ZESZYTY NAUKOWE AKADEMII MARYNARKI WOJENNEJ
ROK LIII NR 3 (190) 2012

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WHIRLING OF ASYMMETRIC SHAFT UNDER CONSTANT LATERAL FORCE

ABSTRACT

The paper deals with parametric vibrations of asymmetric shaft subjected to constant lateral force. As an example of asymmetry, a uniform shaft of rectangular cross section is chosen. During rotation of such a shaft its bending stiffness with respect to fixed coordinate system varies with time which leads to vibrations. In order to avoid solving differential equation with time-dependent coefficient, the motion of the shaft is calculated in the co-rotating coordinate system and then the solution is transformed to the fixed coordinate system by means of simple kinematic relationships. Discrete and continuous undamped models of the vibrating system are considered under assumption that the rotational speed of the shaft is constant. It is shown that the geometric centers of the cross sections of the shaft perform circular motions in the planes perpendicular to the bearing axis so that the shaft is whirling with the frequency twice as high as its angular velocity.

Keywords:
mechanical vibrations, discrete model, continuous model, parametric excitation.

INTRODUCTION

The subject of this paper are parametric vibrations of shafts the cross sections thereof are asymmetric. In particular, it refers to uniform shafts of non-circular cross sections (e.g. to shafts of rectangular cross sections), but also to shafts with a lack of circular symmetry of bending stiffness in some segments, caused by non-homogeneous material, casting defects, technological cuts (grooves, key beds, splineways), misuse and fatigue damages (indentations, cracks), etc. During rotation of such shafts their bending stiffness with respect to a fixed coordinate system varies with time which may lead to vibrations [1, 2]. For example, the constrained motion (limited mechanically to vertical vibration) of a mass mounted on a shaft of rectangular cross section is governed by Mathieu equation [3] of known solution [4]. However, in the general case of linear differential equations with time-dependent coefficients their solutions are not known [5] and numerical, graphical or approximate methods in vibration analysis have to be applied [6].

The present paper is concerned with vibration of an asymmetric shaft loaded by a lateral force of constant magnitude and direction. As an example of asymmetry, a uniform shaft of rectangular cross section is chosen. In order to avoid solving differential equation with time-dependent coefficient, the motion of the shaft is calculated in the co-rotating coordinate system and then the solution is transformed to the fixed coordinate system by means of simple kinematic relations. Discrete and continuous undamped models of the vibrating system are considered under assumptions that the shaft is simply supported (pinned) in bearings at the ends and that the steady-state angular velocity of the shaft, ω, is constant. Other models can be found, for example, in [7, 8] where the influence of bearing damping on stability of asymmetric shafts supported by isotropic bearings are considered, in [9] where a shaft crack is distinguished from other rotating asymmetries or in [10–13] where the vibrations of a shaft carrying an asymmetrical rotor are analysed.

ANALYSIS OF THE DISCRETE MODEL

Consider a uniform shaft of length l subjected to the vertical force F at the middle. In this Section we shall assume that the mass forces of the shaft are negligible in comparison with the spring forces of the shaft. We also assume that the cross section of the shaft has two axes of symmetry, lying on the principal axes of inertia of the cross section, y' and z' (fig. 1). We shall use the fixed coordinate system *OXYZ* and the coordinate system *Oxyz* rotating with the angular velocity ω, where:

X, x — axes lying along the line of centers of the bearings;
Y, Z — horizontal and vertical axes;
y, z — axes parallel to the axes y' and z'.

The shaft is pinned at $X = x = 0$ and at $X = x = l$. Let C denote the geometric center of the cross section of the shaft at $X = x = \frac{1}{2}l$ where the force F is acting. In an angular position of the shaft determined by the angle ωt between the axes y and Y, the components of the force F on the axes y' and z' are $F\sin\omega t$ and $F\cos\omega t$, respectively.

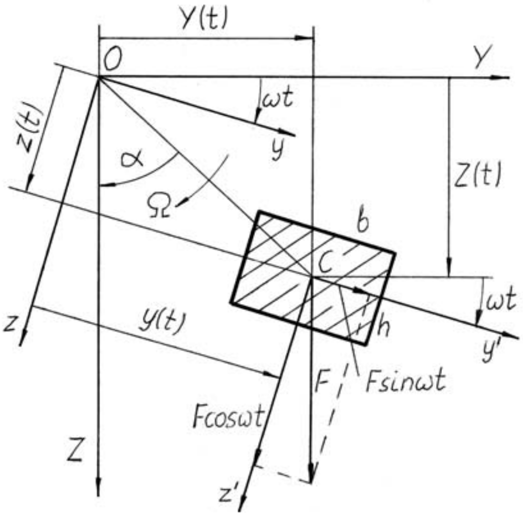

Fig. 1. Components of the force *F* and translation *OC* of the center *C* in displaced condition of the shaft at $X = x = \frac{1}{2}l$

Source: own study.

Denoting k_y and k_z the coefficients of bending stiffness of the shaft at $X = x = \frac{1}{2}l$ in the directions y and z, the equations of motion of the center *C* in the rotating coordinate system can be written as

$$k_y y(t) = F\sin\omega t\ , \quad k_z z(t) = F\cos\omega t\ . \tag{1}$$

Hence

$$y(t) = a\sin\omega t\ , \quad z(t) = c\cos\omega t\ , \tag{2}$$

where

$$\begin{aligned} a &= \frac{F}{k_y}, \quad k_y = \frac{48EI_{z'}}{l^3} \\ c &= \frac{F}{k_z}, \quad k_z = \frac{48EI_{y'}}{l^3} \end{aligned} . \tag{3}$$

In Eqs (3) *E* is the Young's modulus and

$$I_{y'} = \frac{bh^3}{12}\ , \quad I_{z'} = \frac{hb^3}{12} \tag{4}$$

are the axial moments of inertia of the cross section of the shaft.

With the aid of Eqs (2) it is easy to describe the motion of the cross section of the shaft at the middle. For example, the resultant translation *OC* of the geometric center *C*

$$OC = \sqrt{[Y(t)]^2 + [Z(t)]^2}$$

can be now calculated as (see fig. 1)

$$OC = \sqrt{[y(t)]^2 + [z(t)]^2} = \sqrt{\frac{1}{2}\left[a^2 + c^2 - \left(a^2 - c^2\right)\cos 2\omega t\right]}. \tag{5}$$

The angular displacement α of the line *OC* and the velocity $\Omega = \frac{d\alpha}{dt}$ of its angular oscillation are determined by the coordinates of the center *C*. Hereunder these coordinates will be calculated in the fixed coordinate system. For this purpose use will be made of the kinematic relationships following from fig. 1

$$\begin{aligned} Y(t) &= y(t)\cos\omega t - z(t)\sin\omega t \\ Z(t) &= y(t)\sin\omega t + z(t)\cos\omega t \end{aligned}. \tag{6}$$

Substitution of Eqs (2) into Eqs (6) yields

$$\begin{aligned} Y(t) &= A\sin 2\omega t \\ Z(t) &= D - A\cos 2\omega t \end{aligned}, \tag{7}$$

where

$$A = \frac{1}{2}(a - c), \quad D = \frac{1}{2}(a + c). \tag{8}$$

Eqs (7) describe horizontal and vertical components of the translation of the geometric center of the cross section of the shaft at the middle, which result in circular motion of this center and whirling of the shaft with angular velocity twice as high as the angular velocity of the shaft. The radius of the circle is given by

$$|A| = \frac{1}{2}|a - c| \tag{9}$$

and the coordinates of its centre are

$$X = \frac{1}{2}l\,, \quad Y = 0\,, \quad Z = D = \frac{1}{2}(a + c). \tag{10}$$

Thus the extreme vertical deflections of the shaft axis amount to

$$Z = D \pm A = \frac{1}{2}\left[a + c \pm (a - c)\right]. \tag{11}$$

As to the angular oscillation of the line *OC*, the extreme values of the angle α are $\pm\alpha_a$, where

$$\alpha_a \cong arc\text{tg}\frac{A}{D} = arc\text{tg}\frac{a-c}{a+c}. \tag{12}$$

ANALYSIS OF THE CONTINUOUS MODEL

Generally, the information obtained from a discrete model of a system may not be as accurate as that obtained from a continuous model. In particular, the model analyzed in the foregoing does not include mass forces so that Eqs (2) can not reveal any resonance phenomena. However, for any beam modeled as a continuous system there will be an infinite number of normal modes with one natural frequency associated with each normal mode. Therefore in what follows the continuous distribution of the mass and elasticity of the shaft will be taken into account with the aid of the Euler--Bernoulli theory of bending of beams. In the considered case, the instantaneous deflection of the shaft axis will be determined by the coordinates $Y(X,t)$ and $Z(X,t)$ of the geometric centers of the cross sections of the shaft in the fixed coordinate system or, alternatively, by the coordinates $y(x,t)$ and $z(x,t)$ of these centers in the rotating coordinate system. Here $x = X$ and, analogously to Eqs (6)

$$\begin{aligned} Y(x,t) &= y(x,t)\cos\omega t - z(x,t)\sin\omega t \\ Z(x,t) &= y(x,t)\sin\omega t + z(x,t)\cos\omega t \end{aligned}. \tag{13}$$

According to the Euler-Bernoulli theory, the forced lateral vibration $w(x,t)$ of a uniform beam is governed by equation [5, 6]

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho S\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t), \tag{14}$$

where:

$f(x,t)$ — external lateral force per unit length of the beam;
I — axial moment of inertia of the cross section of the beam;
S — cross-sectional area of the beam;
ρ — mass density of the beam.

The solution of Eq. (14) can be determined using the mode superposition principle. For this, the deflection of the beam is assumed as

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) q_n(t), \tag{15}$$

where $W_n(x)$ is the n-th normal mode function satisfying equation

$$EI \frac{d^4 W_n(x)}{dx^4} - \omega_n^2 \rho S W_n(x) = 0 ; \quad n = 1,2,... \tag{16}$$

$q_n(t)$ is the generalized coordinate in the n-th mode, and ω_n is the n-th natural frequency of bending vibration of the beam. These frequencies are computed as

$$\omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho S l^4} \right)^{1/2}, \tag{17}$$

where l is the length of the beam, and the values of $\beta_n l$ for a pinned-pinned beam are [5, 6]

$$\beta_n l = n\pi . \tag{18}$$

For any combination of the boundary conditions, the normal mode functions are orthogonal, that is

$$\int_0^l W_i(x) W_j(x) dx = 0; \quad i, j = 1,2,...; i \neq j . \tag{19}$$

For a pinned-pinned beam the normal mode functions are given by

$$W_n(x) = \sin \beta_n x = \sin \frac{n\pi x}{l} . \tag{20}$$

By substituting Eq. (15) into Eq. (14), we obtain

$$EI \sum_{n=1}^{\infty} \frac{d^4 W_n(x)}{dx^4} q_n(t) + \rho S \sum_{n=1}^{\infty} W_n(x) \frac{d^2 q_n(t)}{dt^2} = f(x,t). \tag{21}$$

With regard to Eq. (16), Eq. (21) can be written as

$$\sum_{n=1}^{\infty} \omega_n^2 W_n(x) q_n(t) + \sum_{n=1}^{\infty} W_n(x) \frac{d^2 q_n(t)}{dt^2} = \frac{1}{\rho S} f(x,t). \tag{22}$$

By multiplying Eq. (22) throughout by $W_n(x)$, integrating from O to l, and using the orthogonality condition, Eq. (19), one gets

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho S k} Q_n(t), \tag{23}$$

where $Q_n(t)$ is called the generalized force corresponding to $q_n(t)$, calculated as

$$Q_n(t) = \int_o^l f(x,t) W_n(x) dx \tag{24}$$

and the constant k is given by

$$k = \int_o^l W_n^2(x) dx. \tag{25}$$

The solution of Eq. (23) can be expressed as

$$q_n(t) = A_n \cos\omega_n t + B_n \sin\omega_n t + \frac{1}{\rho S k \omega_n} \int_O^t Q_n(\tau) \sin\omega_n(t-\tau) d\tau, \tag{26}$$

where the first two terms on the right-hand side of Eq. (26) represent the free vibration resulting from initial conditions and the third term denotes the steady-state vibration resulting from the forcing function $f(x,t)$. If a harmonic force

$$f(x,t) = f_o \sin\omega t \tag{27}$$

is applied at $x = x_o$, the generalized force becomes [6, 14]

$$Q_n(t) = \int_O^l f(x,t) \sin\beta_n x dx = f_o \sin\frac{n\pi x_o}{l} \sin\omega t. \tag{28}$$

We confine ourselves to the steady-state solution of Eq. (23)

$$q_n(t) = \frac{1}{\rho S k \omega_n} \int_O^t Q_n(\tau) \sin\omega_n(t-\tau) d\tau, \tag{29}$$

where, by Eqs (20) and (25),

$$k = \int_O^l \sin^2\beta_n x dx = \frac{1}{2} l. \tag{30}$$

The solution of Eq. (29) reads

$$q_n(t) = \frac{2 f_o}{\rho S l \left(\omega_n^2 - \omega^2\right)} \sin\frac{n\pi x_o}{l} \sin\omega t. \tag{31}$$

Thus the response of the beam, Eq. (15), becomes

$$w(x,t)=\frac{2f_o}{\rho S l}\sum_{n=1}^{\infty}\frac{1}{\omega_n^2-\omega^2}\sin\frac{n\pi x_o}{l}\sin\frac{n\pi x}{l}\sin\omega t\,. \tag{32}$$

In view of the shaft with rectangular cross section, in the first step it is convenient to resolve its deflection $w(x,t)$ into the two components $y(x,t)$ and $z(x,t)$. Then, instead of Eq. (14), we obtain the following equations

$$\left. EI_{z'}\frac{\partial^4 y}{\partial x^4}(x,t)+\rho S\frac{\partial^2 y}{\partial t^2}(x,t)=F\sin\omega t\right|_{x=\frac{1}{2}l},\qquad \left. EI_{y'}\frac{\partial^4 z}{\partial x^4}(x,t)+\rho S\frac{\partial^2 z}{\partial t^2}(x,t)=F\cos\omega t\right|_{x=\frac{1}{2}l} \tag{33}$$

where $I_{y'}$ and $I_{z'}$ are given by Eqs (4). The forces $F\sin\omega t$ and $F\cos\omega t$ in Eqs (33) are acting on the rotating shaft at $x_o = x = \frac{1}{2}l$ in the directions *y* and *z*, and its response, by Eq. (32), reads

$$y(x,t)=\frac{2F}{\rho Sl}\sum_n\frac{1}{\omega_{yn}^2-\omega^2}\sin n\frac{\pi}{2}\sin\frac{n\pi x}{l}\sin\omega t;\quad n=1,3,5,...$$
$$z(x,t)=\frac{2F}{\rho Sl}\sum_n\frac{1}{\omega_{zn}^2-\omega^2}\sin n\frac{\pi}{2}\sin\frac{n\pi x}{l}\cos\omega t;\quad n=1,3,5,... \tag{34}$$

According to Eqs (17) and (18), natural frequencies of bending vibrations of the shaft in the directions *y* and *z* become, respectively,

$$\omega_{yn}=(n\pi)^2\left(\frac{EI_{z'}}{\rho Sl^4}\right)^{1/2};\quad \mathrm{n}=1,3,...$$
$$\omega_{zn}=(n\pi)^2\left(\frac{EI_{y'}}{\rho Sl^4}\right)^{1/2};\quad \mathrm{n}=1,3,... \tag{35}$$

That angular velocity of the shaft which equals one of the frequencies (35) will be called a critical speed of the shaft.

Denoting

$$y_n(x) = \frac{2F}{\rho Sl\left(\omega_{yn}^2 - \omega^2\right)} \sin n\frac{\pi}{2} \sin\frac{n\pi x}{l}$$
$$z_n(x) = \frac{2F}{\rho Sl\left(\omega_{zn}^2 - \omega^2\right)} \sin n\frac{\pi}{2} \sin\frac{n\pi x}{l} \quad . \qquad (36)$$

Eqs (34) can be rewritten as

$$y(x,t) = a(x)\sin\omega t$$
$$z(x,t) = c(x)\cos\omega t \;, \qquad (37)$$

where

$$a(x) = \sum_n y_n(x), \quad c(x) = \sum_n z_n(x) \quad ; \quad n = 1,3,... \qquad (38)$$

The next step will be to transform the solutions (37) to the fixed coordinate system. Making use of Eqs (13) and (37) one gets

$$Y(x,t) = A(x)\sin 2\omega t \qquad (39)$$
$$Z(x,t) = D(x) - A(x)\cos 2\omega t \qquad (40)$$

where

$$A(x) = \frac{1}{2}\left[a(x) - c(x)\right]; \qquad (41)$$

$$D(x) = \frac{1}{2}\left[a(x) + c(x)\right]. \qquad (42)$$

The vertical deflection of the shaft axis, Eq. (40), is the sum of the invariable in time quantity $D(x)$ and

$$Z'(x,t) = -A(x)\cos 2\omega t \,. \qquad (43)$$

By defining a complex quantity $V(x,t)$ as

$$V(x,t) = Y(x,t) + iZ'(x,t), \qquad (44)$$

where $i = (-1)^{1/2}$ and by adding Eq.(39) to Eq. (43) multiplied by *i*, we obtain a single equation of lateral motion of the shaft axis

$$V(x,t) = -iA(x)\left(\cos 2\omega t + i\sin 2\omega t\right) \qquad (45)$$

or, using a vector notation,

$$\overline{V}(x,t) = \overline{v}(x)e^{2i\omega t} \,. \qquad (46)$$

The vector $\bar{v}(x)$ represents a complex number

$$\bar{v}(x) = 0 + i[-A(x)]. \tag{47}$$

The modulus $v(x)$ and argument Θ of the vector $\bar{v}(x)$ are

$$v(x) = \sqrt{0^2 + [-A(x)]^2} = |A(x)|; \tag{48}$$

$$\Theta = -\frac{\pi}{2} \text{ for } A(x) > 0, \quad \Theta = \frac{\pi}{2} \text{ for } A(x) < 0. \tag{49}$$

Hence on a complex plane we have

$$V(x,t) = |A(x)| e^{i(2\omega t \pm \pi/2)}, \tag{50}$$

where 2ω denotes the circular frequency of rotation of the vector $\bar{V}(x,t)$ in the same direction as that of the shaft. It means that each point of the shaft axis performs a circular motion which presents a forward whirl of the shaft axis of the amplitude $|A(x)|$ about the curved axis of equation (42) and of the frequency two times higher than the angular velocity of the shaft. Note that the whirl axis is lying on the plane made by the line of centers of the bearings and by the line of action of the force *F*, i.e. on the vertical plane *XZ*, and that its distance from the line of centers of the bearings is given by Eq. (42). In particular, at the middle of the shaft the whirl amplitude and the distance in question are

$$\left| A\left(\frac{1}{2}l\right) \right| = \frac{F}{\rho S l} \left| \sum_n \left(\frac{1}{\omega_{yn}^2 - \omega^2} - \frac{1}{\omega_{zn}^2 - \omega^2} \right) \right|; \quad n = 1,3,\ldots; \tag{51}$$

$$D\left(\frac{1}{2}l\right) = \frac{F}{\rho S l} \sum_n \left(\frac{1}{\omega_{yn}^2 - \omega^2} + \frac{1}{\omega_{zn}^2 - \omega^2} \right); \quad n = 1,3,\ldots \tag{52}$$

SUMMARY

1. Under constant lateral load, the shaft of rectangular cross section is whirling.
2. For a constant rotational speed of the shaft, each point of the shaft axis performs a circular motion of constant radius about a fixed point.
3. The circular frequency of the whirl is two times higher that the angular velocity of the shaft.

4. There is an infinite number of natural frequencies of bending vibrations of the shaft which correspond to the critical speeds of the shaft. The closer is the angular velocity of the shaft to any of its critical speeds, the larger is the whirl amplitude and the more significant is the distance of the whirl axis from the line of centers of the bearings.
5. The advantage of the fixed coordinate system is that it gives the simplest form of the rotor dynamics equations in most situations. There are, however, cases that are best analyzed in a coordinate system whose axes follow the rotating shaft. This is particularly true when the rotor is not axisymmetric [15]. The presented above approach to the considered problem is a simple example of it.

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DRGANIA KOŁOWE ASYMETRYCZNEGO WAŁU PRZY STAŁEJ SILE POPRZECZNEJ

STRESZCZENIE

Artykuł dotyczy parametrycznych drgań asymetrycznego wału obciążonego stałą siłą poprzeczną. Jako przykład asymetrii przyjęto pryzmatyczny wał o przekroju prostokątnym. W czasie ruchu obrotowego takiego wału jego sztywność giętna względem nieruchomego układu współrzędnych zmienia się w czasie, co prowadzi do drgań. Dla uniknięcia rozwiązywania równania różniczkowego o współczynniku zależnym od czasu wyznaczono ruch wału w wirującym wraz z wałem układzie współrzędnych, a następnie rozwiązanie przetransformowano do nieruchomego układu współrzędnych za pomocą prostych kinematycznych zależności. Rozpatrywany jest dyskretny i ciągły model układu zachowawczego przy założeniu, że prędkość obrotowa wału jest stała. Wykazano, że geometryczne środki przekrojów poprzecznych wału poruszają się po okręgach w płaszczyznach prostopadłych do osi łożysk, tak że wał wykonuje drgania kołowe o częstości dwukrotnie wyższej od jego prędkości kątowej.

Słowa kluczowe:

drgania mechaniczne, model dyskretny, model ciągły, wzbudzenie parametryczne.