, B-splines, other methods: using graph theory or Discrete Cosine Transformation, building skeleton of the object.

Let us assume there is given a finite set of points, called further nodes $(x_i,y_i)\in\mathbf{R}^2$ such as:

1. each node (x_i,y_i) has a fixed step of coordinates x_i or y_i ;
2. nodes (characteristic points) are settled at local extrema (maximum or minimum) of one of coordinates and at least one point between two local extrema.

Assume that the nodes belong to a curve in the plane.

How the whole contour could be reconstructed using this discrete set of nodes?

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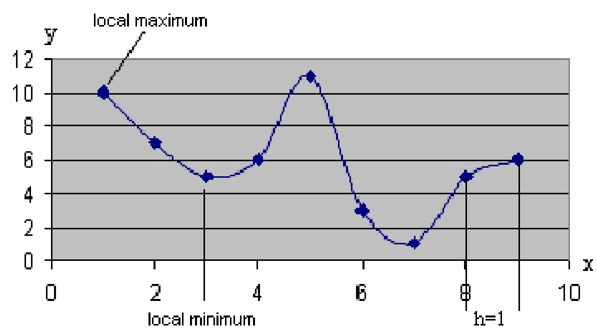

Fig.1. Part of contour equidistance in x_i

$$(h = const. = x_{i+1} - x_i)$$

Proposed method is based on local, orthogonal matrix operators. Values of nodes' coordinates (x_i,y_i) are connected with Hurwitz-Radon (HR) matrices [2] build on N dimensional vector space. It is important that HR-matrices are skew-symmetric and for dimension N=2,4,8 columns and rows of HR-matrices are orthogonal [7] only. Let us list their forms for that dimensions, where entries are real numbers:

$$W=\begin{bmatrix} 0 & w_1 \\ -w_1 & 0 \end{bmatrix}, \qquad W=\begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ -w_1 & 0 & -w_3 & w_2 \\ -w_2 & w_3 & 0 & -w_1 \\ -w_3 & -w_2 & w_1 & 0 \end{bmatrix}, \qquad W=\begin{bmatrix} 0 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 \\ -w_1 & 0 & w_3 & -w_2 & w_5 & -w_4 & -w_7 & w_6 \\ -w_2 & -w_3 & 0 & w_1 & w_6 & w_7 & -w_4 & -w_5 \\ -w_3 & w_2 & -w_1 & 0 & w_7 & -w_6 & w_5 & -w_4 \\ -w_4 & -w_5 & -w_6 & -w_7 & 0 & w_1 & w_2 & w_3 \\ -w_5 & w_4 & -w_7 & w_6 & -w_1 & 0 & -w_3 & w_2 \\ -w_6 & w_7 & w_4 & -w_5 & -w_2 & w_3 & 0 & -w_1 \\ -w_7 & -w_6 & w_5 & w_4 & -w_3 & -w_2 & w_1 & 0 \end{bmatrix}.$$

If one part of monochromatic image's contour is described by a set of nodes $\{(x_i,y_i), i=1,2,\ldots,n\}$ equidistance in coordinates x_i, then HR-matrices combined with identity matrix are used to build an orthogonal and discrete Hurwitz-Radon Operator (OHR). For nodes (x_1,y_1), (x_2,y_2) the OHR of dimension N=2 is constructed:

$$M = \frac{1}{x_1^2 + x_2^2}\begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}\begin{bmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{bmatrix}$$

For nodes (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) equidistance in x_i the OHR of dimension N=4 is constructed:

$$M = \frac{1}{x_1^2 + x_2^2 + x_3^2 + x_4^2}\begin{bmatrix} u_0 & u_1 & u_2 & u_3 \\ -u_1 & u_0 & -u_3 & u_2 \\ -u_2 & u_3 & u_0 & -u_1 \\ -u_3 & -u_2 & u_1 & u_0 \end{bmatrix} \quad \text{For} \quad \begin{aligned} u_0 &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4, \\ u_1 &= -x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3, \\ u_2 &= -x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2, \\ u_3 &= -x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1. \end{aligned} \qquad (1)$$

Kjkj For nodes (x_1,y_1), (x_2,y_2), …, (x_8,y_8) equidistance in x_i the OHR of dimension N=8 is equal to:

$$M=\frac{1}{\sum_{i=1}^{8}x_i^2}\begin{bmatrix} u_0 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ -u_1 & u_0 & u_3 & -u_2 & u_5 & -u_4 & -u_7 & u_6 \\ -u_2 & -u_3 & u_0 & u_1 & u_6 & u_7 & -u_4 & -u_5 \\ -u_3 & u_2 & -u_1 & u_0 & u_7 & -u_6 & u_5 & -u_4 \\ -u_4 & -u_5 & -u_6 & -u_7 & u_0 & u_1 & u_2 & u_3 \\ -u_5 & u_4 & -u_7 & u_6 & -u_1 & u_0 & -u_3 & u_2 \\ -u_6 & u_7 & u_4 & -u_5 & -u_2 & u_3 & u_0 & -u_1 \\ -u_7 & -u_6 & u_5 & u_4 & -u_3 & -u_2 & u_1 & u_0 \end{bmatrix} \qquad u=\begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\ -y_2 & y_1 & -y_4 & y_3 & -y_6 & y_5 & y_8 & -y_7 \\ -y_3 & y_4 & y_1 & -y_2 & -y_7 & -y_8 & y_5 & y_6 \\ -y_4 & -y_3 & y_2 & y_1 & -y_8 & y_7 & -y_6 & y_5 \\ -y_5 & y_6 & y_7 & y_8 & y_1 & -y_2 & -y_3 & -y_4 \\ -y_6 & -y_5 & y_8 & -y_7 & y_2 & y_1 & y_4 & -y_3 \\ -y_7 & -y_8 & -y_5 & y_6 & y_3 & -y_4 & y_1 & y_2 \\ -y_8 & y_7 & -y_6 & -y_5 & y_4 & y_3 & -y_2 & y_1 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \qquad (2)$$

We can see here that the components of the vector $u=(u_0, u_1,…, u_7)^T$, appearing in the matrix M are defined by (2) in the similar way to (1) but in terms of the coordinates of the above 8 nodes. If one part of monochromatic image's contour is described by a set of nodes $\{(x_i,y_i), i=1,2,…,n\}$ equidistance in coordinates y_i, then the HR-matrices combined with identity matrix are used to build an orthogonal and discrete reverse Hurwitz-Radon Operator (reverse OHR). For nodes (x_1,y_1), (x_2,y_2) the reverse OHR of dimension N=2 is constructed:

$$M^{-1}=\frac{1}{y_1^2+y_2^2}\begin{bmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{bmatrix}\begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix}$$

For nodes (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4) equidistance in y_i the reverse OHR of dimension N=4 is constructed for (1):

$$M^{-1}=\frac{1}{y_1^2+y_2^2+y_3^2+y_4^2}\begin{bmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{bmatrix}$$

For nodes (x_1,y_1), (x_2,y_2), …, (x_8,y_8) equidistance in y_i the reverse OHR of dimension N=8 is equal to:

$$M^{-1}=\frac{1}{\sum_{i=1}^{8}y_i^2}\begin{bmatrix} u_0 & -u_1 & -u_2 & -u_3 & -u_4 & -u_5 & -u_6 & -u_7 \\ u_1 & u_0 & -u_3 & u_2 & -u_5 & u_4 & u_7 & -u_6 \\ u_2 & u_3 & u_0 & -u_1 & -u_6 & -u_7 & u_4 & u_5 \\ u_3 & -u_2 & u_1 & u_0 & -u_7 & u_6 & -u_5 & u_4 \\ u_4 & u_5 & u_6 & u_7 & u_0 & -u_1 & -u_2 & -u_3 \\ u_5 & -u_4 & u_7 & -u_6 & u_1 & u_0 & u_3 & -u_2 \\ u_6 & -u_7 & -u_4 & u_5 & u_2 & -u_3 & u_0 & u_1 \\ u_7 & u_6 & -u_5 & -u_4 & u_3 & u_2 & -u_1 & u_0 \end{bmatrix}$$

where the components of the vector $u=(u_0, u_1,…, u_7)^T$ are defined in terms of (2).

2. MHR

How can we compute coordinates of points settled between nodes? On a segment of a line every number "*c*" situated between "*a*" and "*b*" is described by a linear (convex) combination $c = \alpha \cdot a + (1-\alpha) \cdot b$ for $\alpha \in [0;1]$ (Fig.2).

Fig.2. Point "*c*" between "*a*" and "*b*"

When nodes are equidistance in coordinates x_i ($h=const.= x_{i+1} - x_i$), the average HR operator M_2 of dimension N=2,4,8 is constructed as follows:

$$M_2 = \alpha \cdot M_0 + (1-\alpha) \cdot M_1$$

with the operator M_0 built by nodes ($x_1=a, y_1$), (x_3, y_3), …, (x_{2N-1}, y_{2N-1}) and M_1 built by nodes ($x_2=b, y_2$), (x_4, y_4), …, (x_{2N}, y_{2N}).

When the nodes are equidistance in coordinates y_i ($h=const.= y_{i+1} - y_i$), the average reverse HR operator M_2^{-1} of dimension N=2,4,8 is constructed as follows:

$$M_2^{-1} = \alpha \cdot M_0^{-1} + (1-\alpha) \cdot M_1^{-1}$$

with the reverse operator M_0^{-1} built by nodes ($x_1, y_1=a$), (x_3, y_3), …, (x_{2N-1}, y_{2N-1}) and M_1^{-1} built by nodes ($x_2, y_2=b$), (x_4, y_4), …, (x_{2N}, y_{2N}).

Notice that having the operator M_2 it is possible to reconstruct the second coordinates of points (x,y) in terms of the vector C defined with the double step $2h$ as $C=[c, c+2h, \ldots, c+2(N-1)h]^T$. The required formula is $Y(C) = M_2 \cdot C$ in which components of the vector $Y(C)$ give the second coordinate of the points (x,y) corresponding to the first coordinate, given in terms of components of the vector C.

On the other hand, having the operator M_2^{-1} it is possible to reconstruct the first coordinates of points (x,y) in terms of the corresponding second coordinates given by components of the new vector C defined, as previously, with double step $2h$: $C=[c, c+2h, \ldots, c+2(N-1)h]^T$. The final formula is $X(C) = M_2^{-1} \cdot C$ in which components of the vector $X(C)$ give the first coordinate of the points (x,y) corresponding to the second coordinate, given in terms of components of the vector C.

In general case if a monochromatic image (Fig.3) is described by some contours, then each contour must be divided into parts, where to each part corresponds a set of nodes equidistance in one of two coordinates.

Fig.3. Phantom from Computed Tomography [1] with nodes

For example on Fig.4 there is one contour with three parts (three sets of nodes):

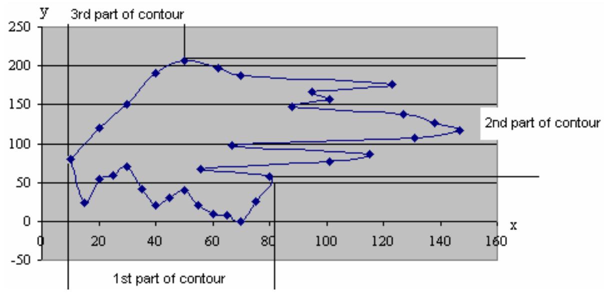

Fig.4. Example of the contour

First set of nodes equidistance in *x* is

x=	y=
10	80
15	23
20	55
25	59
30	70
35	42
40	20
45	31
50	40
55	21
60	10
65	8
70	0
75	26
80	57

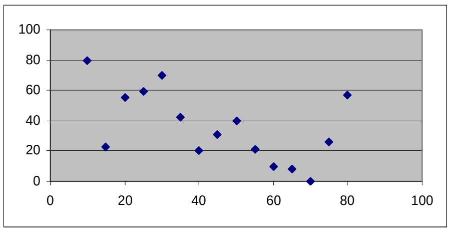

Fig.5. First set of nodes

For nodes with coordinates: x = [10 20 30 40]T and y = [80 55 70 20]T the OHR is constructed as follows:

$$M_0 = \begin{bmatrix} 1.598 & -0.382 & 1.167 & 0.917 \\ 0.382 & 1.598 & -0.917 & 1.167 \\ -1.167 & 0.917 & 1.598 & 0.382 \\ -0.917 & -1.167 & -0.382 & 1.598 \end{bmatrix}$$

For nodes with coordinates: $\mathbf{x} = [15\ 25\ 35\ 45]^T$ and $\mathbf{y} = [23\ 59\ 42\ 31]^T$ the second OHR is constructed as follows:

$$M_1 = \begin{bmatrix} 1.142 & -0.272 & 0.501 & -0.108 \\ 0.272 & 1.142 & 0.108 & 0.501 \\ -0.501 & -0.108 & 1.142 & 0.272 \\ 0.108 & -0.501 & -0.272 & 1.142 \end{bmatrix}$$

The problem is to compute second coordinates of points with their their first coordinates given in terms of the vector **p** = [12 22 32 42]^T. Corresponding average OHR is calculated in accordance with: $M_2 = 0.6 \cdot M_0 + 0.4 \cdot M_1$.

$$M_2 = \begin{bmatrix} 1.416 & -0.338 & 0.9 & 0.507 \\ 0.338 & 1.416 & -0.507 & 0.9 \\ -0.9 & 0.507 & 1.416 & 0.338 \\ -0.507 & -0.9 & -0.338 & 1.416 \end{bmatrix}, \quad M_2 \cdot p = \begin{bmatrix} 59.638 \\ 56.806 \\ 59.865 \\ 22.748 \end{bmatrix}$$

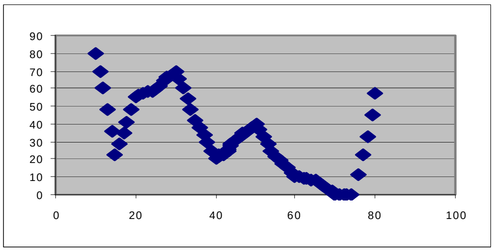

Fig.6. Several reconstructed points for nodes from Fig.5

Second part of the contour (Fig.4) with nodes equidistance in *y* and reconstructed points are:

x=	y=
80	57
56	67
101	77
115	87
67	97
131	107
147	117
138	127
127	137
88	147
101	157
95	167
123	177
70	187
62	197
50	207

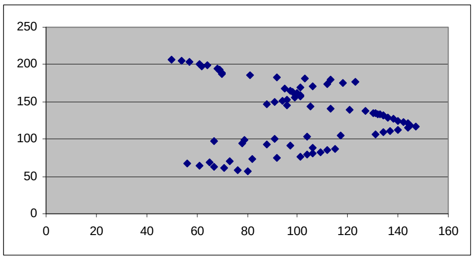

Fig.7. Several reconstructed points for second part of the contour

On Fig.8 we draw some points of the whole reconstructed contour from Fig.4.

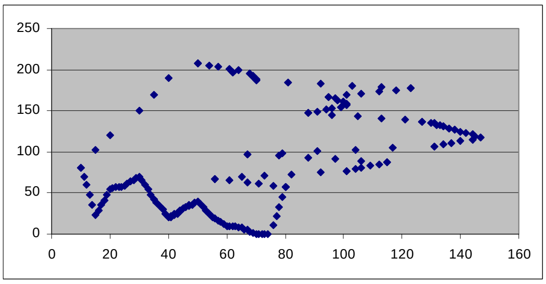

Fig.8. Several points of the contour

3. BÉZIER CURVES AND MHR METHOD

Bézier curves are applied for the problem of contour description by the use of control points [3]. It is important to compare the Bézier method and the present MHR. For example let us consider one curve with seven characteristic points:

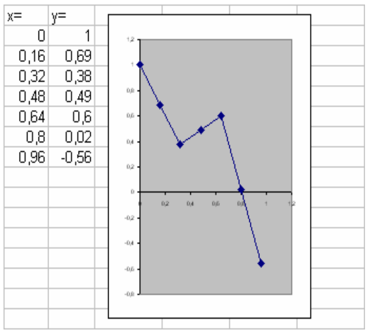

Fig.9. One curve and its nodes

Bézier polynomial for the curve from Fig.9 and the control points (0;1), (0.25;0), (0.5;0), (0.75;2), (1;-1) is (Fig.10) $y = -8x^4 + 4x^3 + 6x^2 - 4x + 1$.

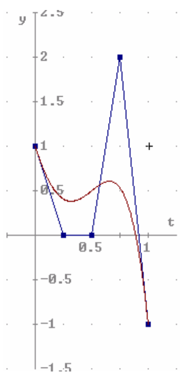

Fig.10. Bézier polynomial and five control points connected with lines

Now it is possible to compare the accuracy of determination of points (x,y) by the use of Bézier method and MHR. In Table 1 below there are given some values of the first coordinate x, then computed values of second coordinate y: original values y for the curve from Fig.9 and finally approximated values y computed by the Bézier method and MHR.

Table 1. Approximation errors for Bézier method and MHR

x=	original	Bezier	MHR
0,24	0,535	0,410	0,585
0,56	0,545	0,560	0,524
0,28	0,458	0,390	0,490
0,60	0,573	0,590	0,560
0,30	0,420	0,380	0,440
0,62	0,590	0,600	0,580
0,20	0,613	0,460	0,650
0,52	0,518	0,520	0,510
0,18	0,650	0,490	0,670
0,50	0,504	0,500	0,500
mean squared error		0,08455	0,02546
average modul of difference		0,05940	0,02150

Definitely errors of MHR are less than those given by the Bézier method. We can also compare costs of computations of approximated values *y* expressed in terms of the number of multiplications and divisions. The results are given in Tabl.2.

Table 2. Computational costs for Bézier method and MHR method

Bezier	90
MHR	58

Complexity of calculations is better in MHR.

4. APPLICATION OF MHR IN MONOCHROMATIC IMAGE COMPRESSION

Image compression is still a very important problem. Three factors must be considered dealing with compression and decompression problems: cost of computation, accuracy of approximation and level of compression (defined as a number of nodes divided by number of points of whole contour). For example in Fig.11 there is given black and white image.

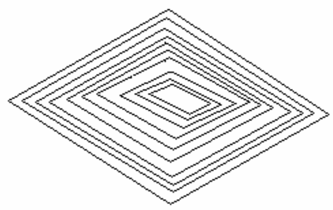

Fig.11. Binary image

Comparing of compression level for two methods: MHR and RLE (*Run Length Encoding*) method [6] we can see the compression (inverse of the classical compression rate) is definitely better in MHR (cost of computation is almost the same), cf. Tabl.3.

Table 3. Level of compression for RLE and MHR

RLE	75%
MHR	9%

Very significant part of the image compression relates to medical image compression [8]. For example image from Fig.3 (256 x 256 pixels) is compressed by 310 nodes, which means 0.5% level of compression (310/65536). MHR allows for better level of compression than some known methods as it is demonstrated in Tabl.4.

Table 4. Comparing of compression level for monochromatic medical images

	Medical image compression
Compression methods:	
RLE,JPEG,JPEG-LS,	25% - 66%
JPEG2000,PNG,CALIC	
MHR method	0.5% - 1%
(contour image)	

Three-dimensional images can also be described by several number of contours [4,9,10:

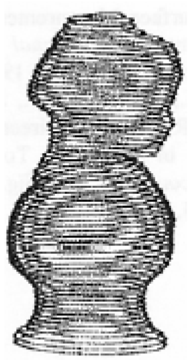

Fig.12. 3D image consists of several contours

5. CONCLUSIONS

Main features of MHR method are listed below [10]:

1) Accuracy of contour reconstruction depending on number of nodes and method of choosing nodes.
2) Compression and decompression of image *NxN* pixels is connected with the cost of computation of rank $O(N^2)$.
3) High level of compression.
4) Geometrical transformations (translations, rotations, scaling) are easy.
5) Method is based on local operators.
6) Possibility to apply the MHR to three-dimensional images.

MHR allows for compression and reconstruction of monochromatic contour images with high accuracy of points' interpolation, low cost of computation and a very good level of compression.

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