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STATIONARY HEAT CONDUCTION IN TRANSVERSALLY GRADED LAMINATES WITH UNIFORM AND NON-UNIFORM DISTRIBUTION OF LAMINAS

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Streszczenie: A problem of stationary heat conduction in laminates with transversally graded structure is analysed. These composites consist of two components, non-periodically distributed as laminas. Averaged (macroscopic) properties of these laminates are slowly and continuously varied along the axis normal to laminas. In this contribution the stationary heat conduction problem is analysed in the framework of the tolerance model, proposed by Jędrysiak and Radzikowska [2,6,3].

Słowa kluczowe: heat conduction, transversally graded laminates, tolerance averaging.

1. INTRODUCTION

Laminates under consideration are made of two components, non-periodically distributed along a direction normal to laminas. Averaged (macroscopic) properties of them are assumed to be continuously varied along this direction, cf. Fig. 1. Their microstructure can be realised in the form of uniform (λ=const), cf. Fig. 2a, or non-uniform ($\lambda=\lambda(x)$) distribution of laminas, cf. Fig. 2b. Thus, these laminates can be treated as made of *a functionally graded material* (FGM), cf. Suresh and Mortensen [7]. These composites are called *transversally graded laminates* (*TGL*), cf. Jędrysiak and Radzikowska [4].
In order to describe thermomechanical phenomena in FG-type composites methods proposed for macroscopically homogeneous materials are usually applied. Some of the fundamental methods are discussed in [7]. It has to be mentioned those approaches based on *the asymptotic homogenization*. Unfortunately, the effect of the microstructure size is usually omitted in governing equations of these models, cf. the book by Cz. Woźniak and Wierzbicki [10] and edited by Cz. Woźniak, Michalak and Jędrysiak [8].

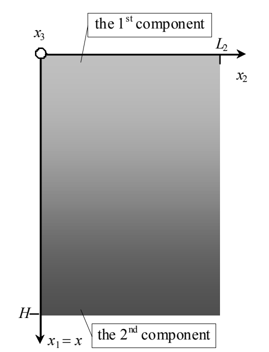

Fig. 1. A fragment of the laminate on the macro-scale

The tolerance modelling is an approach without this drawback, which was proposed for non-stationary problems of periodic composites and structures in [10] and extended on FG-type media in [8] and the book edited by Cz. Woźniak *et al*. [9]. The bibliography of applications of this approach to investigate various problems of FG-type composites can be found in these monographs.
This modelling technique is also applied in the analysis of heat conduction problems, cf. Jędrysiak and Radzikowska [2-4], Michalak, Cz. Woźniak and M. Woźniak [5], Radzikowska and Jędrysiak [6], which are described for FG-type laminates by differential equations with highly oscillating, tolerance-periodic, non-continuous, functional coefficients. Using the tolerance modelling these equations are replaced by the system of differential equations with slowly-varying coefficients. Some applications of this approach for transversally graded structures are also shown in the book by Jędrysiak [1].

The aim of this paper is to show differences between distributions of temperature in transversally graded laminates with uniform and non-uniform distribution of laminas.

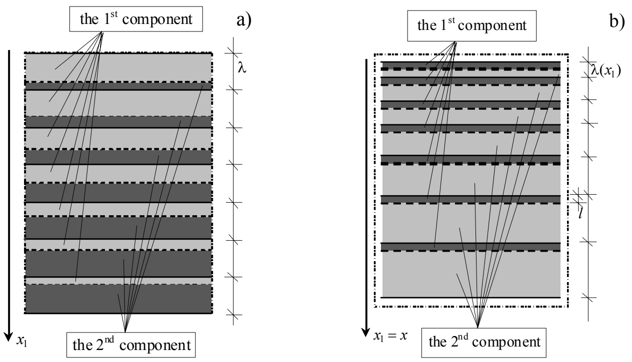

Fig. 2. A fragment of the laminate on the micro-scale:
a) with the uniform distribution of laminas,
b) with the non-uniform distribution of laminas

2. FORMULATION OF THE PROBLEM

Introduce indices related to the Carthesian coordinate system $Ox_1x_2x_3$ by i, j,…, which run on 1,2,3; and indices related to the system Ox_2x_3 by α, β,…, which run on 2,3. Denote: $\mathbf{x}\equiv(x_2,x_3)$, $x\equiv x_1$. Let derivatives of x_i, i=1,2,3, be denoted by $\partial_{i\ldots j}\equiv\partial_i\ldots\partial_j$, of x_α, α=2,3, by $\partial_{\alpha\ldots\beta}\equiv\partial_\alpha\ldots\partial_\beta$, and of x by ∂. Let us assume that the considered layer has thickness H along the x-axis and the dimensions L_α along the x_α-axes. The laminate is made of two materials, distributed in M laminas with thickness λ. We assume that $\lambda<<H$. The heat conduction tensors of these materials have components k'_{ij}, k''_{ij}, i,j=1,2,3. The m^{th} lamina (m=1,…,M) is consisted of two homogeneous sub-laminas having thicknesses λ'_m, λ''_m. For the laminate with non-uniform distribution of laminas the m^{th} lamina has thickness λ_m dependent on x (cf. Fig. 2b) and thicknesses of sub-laminas are equal: $\lambda'_m = l = \text{const}$, $\lambda''_m = \lambda_m - l$ which depends on x. However, for the laminate with uniform distribution of laminas the m^{th} lamina has thickness λ_m=const (independent of x, cf. Fig. 2a) and thicknesses of sub-laminas are equal λ'_m, λ''_m, which (both) depend on x. Let $\nu'_m\equiv\lambda'_m/\lambda_m, \nu''_m\equiv\lambda''_m/\lambda_m$ be material volume fractions in the m^{th} lamina. Since sequence $\{\nu'_m\}$, m=1,…,M, is assumed to be monotone and for every m=1,…,M–1 to satisfy condition $|\nu'_{m+1}-\nu'_m|<<1$, the layer can be treated as made of the functionally graded material. Moreover $\nu'_m+\nu''_m=1$. Hence, the above conditions are satisfied also by sequence $\{\nu''_m\}$. It follows that sequences $\{\nu'_m\}$, $\{\nu''_m\}$, m=1,…,M, can be approximated by continuous functions $\nu'(\cdot)$, $\nu''(\cdot)$. These functions determine the distribution of material properties along the x-axis and are called *the fraction ratios of materials*. Similarly, sequence $\{\lambda_m\}$ of laminas thicknesses can be approximated by function $\lambda(\cdot)$, called *the cell distribution function* (which is constant for laminates with uniform distribution of laminas, cf. [1]). Let us also introduce *the non-homogeneity ratio* ν, defined by $\nu(\cdot)\equiv[\nu'(\cdot)\nu''(\cdot)]^{1/2}$. Functions ν′, ν″, λ are assumed to be slowly-varying in x, cf. [2,3,5]. A fragment of the layer on the macrolevel is shown in Fig. 1 and on the microlevel in Fig. 2.

Let us assume small oscillations of the unknown temperature field T. Denote by p the intensity of heat sources. Hence, heat conduction in a transversally graded composite is described by the Fourier's equation in the form

$$-\partial_i(k_{ij}\partial_j T) = p\,, \tag{1}$$

with coefficients k_{ij}=$k_{ij}(x)$, which can be highly oscillating, tolerance-periodic, non-continuous functions in x. To solve heat conduction problems of this type, equation (1) will be replaced by a system of differential equations with slowly-varying coefficients, by using the tolerance averaging technique, cf. [9,2,5,1].

3. TOLERANCE MODELLING

3.1. Modelling assumptions

In the tolerance modelling there are applied introductory concepts, e.g.: the highly oscillating function, the averaging operator, the slowly-varying function, the fluctuation shape function, which are defined and explained e.g. in [8,9]. One of them is *the averaging operator* defined for an arbitrary integrable function f determined in interval [0,H] in the form

$$<f>(\bar{x}) = \lambda^{-1}\int_{\bar{x}-\lambda/2}^{\bar{x}+\lambda/2} f(x)dx, \qquad \bar{x}\in[\lambda/2, h-\lambda/2]. \tag{2}$$

The other concept is *the fluctuation shape function h*, being continuous. It can be assumed for the considered laminate in form (3):

$$h(x)=\begin{cases} -A\sqrt{3}\dfrac{\nu(\bar{x})}{\nu'(\bar{x})}[2\dfrac{x}{\lambda}+\nu''(\bar{x})] & \text{for } x\in\left(\bar{x}-\frac{1}{2}\lambda, \bar{x}-\frac{1}{2}\lambda+\lambda\nu'(\bar{x})\right) \\ A\sqrt{3}\dfrac{\nu(\bar{x})}{\nu''(\bar{x})}[2\dfrac{x}{\lambda}-\nu'(\bar{x})] & \text{for } x\in\left(\bar{x}+\frac{1}{2}\lambda-\lambda\nu''(\bar{x}), \bar{x}+\frac{1}{2}\lambda\right), \end{cases}$$

where $\bar{x}$ is a centre of interval $[-\lambda/2, \lambda/2]$; A is an amplitude, being of an order of the microstructure parameter in the problem under consideration. It will be assumed that the microstructure parameter is equal λ=const for the TGL with uniform distribution of laminas (hence A=λ), but for the TGL with non-uniform distribution – it is equal l=const (i.e. A=l).

The fundamental assumption of the tolerance modelling, cf. [8,9,3], is *the micro-macro decomposition*, in which temperature $T=T(x,\mathbf{x},t)$, $x\in[0,H]$, is decomposed as

$$T(x,\mathbf{x},t)=\theta(x,\mathbf{x},t)+h(x)\vartheta(x,\mathbf{x},t), \qquad (4)$$

with *macrotemperature* $\theta(\cdot,\mathbf{x},t)$ and *amplitude fluctuation* $\vartheta(\cdot,\mathbf{x},t)$, which are new basic unknowns, being slowly-varying functions in x.

Moreover, in the modelling procedure we use *the tolerance averaging approximation*, i.e. terms of an order $O(\delta)$ are neglected as negligibly small in the comparing to 1 (where δ is a tolerance parameter).

3.2. Tolerance model equations

The modelling procedure presented for FG-type composites in [8,9] and applied for heat conductions in laminates in [2-4,6] leads from equation (1) to the governing equations of *the tolerance model of heat conduction in transversally graded laminates*:

$$\begin{aligned}&<k_{\alpha j}>\partial_{j\alpha}\theta+\partial(<k_{1j}>\partial_j\theta)+\\&+\partial(<k_{11}\partial h>\vartheta)+<k_{\alpha 1}\partial h>\partial_\alpha\vartheta=<p>,\\&<k_{1j}\partial h>\partial_j\theta+<k_{11}(\partial h)^2>\vartheta-\underline{<k_{\alpha\beta}h^2>}\partial_{\alpha\beta}\vartheta=\underline{<ph>},\end{aligned} \qquad (5)$$

where underlined terms depend on the microstructure parameter. Coefficients of equations (5) are slowly-varying functions in x. It can be observed that the above tolerance model equations take into account the effect of the microstructure size.

4. APPLICATION – A STATIONARY HEAT TRANSFER ACROSS LAMINAS

4.1. Analytical solutions

In order to compare distributions of temperature in transversally graded laminates with uniform and non-uniform distribution of laminas it is considered the stationary heat conduction across laminas, i.e. along the x-axis. The basic unknowns are functions of argument x, i.e. $\theta=\theta(x)$, $\vartheta=\vartheta(x)$. We also neglect heat sources p. Denoting:

$$\begin{aligned}&K\equiv<k_{11}>,\quad \tilde{K}\equiv<k_{11}\partial h>,\quad \breve{K}\equiv<k_{11}(\partial h)^2>,\\&K^{eff}\equiv K-\tilde{K}^2\breve{K}^{-1},\end{aligned} \qquad (6)$$

from equations (5) we obtain one differential equation for macrotemperature θ and the formula for amplitude fluctuation ϑ in the form:

$$\partial(K^{eff}\partial\theta)=0, \qquad \vartheta=-\tilde{K}\breve{K}^{-1}\partial\theta, \qquad (7)$$

with slowly-varying functional coefficients. Since all these coefficients are determined by the known functions, a solution to equation $(7)_1$ can be calculated as in papers [2,3]. For the layer under consideration parameters k', k'' are constant. Hence, using formula $(6)_4$ and the fluctuation shape function (3) the effective heat conduction coefficient K^{eff} is equal $K^{eff}(x)=k'k''[k'+(k''-k')\nu'(x)]^{-1}$.

Denoting $N(x)=\int\nu'(x)dx$ macrotemperature θ, being the solution to $(7)_1$, is given by

$$\theta(x)=A[x(k'')^{-1}+(k''-k')(k'k'')^{-1}N(x)]+B, \qquad (8)$$

where A, B are constants calculated from boundary conditions. Assuming the boundary conditions in the form

$$x=0:\quad \theta(0)=\theta_0; \qquad x=H:\ \theta(H)=0, \qquad (9)$$

these constants are equal:

$$A=\theta_0\frac{k'k''}{(k''-k')[N(0)-N(H)]-k'H}; \qquad (10)$$

$$B=-\theta_0\frac{k'H+(k''-k')N(H)}{(k''-k')[N(0)-N(H)]-k'H}. \qquad (11)$$

Using formula $(7)_2$ with (8) and substituting the resulting equation and (8) into formula (4) the temperature takes the form:

$$T(x)=\left\{\frac{x}{k''}+\frac{k''-k'}{k'k''}[N(x)+\tfrac{\sqrt{3}}{6}h(x)\nu(x)]\right\}A+B. \qquad (12)$$

Formula (12) with constants A, B determined by (10), (11) describes the distribution of the temperature in both the type of transversally graded laminates under consideration.

4.2. Results

Let us assume for the TGL layer with non-uniform distribution of laminas that *the cell distribution function* $\lambda(x)$ is linear:

$$\lambda(\bar{x})=\bar{x}\tfrac{2(H-Ml)}{H(M-1)}+l, \qquad (13)$$

where H is the layer thickness coupled with the microstructure parameter l by the relation $H=(2M\text{-}1)l$ (M is the number of laminas). The fraction ratios of materials have the form:

$$\nu'(\bar{x})=l(\lambda(\bar{x}))^{-1}, \qquad \nu''(\bar{x})=1-\nu'(\bar{x}). \qquad (14)$$

In order to compare and evaluate results we consider four cases of the layer:

1) the first case (ϕ=1) – the layer with the non-uniform distribution of laminas, given by function (13), for which functions ν', ν'' are determined by (14);

2) the second case (ϕ=2) – the layer with the uniform distribution of laminas, λ=const, for which functions ν', ν'' are also determined by (14);

3) the third case (ϕ=3) – the layer with the uniform distribution of laminas, λ=const, for which functions ν', ν'' are linear:

$$\nu'(x)=1-x/H, \quad \nu''(x)=1-\nu'(x)=x/H; \qquad (15)$$

4) the fourth case (ϕ=4) – the periodic layer, $\lambda(x)=2l$.

Some results are shown in Fig. 3-4. In Fig. 3 there are presented plots of distributions of temperature T across laminas, obtained in the framework of the tolerance model for four aforementioned cases of the layer – ϕ=1,2,3,4, in interval $[0,H]$. Fig. 4 shows diagrams of temperature T for three of these cases – ϕ=1,2,4, in interval $[0.36H,0.45H]$. Calculations are made for the layer consisted of M=20 laminas, hence: for ϕ=1 – l/H=0.026, for ϕ=2,3,4 – λ/H=0.05. It is also assumed that ratios of heat conduction coefficients of both the materials are equal: k''/k'=0.33 (a) or k''/k'=0.2 (b).

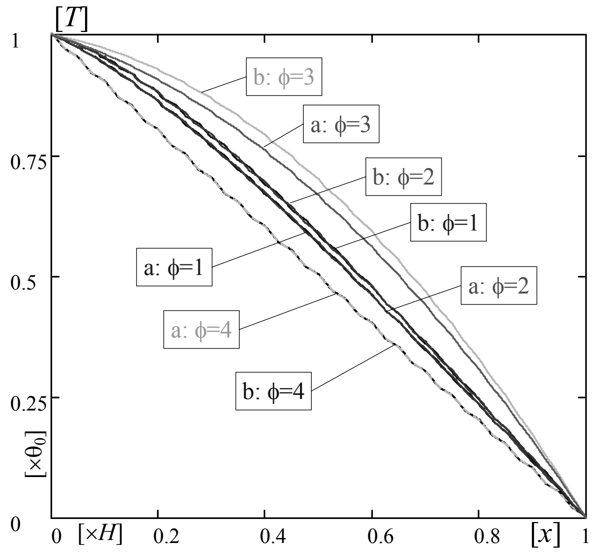

Fig. 3. Plots of temperature T along the layer thickness H for $x\in[0,H]$ (a – for k''/k'=0.33, b – for k''/k'=0.2)

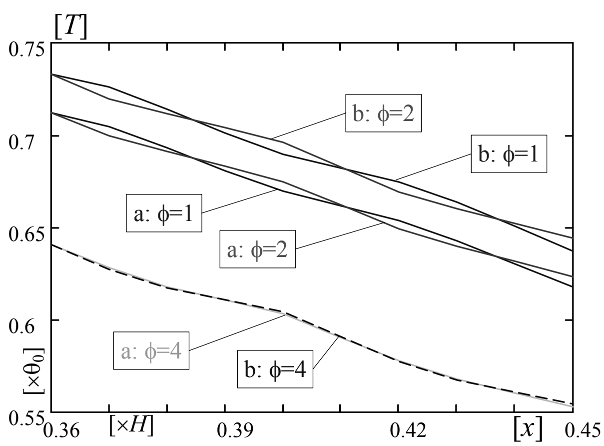

Fig. 4. Plots of temperature T for $x\in[0.36H,0.45H]$ (a – for k''/k'=0.33, b – for k''/k'=0.2)

5. FINAL REMARKS

Analysing obtained results some remarks can be formulated:

- Differences between values of temperature T in the layers with non-uniform (ϕ=1) and uniform (ϕ=2) distribution of laminas, characterised by the same fraction ratios of materials ν', ν'', (14), are very small (i.e. they are observed on the microlevel), cf. Fig. 4.
- Temperatures in the layers with uniform distribution of laminas, described by linear fraction ratios of materials (ϕ=3), are higher than those in the layers with non-uniform distribution of laminas, characterised by linear cell distribution function (ϕ=1), cf. Fig. 3.
- Differences between temperatures in the periodic layers calculated for various ratios k''/k' are very small; they are smaller than in the TGL layers.

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References

[1] Jędrysiak J. Thermomechanics of laminates, plates and shells with functionally graded structure. *Wydawnictwo Politechniki Łódzkiej, Łódź 2010, (in Polish)*

[2] Jędrysiak J., Radzikowska A. *On the modelling of heat conduction in a non-periodically laminated layer*. J. Theor. Appl. Mech. 45 (2007), 239-257

[3] Jędrysiak J., Radzikowska A. *Stationary heat conduction in a laminate with functionally graded macrostructure (FGM)*. Build. Phys. Theory Pract. 3 (2008), 23-26

[4] Jędrysiak J., Radzikowska A. *Some problems of heat conduction for transversally graded laminates with non-uniform distribution of laminas*. Arch. Civ. Mech. Engng. (2010), (in the course of publication)

[5] Michalak B., Woźniak Cz., Woźniak M. *Modelling and analysis of certain functionally graded heat conductor*. Arch. Appl. Mech. 77 (2007), 823-834

[6] Radzikowska A., Jędrysiak J. *The heat conduction in non-periodic composite (FGM)*. Building Physics in Theory and Practice 2 (2007), 239-242, (in Polish)

[7] Suresh S., Mortensen A. *Fundamentals of functionally graded materials*. The University Press, Cambridge 1998

[8] Woźniak Cz., Michalak B., Jędrysiak J. [eds.] Thermomechanics of Heterogeneous Solids and Structures. Tolerance Averaging Approach. *Wydawnictwo Politechniki Łódzkiej, Łódź 2008*

[9] Woźniak Cz. et al. [eds.] Mathematical Modelling and Analysis in Continuum Mechanics of Microstructured Media. *Wydawnictwo Politechniki Śląskiej, Gliwice 2010*

[10] Woźniak Cz., Wierzbicki E. *Averaging Techniques in Thermomechanics of Composite Solids*. Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2000