HARMONIC GENERATION FROM A FINITE AMPLITUDE WAVE CIRCULAR SOURCE

ANNA BARANOWSKA

Gdansk University of Technology, Department of Mathematical and Numerical Analysis
G. Narutowicza 11/12, 80-952 Gdansk, Poland
anbar@mif.pg.gda.pl

The finite amplitude wave propagation problem for circular source was considered. The paper presents mathematical model and some results of theoretical investigations. The problem was considered as an axial symmetric one. The mathematical model was built on the basis of the Khokhlov – Zabolotskaya – Kuznetsov (KZK) equation. To solve this equation the Fourier series expansion and finite-difference method were applied. The pressure harmonic amplitudes as a function of distance form the source and waveform were examined. Influence of discrete model parameters on accuracy of numerical calculations was investigated.

INTRODUCTION

The wave distortion is observed during single frequency finite amplitude wave propagation. The waveform change is equivalent with spectrum change. It means that many harmonics appear during wave propagation. Therefore the harmonic analysis is very often used to investigate wave distortion.

Mathematical model of the finite amplitude wave propagation problem is often built on the basis of the KZK equation which allows to include nonlinearity, dissipation of medium and sound beam diffraction. It describes pressure changes along sound beam. No analytical exact solution for this equation has been found yet. Consequently, it is necessary to solve this equation approximately. The analytic, half-analytic methods and numerical one are used to solve the KZK equation. The method of successive approximations to find this equation solution can be used when the nonlinear effects are not very big [2]. Generally, the KZK equation needs to be solved numerically. The finite-difference method and boundary-element method can be used to solve this equation [1].

The aim of this paper is numerical analysis of the finite amplitude wave propagation. To solve the problem solution of the transformed KZK equation was sought in terms of the Fourier series expansion. The finite-differential method has been used to calculate harmonic components. Some problems connected with numerical solution of this equation were discussed.

1. MATHEMATICAL MODEL

We assume that circular piston with a fixed radius *a* which is placed in plane y*Oz*, is the source of finite amplitude wave. The wave is propagated in the *x* direction. It means that this axis corresponds with sound beam axis.

The mathematical model is built on the basis of the KZK equation:

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p'}{\partial x}-\frac{\varepsilon}{\rho_0 c_0^3}p'\frac{\partial p'}{\partial\tau}-\frac{b}{2\rho_0 c_0^3}\frac{\partial^2 p'}{\partial\tau^2}\right)=\frac{c_0}{2}\left(\frac{\partial^2 p'}{\partial r^2}+\frac{1}{r}\frac{\partial p'}{\partial r}\right) \tag{1}$$

where $p'=p-p_0$ denotes an acoustic pressure, variable $\tau = t - x/c_0$ is the time in the coordinate system fixed in the phase of the propagating wave, ρ_0 - medium density at rest, c_0 - speed of sound, b – dissipation coefficient of the medium, ε – nonlinear coefficient, $r=\sqrt{y^2+z^2}$.

Solution of Eq. (1) is looked for inside hypothetical cylinder for $x\in[0,X_{\max}]$, $r\in[0,R_{\max}]$ where X_{max} denote the biggest investigated distance from the source and R_{max} is cylinder radius (Fig. 1).

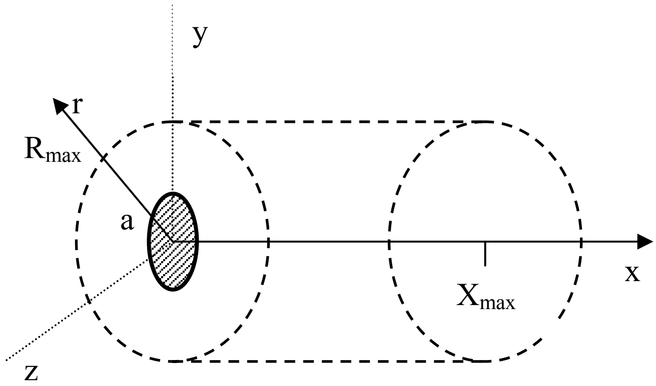


Fig.1 The geometry of the problem

If the source radiates a single frequency wave, the boundary condition for *x=0* can be written in following form:

$$p'(x=0,r,\tau)=-p_0\sin\omega\tau$$

for $r\le a$ and $p'(x=0,r,\tau)=0$ for $r>a$. Parameter p_0 denotes primary wave amplitude and angular frequency is defined by $\omega=2\pi f$. Additionally we assumed that

$$\frac{\partial p'}{\partial r}_{|r=0}=0$$

$$\frac{\partial p'}{\partial r}_{|r=R_{\max}}=0$$

and function p' is a periodic function of the coordinate τ.

2. NUMERICAL SOLUTION

To solve the KZK equation numerically the non-dimensional coordinates are defined:

$$\theta = \omega\tau,\ X = x / R_0,\ R = r / a$$

where $R_0 = 2ka^2, k = \omega / c_0$. Substituting these variables into Eq. (1), the transformed nonlinear equation is obtained:

$$\frac{\partial^2 P}{\partial\theta\partial X} - E\frac{\partial^2 P^2}{\partial\theta^2} - B\frac{\partial^3 P}{\partial\theta^3} = \frac{\partial^2 P}{\partial R^2} + \frac{1}{R}\frac{\partial P}{\partial R} \tag{2}$$

where $P = \frac{p'}{p_0}$, $E = \frac{\varepsilon p_0 \omega R_0}{2\rho_0 c_0^3}$, $B = \frac{b\omega^2 R_0}{2\rho_0 c_0^3}$.

Solution of Eq. (2) is looked for in form

$$P(X,R,\theta) = \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n(X,R)e^{in\theta} + c.c.\right). \tag{3}$$

Substituting (3) into Eq. (2) after calculations we obtain following partial differential equations for harmonic components:

$$in\frac{\partial A_n}{\partial X} + EG_n[A] + iBn^3 A_n - \left(\frac{\partial^2 A_n}{\partial R^2} + \frac{1}{R}\frac{\partial A_n}{\partial R}\right) = 0,\quad n=1,2,\ldots,N \tag{4}$$

were

$$G_n[A] = \sum_{m=1}^{n-1} A_{n-m}A_m(n-m)n - \sum_{m=n+1}^{N} A_m A_{m-n}^*(m-n)n + \sum_{m=1}^{N-n} A_m^* A_{n+m}(n+m)n$$

Including boundary conditions we can calculate harmonic amplitudes A_n (n=1,2,…N).

The calculations presented in this paper were done for N=3. It is possible to solve the problem for larger number of harmonic components, but for our values of physical parameters it is not necessary. Finally we solve system of equations:

$$i\frac{\partial A_1}{\partial X} + E\left(A_1^* A_2 + A_3 A_2^*\right) + iBA_1 - \left(\frac{\partial^2 A_1}{\partial R^2} + \frac{1}{R}\frac{\partial A_1}{\partial R}\right) = 0$$

$$i2\frac{\partial A_2}{\partial X} + E\left(2A_1^2 + 4A_1^* A_3\right) + i8BA_2 - \left(\frac{\partial^2 A_2}{\partial R^2} + \frac{1}{R}\frac{\partial A_2}{\partial R}\right) = 0$$

$$i3\frac{\partial A_3}{\partial X} + 9EA_1A_2 + i27BA_3 - \left(\frac{\partial^2 A_3}{\partial R^2} + \frac{1}{R}\frac{\partial A_3}{\partial R}\right) = 0$$

Pressure changes along sound beam can be analyzed substituting harmonic components into (3).

To solve Eqs. (4) numerically functions $A_n(X,R)$ are discretized. The rectangular net is constructed in domain $D=\{(X,R): X\in[0,X_1],\, R\in[0,R_1]\}$:

$$X_m = m\Delta X,\; R_k = k\Delta R$$
$$\Delta X = \frac{X_1}{N_X},\; \Delta R = \frac{R_1}{N_R}$$

where m=0,1,…,N_X, k=0,1,…,N_R. After approximation of derivatives in Eqs. (4) we obtain following difference equations:

$$in\frac{A_n^{(m+1,k)} - A_n^{(m,k)}}{\Delta X} + EG_n^{(m,k)}[A] + iBn^3 A_n^{(m+1,k)} - \Delta_\perp A_n^{(m+1,k)} = 0,\; n=1,2,\ldots,N \quad (5)$$

where operator $\Delta_\perp = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}$. Finally pressure amplitudes along sound beam are the result of computer calculations.

3. NUMERICAL INVESTIGATIONS

The numerical calculations were carried out assuming that circular piston with radius a=25 mm was the source of harmonic wave which is propagated in water where medium density at rest ρ_0=1000 kg/m^3, speed of sound c_0=1500 m/s, nonlinear coefficient ε=3.5. Moreover it was assumed that primary wave frequency was equal f=600 kHz.

As a result of numerical calculations we obtain pressure harmonic components. Figure 2 presents normalized first, second and third pressure harmonic amplitudes as a function of axial distance from the source. During numerical calculations it was assumed that the fundamental pressure p_0=150 kPa.

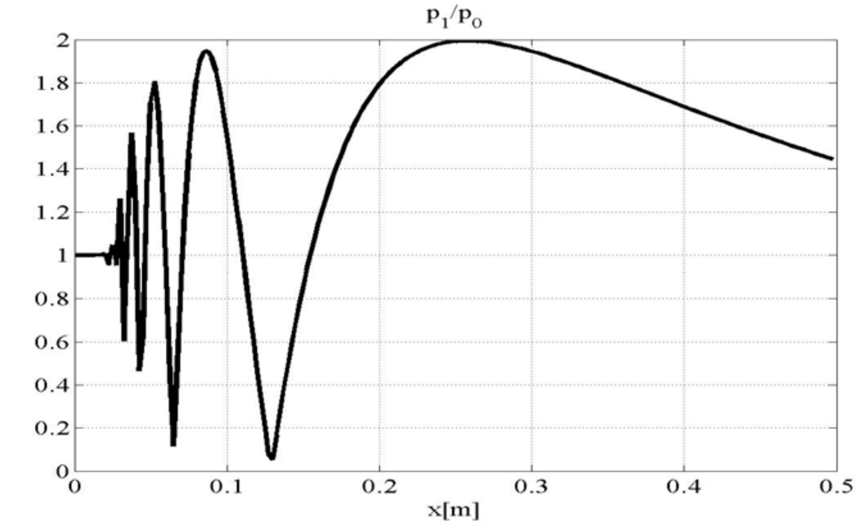
The correct choice of numerical parameters (step sizes, size of the space) is very important during numerical calculations.

On-axis first harmonic pressure amplitude as a function of distance from the source obtained for two different values of cylinder radius R_{max} is presented in Fig. 3. Curve number 1 was obtained for R_{max}=3a and curve number 2 for R_{max}=5a respectively. In this example amplitude of primary wave was equal p_0=10 kPa. This figure shows unphysical oscillations of the pressure amplitude for distance larger then x=0.2 m from the source when R_{max}=3a. This calculating error observed for big distances from the source is connected with the sound beam diffraction.

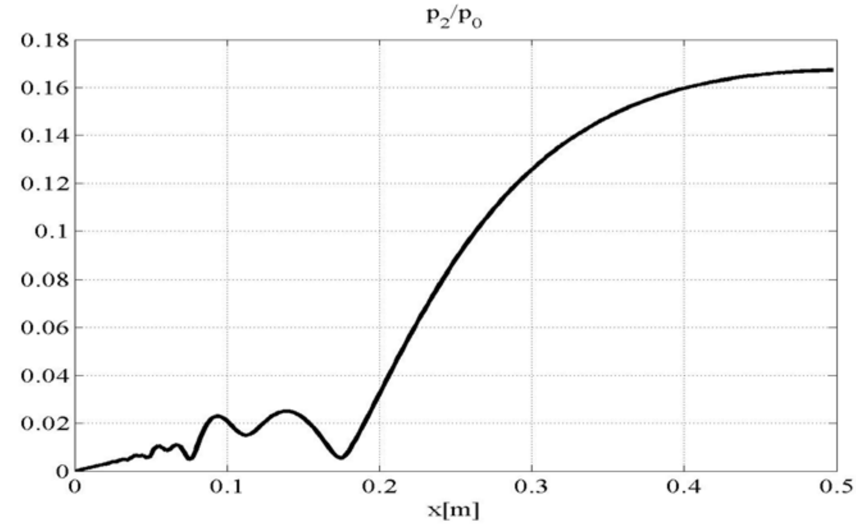
The results of numerical investigations of the space step sizes ΔX and ΔR influence on the calculation accuracy are presented in Figs. 4 and 5. Figure 4 shows the results obtained for two different sizes of the step size ΔX. Curve number 1 was obtained for $\Delta X = 5\cdot 10^{-5}$ and curve number 2 was obtained for $\Delta X = 1.25\cdot 10^{-5}$. The results of calculations obtained for two different sizes of step size ΔR presents Fig. 5. Curve number 1 was obtained for ΔR=0.02. Curve number 2 presents similar result of computer calculations obtained for twice smaller size of this step size.

Next figures present pressure changes of the second harmonic component as a function of distance from the source obtained for different values of radius R_{max} and step size ΔR respectively. The results of numerical calculations presented in Figs. 6 and 7 were done for the same values of numerical and physical parameters like presented in Figs. 3 and 5.

a)



b)



c)

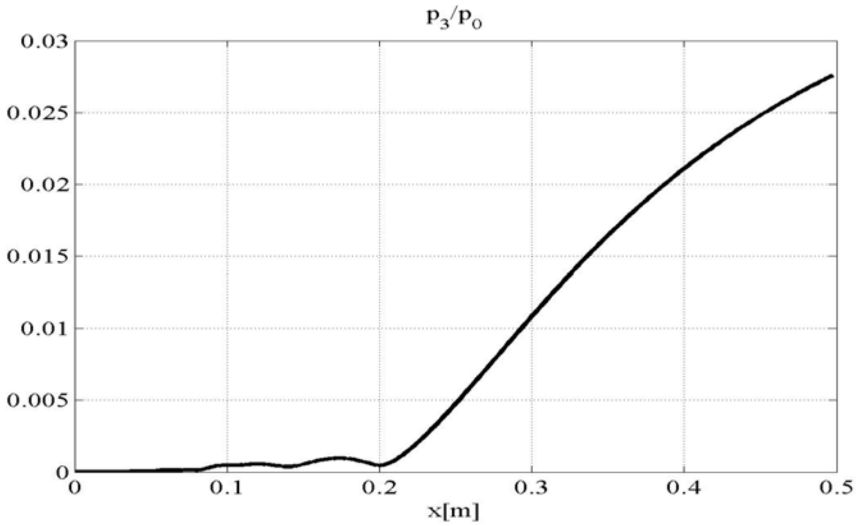


Fig.2 On-axis first (a), second (b) and third (c) harmonic pressure amplitudes as a function of distance from the source

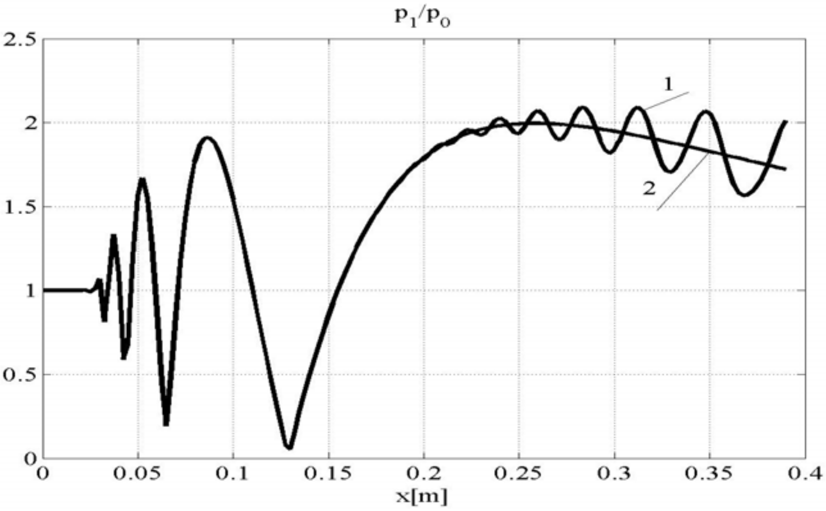


Fig.3 On-axis first harmonic pressure amplitude as a function of distance from the source for different values of radius R_{max}: 1 - R_{max}=3a, 2 - R_{max}=5a

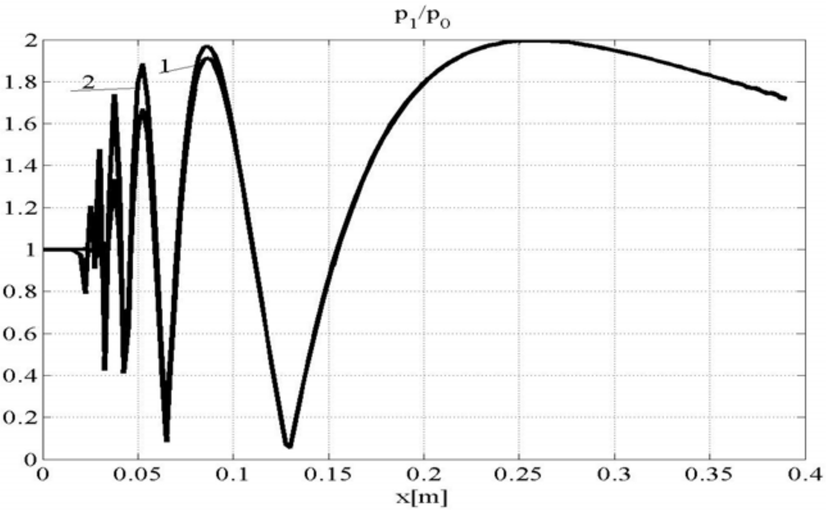


Fig.4 On-axis first harmonic pressure amplitude as a function of distance from the source for different values of step size ΔX: 1 - $\Delta X = 5 \cdot 10^{-5}$, 2 - $\Delta X = 1.25 \cdot 10^{-5}$

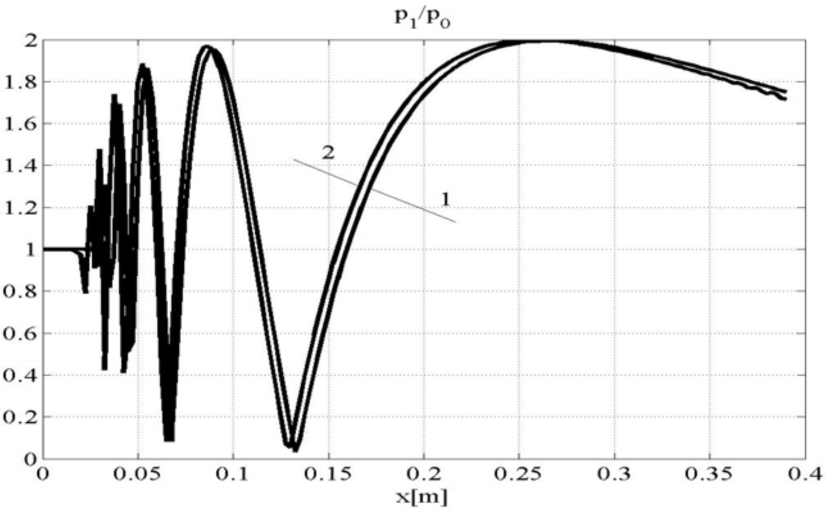


Fig.5 On-axis first harmonic pressure amplitude as a function of distance from the source for different values of step size ΔR: 1 - ΔR=0.02, 2 - ΔR=0.01

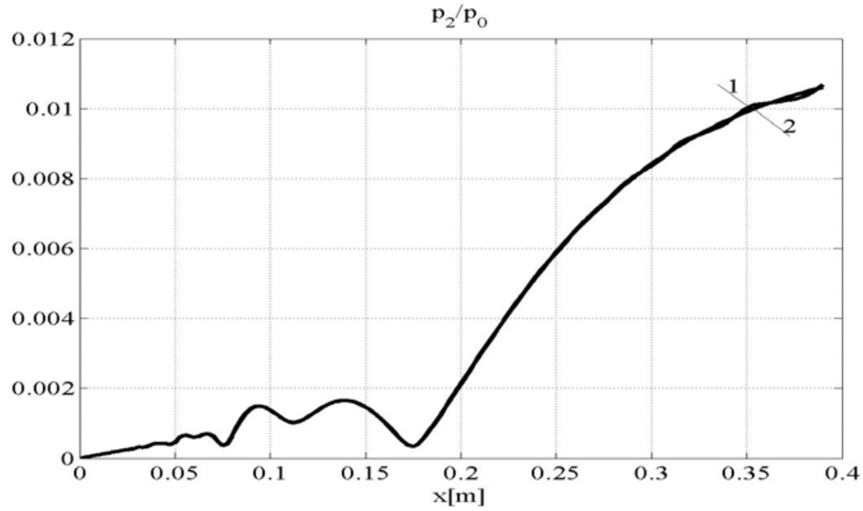


Fig.6 On-axis second harmonic pressure amplitude as a function of distance from the source for different values of radius R_{max}: 1 - R_{max}=3a, 2 - R_{max}=5a

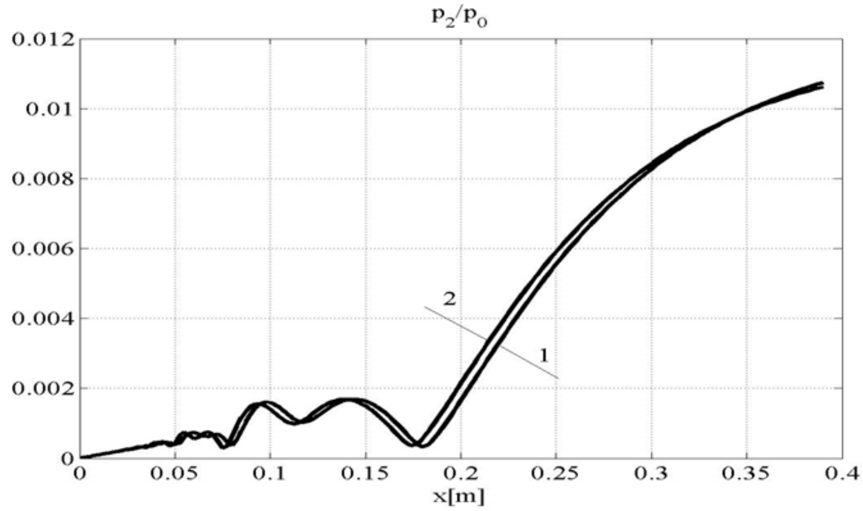


Fig.7 On-axis second harmonic pressure amplitude as a function of distance from the source for different values of step size ΔR: 1 - ΔR=0.02, 2 - ΔR=0.01

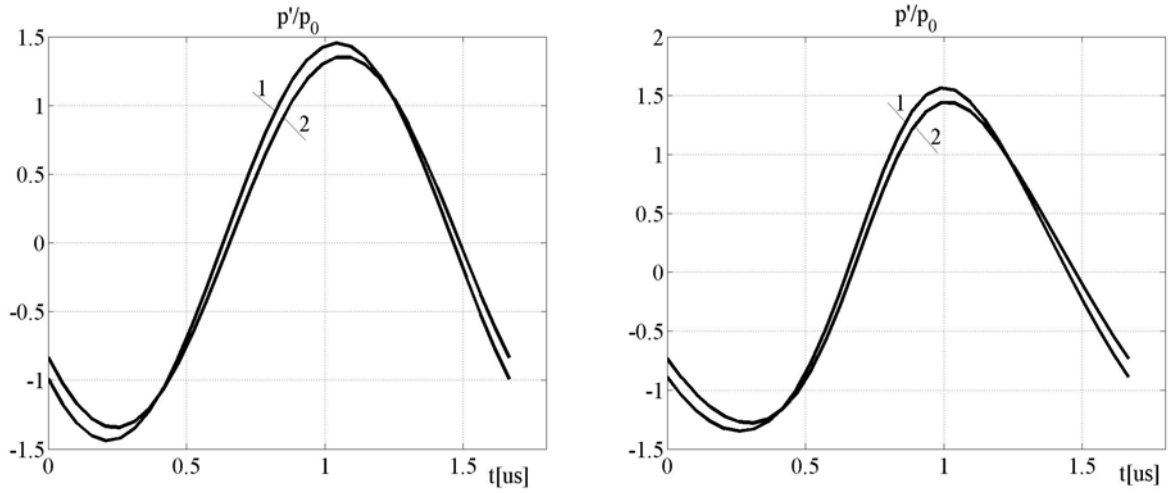


Fig.8 Normalized on-axis pressure as a function of time for different values of radius R_{max}: 1 - R_{max}=4a, 2 - R_{max}=8a

As a result of numerical calculations of Eqs. (5) we obtain the harmonic pressure amplitudes A_n. Substitution them into Eq. (3) yields the pressure changes along sound beam. Figure 8 presents normalized pressure as a function of time at distance x=0.5 m from the source calculated for two different values of radius R_{max}. The results of calculations obtained for p_0=10 kPa shows left figure, similar results obtained for p_0=150 kPa presents right figure.

The results of numerical calculations presented till now were carried out assuming that finite amplitude wave is propagated in non-dissipative medium (b=0). An example of computer calculation obtained for b=0.04 shows Fig. 9. Curve number 1 in right figure presents primary wave shape and curve number 2 presents waveform obtained at distance x=0.35 m (p_0=150 kPa).

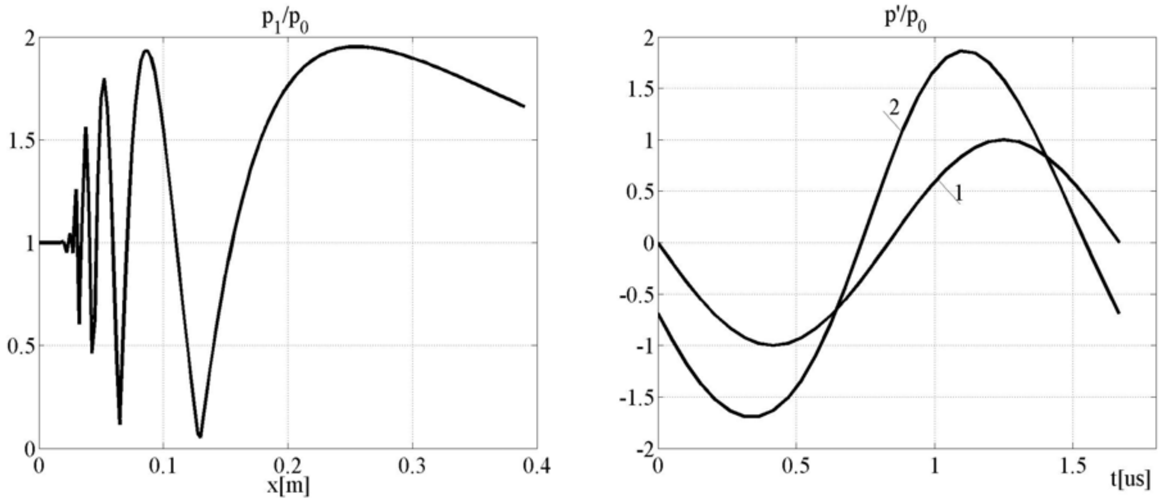


Fig.9 On-axis first harmonic pressure amplitude as a function of distance from the source and normalized pressure as a function of time

4. CONCLUSIONS

The finite amplitude waves propagation problem for circular source was considered. The mathematical model, which was worked out on the basis of the KZK equation, and some results of numerical investigations have been presented. The numerical calculations were carried out using own computer program that was worked out on the basis of obtained mathematical model.

The calculation accuracy depends on values of numerical parameters. The wave propagates in half-infinitive space but the solution of the problem is looked for inside bounded space. Due to sound beam diffraction this space must be suitably big for investigated distances from the source. The accuracy of calculations depends on space step sizes, too.

Proposed method can be used to analyze the wave propagation for different values of source and medium parameters. The computer program can be easy modify for different pressure distributions on the circular sources. Mathematical and numerical models can be modify for sources without axial symmetry.

REFERENCES

[1] A. Baranowska, Influence of discrete model parameters on the KZK equation solution using the finite difference method, Proceedings of the International Symposium on Hydroacoustics and Ultrasonics, 59-64, 1997.

[2] V.E. Kunicyn, O.V. Rudenko, The sound harmonic generation in piston field, Sov. Phys. Acoust., 24, 4, 549-555, 1978.