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STATIONARY HEAT CONDUCTION IN A LAMINATE WITH FUNCTIONALLY GRADED MACROSTRUCTURE (FGM)

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Abstract: In this contribution a problem of heat conduction in a laminate with functionally graded macrostructure is considered. This composite is made of two components, which are non-periodically distributed in laminae with constant thickness. Macroscopic (effective) properties of this composite are continuously varied along an axis perpendicular to laminae. In this note an averaged model of heat conduction is applied. This model is based on concepts of the tolerance averaging technique, cf. Woźniak and Wierzbicki [9].

Key words: heat conduction, laminate, functionally graded materials, tolerance averaging.

1. INTRODUCTION

The object of considerations is a laminate made of two conductors, non-periodically distributed along a direction normal to laminae. It is assumed that macroscopic (averaged) properties of such composite vary continuously along this direction, cf. Fig. 1a. However, the microstructure of this composite is shown in Fig. 1b. This laminate can be referred to *a functionally graded material* (FGM), cf. Suresh and Mortensen [7].
Thermomechanical phenomena in FGM-type composites can be investigated only in the framework of micromechanical models with idealised geometries. To describe these composites methods proposed for macroscopically homogeneous materials are usually applied.
Some fundamental methods used to determine properties of FGM-type composites are discussed in the book [7]. Between various models it can be mentioned those based on *the asymptotic homogenization*. However, governing equations of these models neglect usually the effect of the microstructure size on the overall behaviour of laminates.
Other technique to the modelling various problems of FGM-type composites is proposed by Aboudi *et al*. [1] and called *the higher order theory*. This theory describes some effects of the microstructure.
An alternative approach used to analyse FGM-type composites is *the tolerance averaging technique*. This approach has been proposed to the modelling non-stationary problems of periodic composites and structures and is discussed in the book [9] by Woźniak and Wierzbicki. In the last years this technique was adopted to analyse various problems of FGM-type composites, cf. Jędrysiak *et al*. [3], Rychlewska *et al*. [6], Szymczyk and Woźniak [8].

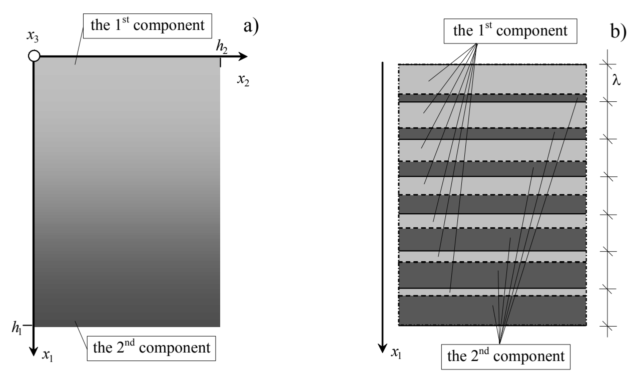

Fig. 1. A fragment of the laminate: a) on the macrolevel, b) on the microlevel

This modelling approach was also applied to investigate heat conduction problems, cf. Jędrysiak and Radzikowska [2], Michalak *et al*. [4], Radzikowska and Jędrysiak [5], described by the known differential equation of the Fourier's model. For FGM-type laminates this heat conduction equation has highly-oscillating, non-continuous, functional coefficients. The tolerance averaging technique applied to this equation leads to the system of differential equations with continuous, slowly-varying coefficients.
The aim of this paper is a comparison between results obtained in the framework of the tolerance averaging

technique and those within the higher order theory for a stationary heat conduction across laminae.

2. PRELIMINARIES

Let indices $i, j, \ldots$ run on 1, 2, 3 and be related to the Carthesian coordinate system $Ox_1x_2x_3$. Moreover, indices α, β, … run on 2, 3 and are related to the system Ox_2x_3. Let us denote: $\mathbf{x}\equiv(x_2,x_3)$, $x\equiv x_1$ and t as a time coordinate. There are also denoted by $\partial_{i\ldots j}\equiv\partial_i\ldots\partial_j$ derivatives of x_i, i=1, 2, 3. The layer under consideration has thickness h along the x-axis and is made of two materials, distributed in m laminae with thickness λ. It is assumed that $\lambda<<h$; hence, thickness λ is called *the microstructure parameter*. The n-th lamina (n=1,…,m) is consisted of two homogeneous sub-laminae, with thicknesses λ'_n, λ''_n dependent on x, cf. Fig. 1b. Properties of these two components are determined by: specific heats c', c'' and heat conduction tensors k'_{ij}, k''_{ij}, i, j=1, 2, 3. Let $\nu'_n\equiv\lambda'_n/\lambda, \nu''_n\equiv\lambda''_n/\lambda$ be material volume fractions in the n-th lamina. Assuming sequence $\{\nu'_n\}$, n=1,…,m, to be monotone and for every n=1,…,m–1 to satisfy condition $|\nu'_{n+1}-\nu'_n|<<1$, the considered layer can be treated as made of the functionally graded material. Because $\nu'_n+\nu''_n=1$ the above conditions are satisfied also by sequence $\{\nu''_n\}$. Hence, we can approximate sequences $\{\nu'_n\}$, $\{\nu''_n\}$, n=1,…,m, by continuous functions $\nu'(\cdot)$, $\nu''(\cdot)$. These functions determine the distribution of material properties along the x-axis. Let us define *the non-homogeneity ratio* ν by $\nu(\cdot)\equiv[\nu'(\cdot)\nu''(\cdot)]^{1/2}$. It is assumed that functions ν', ν'' are slowly-varying (cf. [2, 5]). A fragment of the layer on the macrolevel is shown in Fig. 1a and on the microlevel in Fig. 1b.

It is assumed that oscillations of the unknown temperature field T are small and the intensity of heat sources are neglected. The Fourier's equation of the heat conduction in a transversally graded composite has the form:

$$-\partial_i(k_{ij}\partial_j T)+c\dot{T}=0 \qquad (1)$$

In the above equation coefficients $k_{ij}=k_{ij}(x)$, $c=c(x)$ can be highly-oscillating, non-continuous functions in x. In order to find solutions to this problem equation (1) can be replaced by a system of differential equations with continuous, slowly-varying coefficients, by using the tolerance averaging technique, cf. [9, 2, 5].

3. MODELLING TECHNIQUE

In the modelling procedure we use introductory concepts of the tolerance averaging technique, e.g.: the highly-oscillating function, the averaging operator, the slowly-varying function, the fluctuation shape function. These concepts were introduced in the book [9] and adopted for functionally graded materials in [2-6, 8]. Some of them are reminded below.

The averaging operator for an arbitrary integrable function f determined in interval $[0,h]$ is defined as:

$$<f>(\bar{x})=l^{-1}\int_{\bar{x}-\lambda/2}^{\bar{x}+\lambda/2} f(x)dx, \qquad \bar{x}\in[\lambda/2, h-\lambda/2] \qquad (2)$$

The fluctuation shape function φ is assumed to be continuous function, which values are of an order $O(\lambda)$. Moreover, it is linear across the thickness of every sub-lamina. For the laminate under consideration the fluctuation shape function is assumed in the form:

$$\varphi(x)=\begin{cases}-\lambda\sqrt{3}\dfrac{\nu(\bar{x})}{\nu''(\bar{x})}[2\dfrac{x}{\lambda}+\nu'(\bar{x})] & \text{for } x\in\left(-\dfrac{\lambda}{2}, -\dfrac{\lambda}{2}+\lambda\nu''(\bar{x})\right)\\ \lambda\sqrt{3}\dfrac{\nu(\bar{x})}{\nu'(\bar{x})}[2\dfrac{x}{\lambda}-\nu''(\bar{x})] & \text{for } x\in\left(\dfrac{\lambda}{2}-\lambda\nu'(\bar{x}), \dfrac{\lambda}{2}\right)\end{cases} \qquad (3)$$

where $\bar{x}$ is a centre of the interval $[-\lambda/2, \lambda/2]$. A diagram of this functions is shown in Fig. 2.

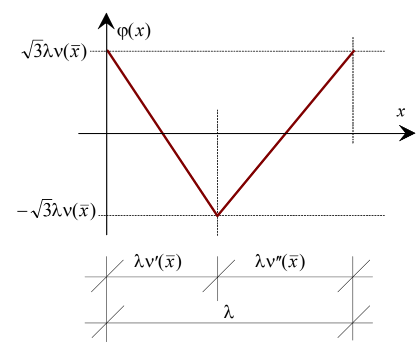

Fig. 2. Scheme of the fluctuation shape function

It can be shown that the averaged value of the fluctuation shape function φ is equal zero, because the non-homogeneity ratio ν is the slowly-varying function.

Now, let us introduce the fundamental assumptions of the tolerance averaging technique, cf. [9, 2, 5].

In *the micro-macro decomposition* we assume that temperature $T=T(x,\mathbf{x},t)$, $x\in[0,h]$, is decomposed in the form:

$$T(x,\mathbf{x},t)=W(x,\mathbf{x},t)+\varphi(x)Q(x,\mathbf{x},t) \qquad (4)$$

where $W(\cdot,\mathbf{x},t)$ is called *the averaged temperature*, $Q(\cdot,\mathbf{x},t)$ is called *the amplitude fluctuation*. Functions W, Q are new basic unknowns, being slowly-varying functions in x.

The second assumption – *the tolerance averaging approximation*, stands that for arbitrary slowly-varying function F the approximation $F+O(Fd)\cong F$ can be applied, i.e. terms of an order $O(d)$ are negligibly small in the comparing to 1.

Using the modelling procedure shown for FGM-type composites in [2, 4, 5], which is similar to that applied for periodic composites, cf. [9], after some manipulations we obtain the model equations of the heat conduction in transversally graded laminates.

4. TOLERANCE MODEL EQUATIONS

The governing equations of *the tolerance model of the heat conduction in laminates with functionally graded macrostructure* can be written in the following form:

$$\begin{aligned}&<k_{\alpha j}>\partial_{j\alpha}W+\partial_1(<k_{1j}>\partial_j W)-\\&-<c>\dot{Q}+\partial_1(<k_{11}\partial_1\varphi>Q)+<k_{\alpha 1}\partial_1\varphi>\partial_\alpha Q=0\\&<k_{1j}\partial_1\varphi>\partial_j W+\\&+<k_{11}\partial_1\varphi\partial_1\varphi>Q+\underline{<c\varphi\varphi>}\dot{Q}-\underline{<k_{\alpha\beta}\varphi\varphi>}\partial_{\alpha\beta}Q=0\end{aligned} \quad (5)$$

One of characteristic features of equations (5) is that coefficients in brackets <> are slowly-varying functions in x. Underlined terms depend on the microstructure parameter λ. Hence, the tolerance model equations (5) describe the effect of the microstructure size.

5. APPLICATIONS TO STATIONARY HEAT CONDUCTION

5.1. Tolerance model

As an example let us consider the stationary heat conduction in a transversally graded layer only across laminae, i.e. along the x-axis. Hence, the basic unknowns are functions of argument x, i.e. $W=W(x)$, $Q=Q(x)$. Let us denote by k', k'' heat conduction coefficients in sub-laminae, and also $\partial=\partial_1$. Introducing notations:

$$\begin{aligned}&K\equiv<k_{11}>,\quad \widetilde{K}\equiv<k_{11}\partial\varphi>,\quad \breve{K}\equiv<k_{11}\partial\varphi\partial\varphi>\\&K^{eff}(x)\equiv K(x)-[\widetilde{K}(x)]^2[\breve{K}(x)]^{-1}\end{aligned} \quad (6)$$

then calculating from equation $(5)_2$ the amplitude fluctuation Q and substituting into equation $(5)_1$, equations (5) can be written in the form:

$$\partial(K^{eff}\partial W)=0,\qquad Q=-\widetilde{K}\breve{K}^{-1}\partial W \quad (7)$$

Equations (7) have functional coefficients. Because the distribution functions of material properties ν', ν'' are slowly-varying and known and the fluctuation shape function φ is also known (cf. (3)), a solution to equation $(7)_1$ can be calculated by integrating this equation. This solution can be written in the form:

$$W(x)=A\int[K^{eff}(x)]^{-1}dx+B \quad (8)$$

where A, B are constants determined by boundary conditions. For the layer under consideration with constant parameters k', k'' and the fluctuation shape function (3) the averaged heat conduction coefficient $(6)_4$ is equal $K^{eff}(x)=k'k''[k'+(k''-k')\nu'(x)]^{-1}$. Denoting $P(x)=\int\nu'(x)dx$ the averaged temperature (8) can be written as:

$$W(x)=A[x(k'')^{-1}+(k''-k')(k'k'')^{-1}P(x)]+B \quad (9)$$

Assuming the boundary conditions in the form

$$x=0:\ W(0)=T_0;\qquad x=h:\ W(h)=0 \quad (10)$$

constants A, B are:

$$\begin{aligned}A&=T_0\frac{k'k''}{(k''-k')[P(0)-P(h)]-k'h}\\B&=-T_0\frac{k'h+(k''-k')P(h)}{(k''-k')[P(0)-P(h)]-k'h}\end{aligned} \quad (11)$$

Combining (9) with $(7)_2$ we obtain a formula for the amplitude fluctuation Q. Then substituting the resulting equation and solutions (9) into equation (4) the temperature can be written as:

$$T(x)=\left\{\frac{x}{k''}+\frac{k''-k'}{k'k''}[P(x)+\tfrac{\sqrt{3}}{6}\varphi(x)\nu(x)]\right\}A+B \quad (12)$$

with constants A, B determined by (11).
On the contrary to the problem considered in [5] we obtain here analytical solutions.

5.2. Higher order theory

In order to evaluate obtained results let us consider the stationary heat conduction from Subsection 5.1 in the framework of *the higher order theory*. This modelling approach was applied in a series of papers and discussed and summarized in the paper [1] by Aboudi *et al*. Following this paper the procedure of this theory will be shown.
Denoting by x', x'' local coordinates in sub-laminae in the n-th lamina (n=1,…,m) temperature distributions in both the materials in the n-th lamina are postulated as:

$$\begin{aligned}T'(x')&=T_0'+x'T_1'+\tfrac{1}{2}[3(x')^2-(\tfrac{1}{2}\lambda_n')^2]T_2'\\T''(x'')&=T_0''+x''T_1''+\tfrac{1}{2}[3(x'')^2-(\tfrac{1}{2}\lambda_n'')^2]T_2''\end{aligned} \quad (13)$$

where $T_0', T_1', T_2', T_0'', T_1'', T_2''$ are unknown constants. For the case of m laminae there are $6m$ unknown parameters being the basic unknowns of the model.
Let us denote by k', k'' heat conduction coefficients in sub-laminae and $\partial\equiv\partial/\partial x$ (x=x', x''). In order to calculate the above $6m$ unknowns the following relations are used:

- the heat conduction equations for the n-th lamina

$$-\partial[k'\partial T'(x')]=0,\qquad -\partial[k''\partial T''(x'')]=0 \quad (14)$$

- the continuity conditions of heat fluxes for the n-th lamina

$$\begin{aligned} &\partial(k'\partial T')\big|^{n}_{x'=\frac{1}{2}\lambda'_n} = \partial(k''\partial T'')\big|^{n}_{x''=-\frac{1}{2}\lambda''_n} \\ &\partial(k'\partial T')\big|^{n+1}_{x'=-\frac{1}{2}\lambda'_{n+1}} = \partial(k''\partial T'')\big|^{n}_{x''=\frac{1}{2}\lambda''_n} \end{aligned} \qquad (15)$$

- the continuity conditions of the temperature for the n-th lamina

$$T'\big|^{n}_{x'=\frac{1}{2}\lambda'_n} = T''\big|^{n}_{x''=-\frac{1}{2}\lambda''_n}, \qquad T'\big|^{n+1}_{x'=-\frac{1}{2}\lambda'_{n+1}} = T''\big|^{n}_{x''=\frac{1}{2}\lambda''_n} \qquad (16)$$

- the boundary conditions (here assumed as (10))

$$\begin{aligned} &T'\big|^{1} = T_T = T_0, \quad x' = -\tfrac{1}{2}\lambda'_1 \qquad (\text{i.e. } x=0) \\ &T''\big|^{m} = T_B = 0, \quad x'' = \tfrac{1}{2}\lambda''_m \qquad (\text{i.e. } x=h) \end{aligned} \qquad (17)$$

where T_T and T_B are temperatures on the top and the bottom boundary, respectively.
Substituting equations (13) into equations (14)-(17) we obtain a system of $6m$ algebraic linear equations for $6m$ basic unknowns.

5.3. Results

Let us consider linear distributions of materials as an example. Hence, material distribution functions are assumed in the form:

1) the first case of the fraction ratios of materials

$$\nu'(x)=x/H, \qquad \nu''(x)=1-\nu'(x)=1-x/H \qquad (18)$$

2) the second case of the fraction ratios of materials

$$\nu'(x)=1-x/H, \qquad \nu''(x)=1-\nu'(x)=x/H \qquad (19)$$

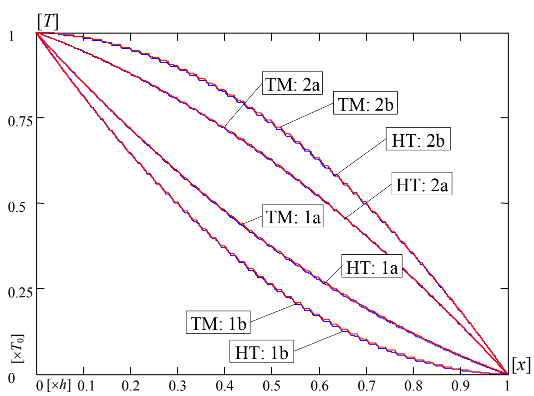

Fig. 3. Diagrams of temperature T along the thickness h of the layer for $x\in[0,h]$ (TM – the tolerance model, HT – the higher order theory; 1 – for formulae (18), 2 – for formulae (19); a – for k''/k'=0.33, b – for k''/k'=0.025)

Some results of calculations are shown in Fig. 3. In this figure there are presented plots of distributions of temperature T across laminae, calculated in the framework of the tolerance model (TM) and the higher order theory (HT). Calculations are made for the composite consisted of m=50 laminae, hence λ/h=0.02. Moreover, it is assumed that ratios of heat conduction coefficients of both the materials are equal: k''/k'=0.33 or k''/k'=0.025.

6. REMARKS

Analysing results obtained for the example of the stationary heat conduction we can observe that:

- in the framework of the tolerance model we have not to solve a system of many algebraic equations in order to obtain a solution to the problem on the contrary to the higher order theory;
- for the problem under consideration the tolerance model leads to the explicit solution determined by a function;
- values of the temperature calculated within the tolerance model are higher than those from the higher order theory but differences between them are very small, cf. Fig. 3.

More detailed analysis will be presented in the forthcoming paper.

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