ANALYSIS OF THE KZK EQUATION SOLUTION FOR FIXED PRESSURE DISTRIBUTIONS AT THE PISTON

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The paper presents the results of the numerical investigations of the finite amplitude waves interaction problem for circular piston with Gaussian pressure amplitude. The mathematical model and some results of numerical investigations are presented. The mathematical model was built on the basis on the Khokhlov – Zabolotskaya – Kuznetsov equation (KZK equation). To solve the problem the finite-difference method was applied. The on-axis pressure amplitude as a function of distance from the source for different frequency waves and their pressure amplitude distributions at horizontal section were investigated. The calculations were done for different values of source and medium parameters. The results of computer calculations were compared with analytical solutions of the KZK equation in special cases.

INTRODUTION

The mathematical model of the finite amplitude waves interaction problem is built using nonlinear differential equations. The KZK equation is often used in theoretical investigations of this problem. This equation describes the pressure changes along sound beam. It allows including nonlinearity, dissipation of medium and sound beam diffraction. This is not known exact analytical solution of this equation till now. There are known only asymptotic solutions of it. The method of successive approximations can be used to find the KZK equation solution [2] when the nonlinear effects are not very big at investigated space ($Re_a<1$, where Re_a – Reynold's number). In general cases the KZK equation is solved numerically. The finite-difference method is one of the numerical methods which are used to solve this problem [1].

The aim of this paper was numerical analysis of the finite amplitude waves interaction. The problem was considered as an axial symmetric one. It was assumed that the circular piston is the source of two different frequency finite amplitude waves. The influence of values of source parameters on the pressure distribution along sound beam was studied.

1. THEORY OF THE PROBLEM

The main aim of this paper was theoretical analysis of pressure changes along the sound beam for the circular piston with fixed radius which is the source of two finite amplitude waves. To realize this aim we assume that the piston is placed in plane *y0x* in such way that axis *x* corresponds with beam axis. Due to assumption of axial symmetry of the source and pressure distribution on the source it is comfortably to solve the problem in cylindrical coordinates, i.e. calculated pressure *p'* is function on time and space coordinate *(x,r)* where $r = \sqrt{y^2 + z^2}$.

The mathematical model is built on the basis on KZK equation:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial p'}{\partial x} - \frac{\varepsilon}{\rho_o c_o^3} p' \frac{\partial p'}{\partial \tau} - \frac{b}{2\rho_o c_o^3}\frac{\partial^2 p'}{\partial \tau^2}\right) = \frac{c_o}{2}\left(\frac{\partial^2 p'}{\partial r^2} + \frac{1}{r}\frac{\partial p'}{\partial r}\right) \tag{1}$$

where $p'=p-p_0$ denotes an acoustic pressure, variable $\tau=t-x/c_o$ is the time in the coordinate system fixed in the zero phase of the propagating wave, ρ_0 - medium density at rest, c_o - speed of sound, b - dissipation coefficient of the medium, ε - nonlinear coefficient.

The pressure distribution on the piston is defined by:

$$p(x = 0, r, \tau) = p_{01}\exp(-L_{01}r^2 / a^2)\sin\omega_1\tau + p_{02}\exp(-L_{02}r^2 / a^2)\sin\omega_1\tau \tag{2}$$

for $r \le a$ and $p'(x = 0, r, \tau) = 0$ for $r>a$, where $\omega_1 = 2\pi f_1$, $\omega_2 = 2\pi f_2$ are angular frequencies of primary waves respectively and a denotes the radius of piston. Parameters L_{01} and L_{02} are certain real constants. The normalized pressure distribution on the piston illustrates Fig. 1. Figure 1a presents on-axis pressure as a function of time for f_1=1.2 MHz and f_2=1 MHz. The normalized pressure amplitude for primary waves as a function of distance from the beam axis for $L_{01}=L_{02}$=1 (line number 1) and $L_{01}=L_{02}$=4 (line number 2) are shown in Fig. 1b.

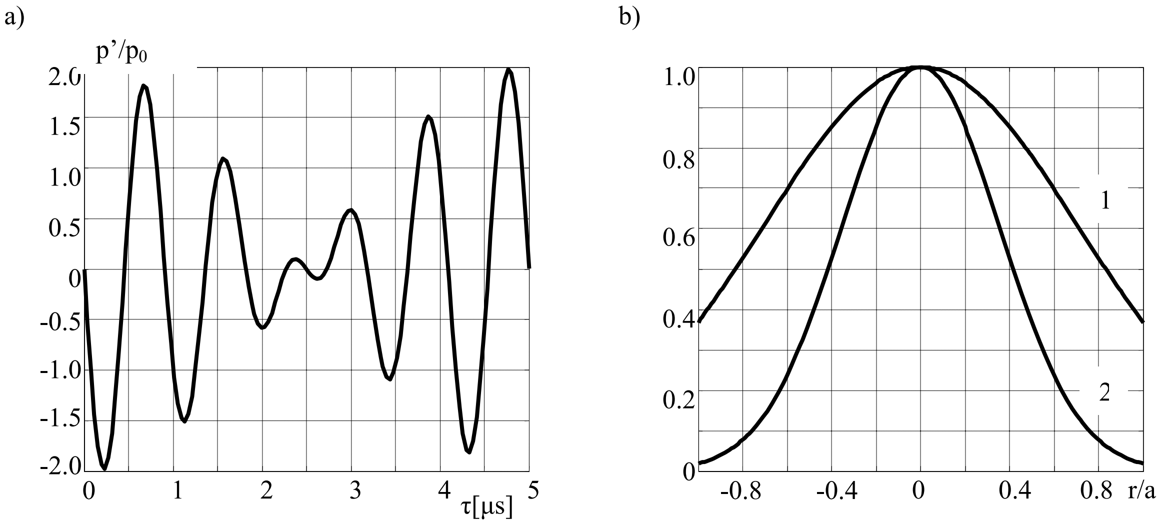

Fig.1 The normalized on-axis pressure as a function of time (a) and normalized pressure amplitude for primary waves as a function of distance from the beam axis (b)

The solution of Eq. (1) is looked for inside a cylinder with radius R_1 for distances to X_1 from the source in time interval $\tau \in [0, T_1]$. Finally, solution of the KZK equation is looked for in a domain D:

$$D = \{(x, r, \tau) \in R^3 : x \in [0, X_1],\ r \in [0, R_1],\ \tau \in [0, T_1]\}.$$

Additionally it is assumed that pressure $p'(x, r, \tau) = 0$ for $r > R_1$ and p' is periodic function of the coordinate τ.

The knowledge of pressure amplitude changes for difference frequency wave is very important in practical applications. Assuming that pressure distribution on the source is defined by formula

$$p(x = 0, r, \tau) = p_{01} \exp(-r^2 / a_1^2) \sin \omega_1 \tau + p_{02} \exp(-r^2 / a_1^2) \sin \omega_1 \tau \qquad (3)$$

and the wave distortion is not very large, the pressure amplitude changes of this wave along the sound beam can be analyzed using formula [2]:

$$p_-(x, r) = \frac{i p_{01} p_{02} \varepsilon (\omega_1 - \omega_2)}{2 c_0^3 \rho_0} \int_0^x \frac{1}{1 - iy / L_2} \exp\left(\frac{y - x}{L_1} - \frac{r^2 / a_2^2}{1 - iy / L_2} \right) dy \qquad (4)$$

where $L_1 = \dfrac{2 c_0^3 \rho_0}{b(\omega_1^2 + \omega_2^2)}$, $L_2 = \dfrac{a_2^2 (\omega_1 - \omega_2)}{2 c_0}$, $a_2 = \dfrac{a_1}{\sqrt{2}}$. This formula can be used for distances from the piston $x << a^2 \omega_i / 2\pi c_0$ where additionally the condition $2\varepsilon p_0 / b\omega_i << 1$ (i=1, 2) is satisfied.

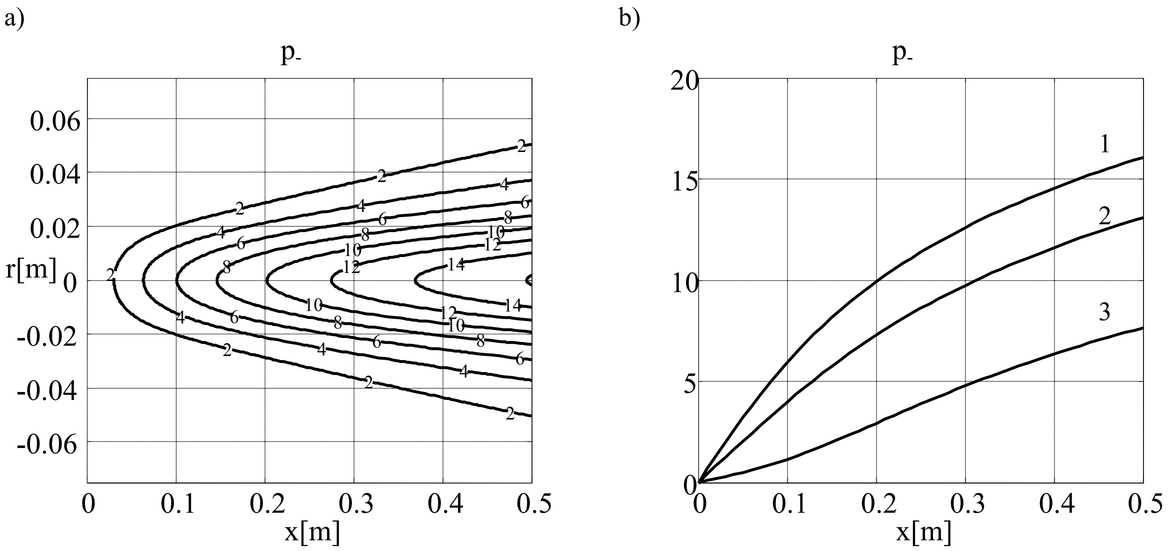

Fig.2 2. Pressure amplitude distribution for difference frequency wave at horizontal section (a) and pressure amplitude for this wave as a function of distance from the source (b):
1 - r=0, 2 - r=a/2, 3 – r=a

The pressure amplitude distribution for difference frequency wave at horizontal section and pressure amplitude for this wave as a function of distance from the source for fixed distances from the beam axis are presented in Fig. 2. The pressure amplitude changes were calculated using formula (4) for source parameters equal f_1=1.2 MHz, f_2=1MHz, p_0=10 kPa, a=25 mm and medium parameters c_0=1000 m/s, ρ_0=1500 kg/m^3, ε=3.5, b=0.004.

The finite-difference method is used to solve the problem numerically. To solve Eq. (1) numerically function $p'(x,r,\tau)$ is discretized in both space and time. The rectangular net is constructed in domain D. The pressure changes along the sound beam are obtained after computer calculations.

The waveform changes are equivalent with spectrum changes during waves propagation in water. The harmonic analysis is very often used to investigate wave distortion. The fast Fourier transform is used to calculate spectrum.

2. NUMERICAL INVESTIGATIONS

Formula (4) allows investigating changes of pressure amplitude only for difference frequency wave. To investigate the pressure amplitude for different frequency waves it is necessary to solve equation KZK numerically.

Figure 3 shows normalized on-axis pressure amplitude for difference frequency wave as a function of distance from the source. In this situation radius of circular piston was equal a=50 mm and pressure distribution was defined by formula (2) where frequencies of primary waves were equal f_1=1.2 MHz and f_2=1 MHz, pressure $p_{01}=p_{02}=p_0$=10 kPa, parameters $L_{01}=L_{02}$=1. The numerical calculations were carried out assuming that the waves are propagated in water where speed of sound c_0=1500 m/s, medium density ρ_0=1000kg/m^3, nonlinear coefficient ε=3.5 and dissipation coefficient of the medium b=0.04.

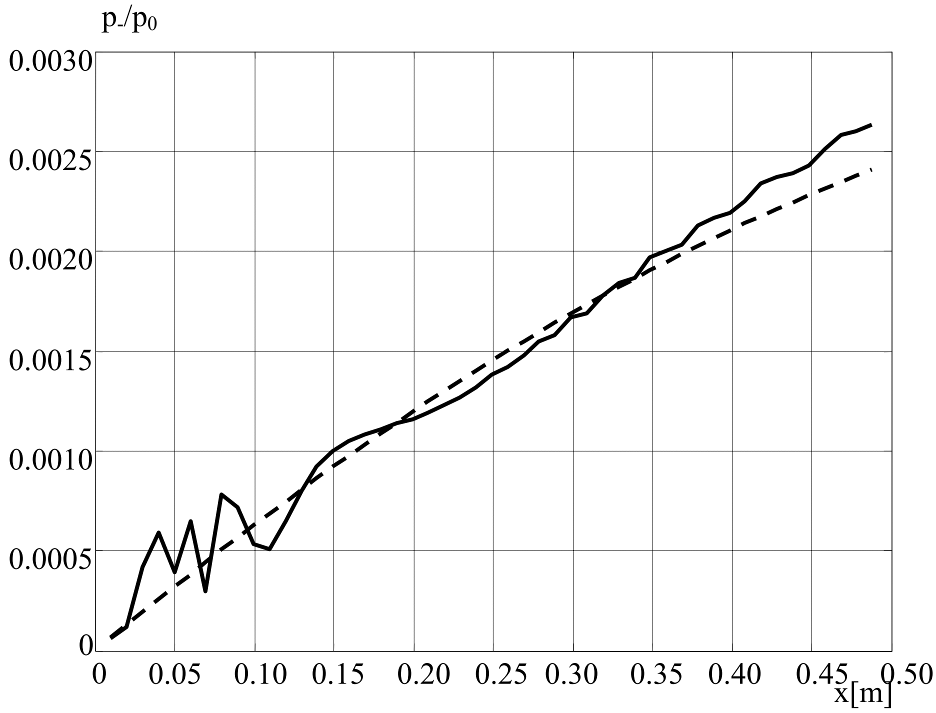

Fig.3 Normalized on-axis pressure amplitude for difference frequency wave as a function of distance from the source calculated numerically (solid line) and analytically (dashed line)

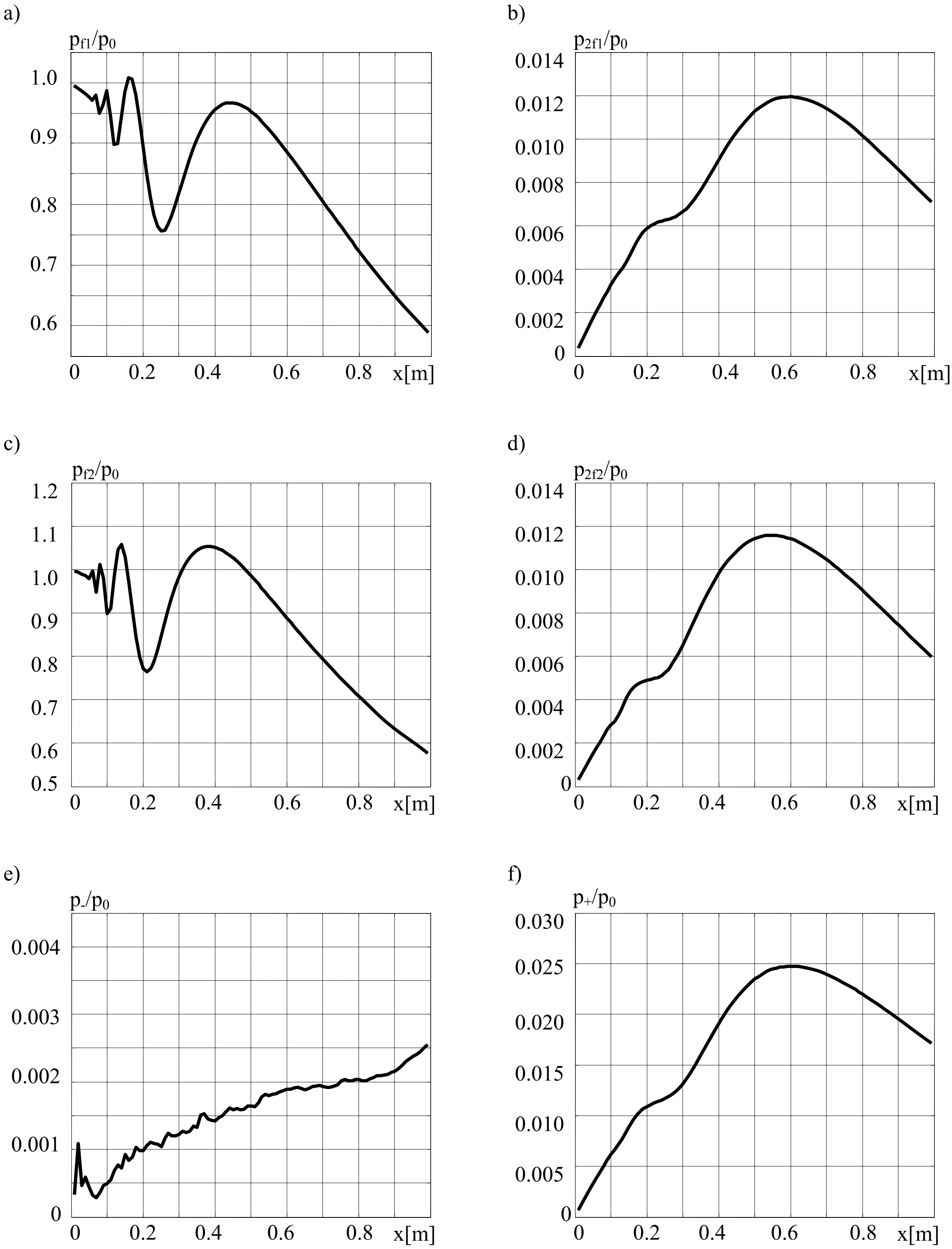

Fig.4 Normalized on-axis pressure amplitude for different frequency waves as a function of distance from the source: a – f_1 , b – $2f_1$, c – f_2, d – $2f_2$, e - f_-, f – f_+

The dashed line was calculated using formula (4) and the solid line presents numerically calculated pressure amplitude changes on the beam axis. Comparison of both curves confirms that proposed mathematical model was worked out correctly.

Figure 4 presents the normalized on-axis pressure amplitude for different frequency waves as a function of distance from the source. Calculations were carried out for the same values of source and medium parameters as earlier except radius of piston. Now, it was assumed that this radius was equal a=25 mm.

The comparison of results presented in Fig. 3 and Fig. 4 shows that value of piston radius has influence on the values of pressure amplitude. Other source parameters have influence on it, too.

The on - axis pressure amplitude for difference frequency wave as a function of distance from the source for different values of pressure p_0 presents Fig. 5. Of course, the pressure p_0=1 MPa is very large but Fig. 5 shows the influence on this parameter for pressure amplitude distinctly. Figure 6 presents similar results obtained for piston generated waves which frequencies are equal f_1=1.2 MHz, f_2=1 MHz and f_1=1.5 MHz, f_2=1.2 MHz respectively, pressure p_0=150 kPa.

It was assumed that the parameters L_{01} and L_{02} were equal 1 in formula (2) till now. Numerical investigations were done for different values of these parameters. Figure 7 presents on-axis pressure amplitude for primary waves as a function of distance from the source calculated for parameters L_{01}=L_{02}=4. Figure 8 shows similar results for second harmonics of primary waves, sum and difference frequency wave. In this situation it was assumed that frequencies of the primary waves were equal f_1=1.2 MHz, f_2=1 MHz respectively, pressure p_0=10 kPa, source radius a=25 mm.

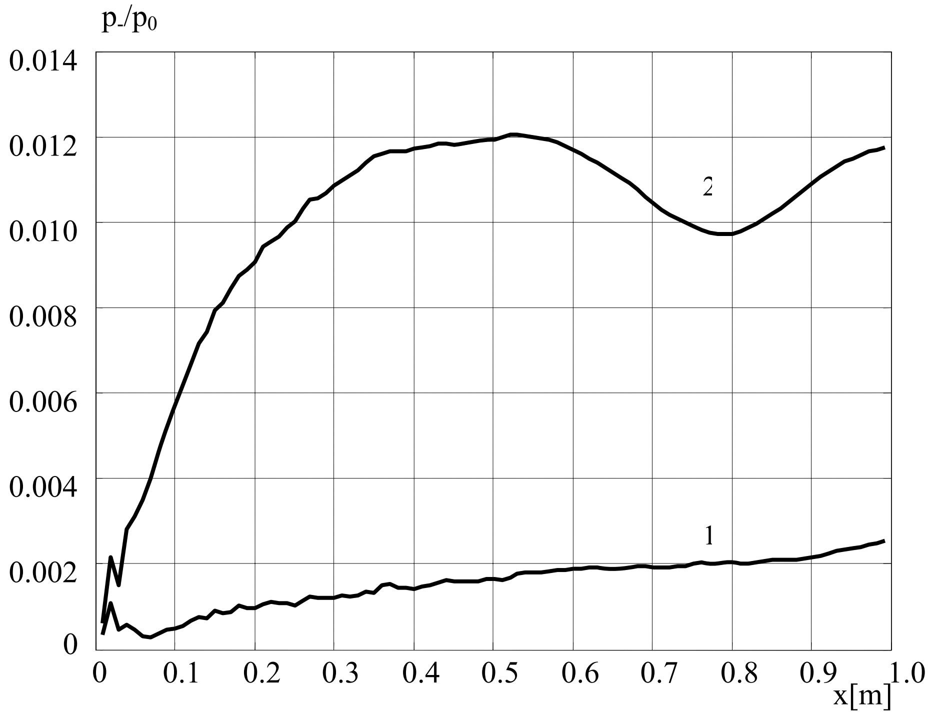

Fig.5 Normalized on-axis pressure amplitude for difference frequency wave as a function of distance from the source for different values of pressure p_0:
1 – p_0=10 kPa, 2 – p_0=1 MPa

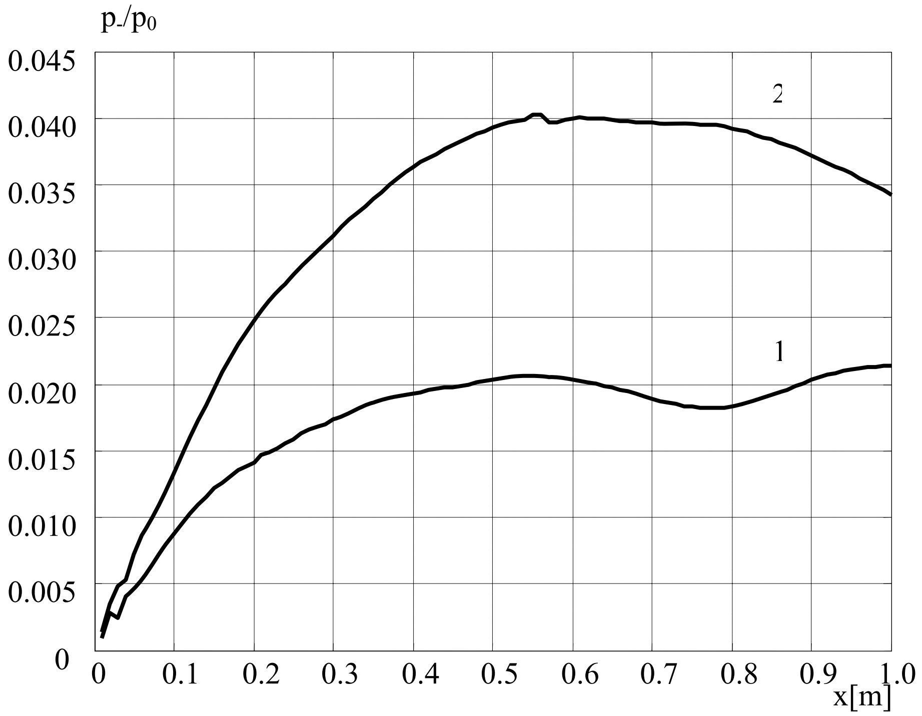

Fig.6 Normalized on-axis pressure amplitude for difference frequency wave as a function of distance from the source for different values of primary wave frequencies: 1 – f_1=1.2 MHz, f_2=1 MHz; 2 - f_1=1.5 MHz, f_2=1.2 MHz

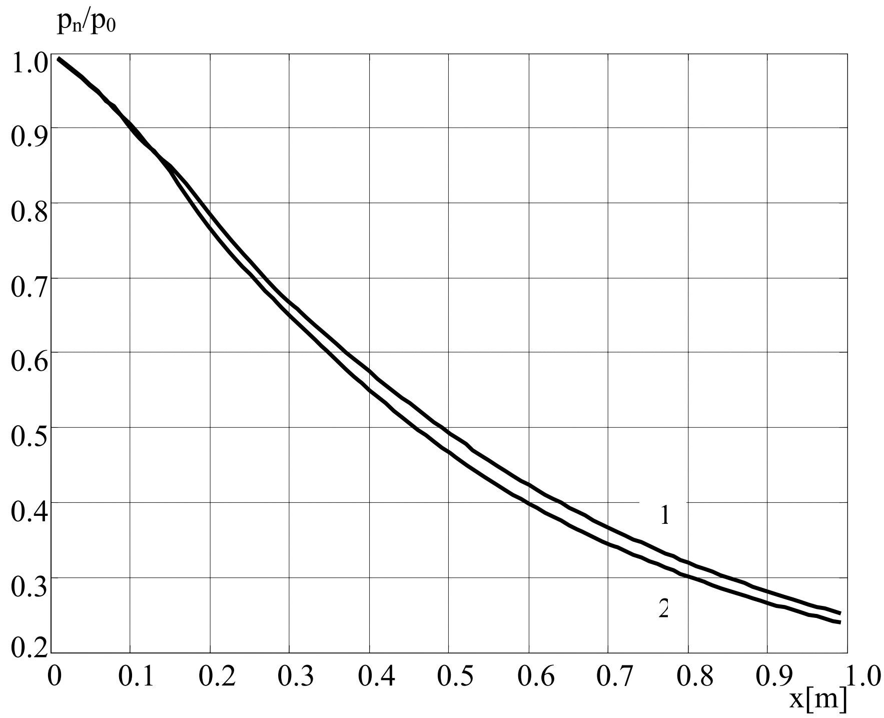

Fig.7 Normalized on-axis pressure amplitude for primary waves as a function of distance from the source: 1 – f_1, 2 - f_2

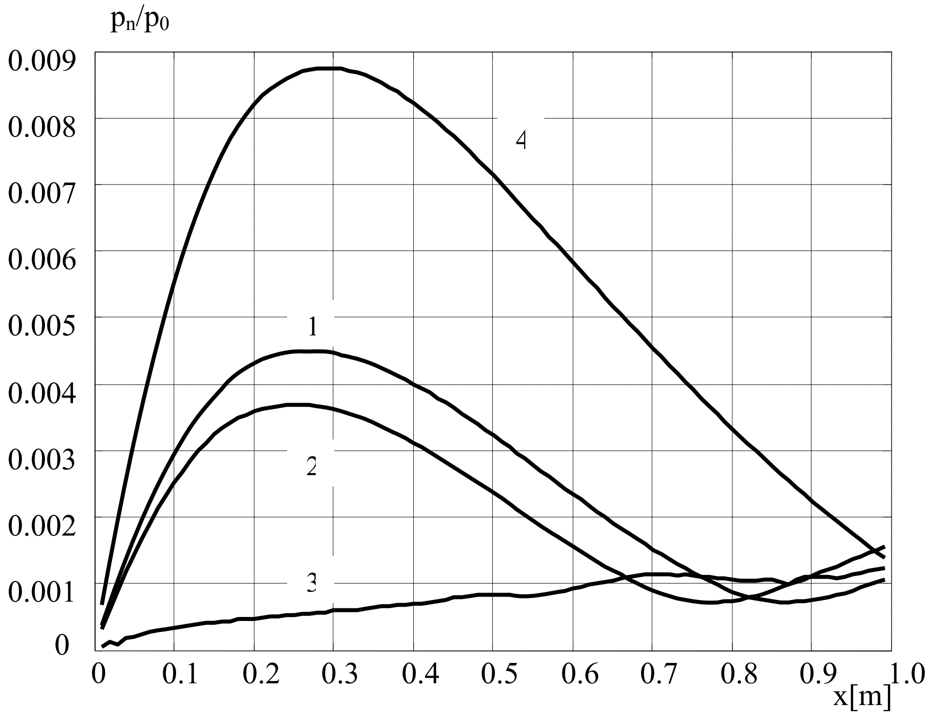

Fig.8 Normalized on-axis pressure amplitude for different frequency waves as a function of distance from the source: 1 – $2f_1$, 2 -$2f_2$, 3 –f_-, 4 –f_+

3. CONCLUSION

The finite amplitude waves interaction problem for circular piston with Gaussian pressure amplitudes was considered. Mathematical model and some results of computer calculations were presented. Mathematical model was built on the basis on the KZK equation. The finite-difference method was used to solve the problem numerically.

Paper presents the analytical formula of the pressure amplitude for difference frequency wave when the waves distortion is not very large. Comparison of the results of numerical calculations with suitable analytical curves confirms correctness of the mathematical model and computer programs.

The analysis of the results of numerical calculations shows that values of primary wave frequency and its amplitude have influence on values of pressure along sound beam. The correct choices of source size and source pressure distribution are very important, too.

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REFERENCES

[1] N.S. Bakhvalov, Ya. Zhileikin, E.A. Zabolotskaya, Nonlinear theory of sound beams, Nauka, Moscov 1982 (in Russian).

[2] V.K. Novikov, O.V. Rudenko, C.I. Coluyan, Equation of the parametric ultrasound source, Sov. Phys. Acoust. 21, 4, 591-596, 1973 (in Russian).