QUADRATIC PHASE COUPLING PHENOMENON AND ITS PROPERTIES

WIESŁAW KICIŃSKI, ARTUR SZCZEPAŃSKI

Akademia Marynarki Wojennej
ul. Śmidowicza 69, 81-103 Gdynia, Poland
wkicinski@amw.gdynia.pl, aszczepanski@amw.gdynia.pl

The analysis of the nonlinear signals or systems is insufficient using the power spectrum and autocorrelation function. We do not have complete information about such signals or systems, because of the loss of the phase information in the power spectrum and autocorrelation function. The analysis of the nonlinear phenomena is more effective using higher order statistics (HOS). In this paper, the basic definitions and properties of HOS are discussed. It is defined the quadratic phase coupling phenomenon (QPC) and numerical results aimed at analysing phase coupling depending on the frequency and amplitude changes are presented.

INTRODUCTION

The abbreviation HOS means Higher Order Statistics and it is assumed to define the term as nth-order moments or cumulants (nonlinear combination of moments) of random signals. In the frequency domain it corresponds to them Higher Order Spectra (also known as polyspectra), which are, by definition, multidimensional Fourier transforms of higher order statistics (moments or cumulants). Particular cases of higher order spectra are the third-order spectrum called the bispectrum and the fourth-order spectrum, called trispectrum, which are the Fourier transforms of the third-order and fourth-order statistics adequately. Thus the power spectrum is a part of the class of higher order spectra, i.e. it is a second-order spectrum. The power spectrum or power spectral density and auto-correlation function provide very useful information in the design and analysis of the linear predictive systems. However they do not give complete information, because they ignore the existence of deterministic mechanisms, which generate signals with so-called flat or near-to-flat densities. For example, music and speech signals are created mechanically by systems with non-linear characteristic of dynamics. Therefore, prediction and coding quality of such signals can be perfected by utilize of additional information contained in the signal and thus we should use the higher order statistics [11].

As it was mentioned above the higher order statistics and spectra can be defined in terms of moments and cumulants. The estimation of the higher order spectra on the basis of moments is more useful in the analysis of deterministic signals, both periodic and transient, whereas the estimation of the higher order spectra on the basis of cumulants is more useful in the analysis of stochastic signals. In general case there are several arguments to use higher order statistics in signal processing. First of all they are „resistant" to Gaussian noise. From the Gaussian property of signal it follows that all cumulants of order greater than two have value zero. Therefore, if a non-Gaussian signal is received with additive Gaussian noise, its transformation to the higher order cumulant domain will eliminate the noise. In this connection we can employ the higher order statistics to detection and estimate parameters of the signals. In particular, cumulant spectra can become high signal-to-noise ratio (SNR) domains in which one may perform detection, parameter estimation or even entire signal reconstruction [4]. Moreover the higher order statistics make possible the identification of non-minimum phase systems and reconstruction of non-minimum phase signals. It results from the property of HOS (in terms of moments and cumulants); they preserve phase information of signals. Additionally it becomes possible the detection and analysis of non-linear property of signal and identification of non-linear systems, because the most of real observed signals have non-Gaussian character and thus their higher order spectra are non-zeros. The higher order statistics have application in a lot of science fields such as communications, processing of seismic data, plasma physics, optics, hydroacoustics, etc.

1. DEFINITION AND PROPERITIES OF HOS

As it was mentioned above, the particular cases of higher order statistics are second, third and fourth order and their adequate Fourier transforms: power spectrum, bispectrum and trispectrum. Hence the discussion will be limited only for them. Consider the real, stationary, discrete-time signal $x(t)$, for $t = 0, \pm 1, \pm 2, \pm 3, \ldots$ and if its moments up to n order exist, then in general case [4]

$$m_n^x(\tau_1, \tau_2, ..., \tau_{n-1}) = E\left[\, x(t)\, x(t+\tau_1) \ldots x(t+\tau_{n-1}) \,\right] \tag{1}$$

is the nth-order moment function of the signal and $\tau_1, \tau_2, \ldots, \tau_{n-1}$, ($\tau_i = 0, \pm 1, \pm 2, \ldots$ for all i) are the temporal (or spatial) lags. $E[\,]$ is an operator of statistical expectation. Evidently $m_2(\tau_1)$ is the 2nd-order moment function (autocorrelation function) of $x(t)$ and $m_3(\tau_1, \tau_2)$ and $m_4(\tau_1, \tau_2, \tau_3)$ are the 3rd- and 4th-order moment functions, adequately.

If we assume that $x(t)$ has zero-mean value than cumulant functions (which are defined as combinations of moments) can be written as (for n = 2, 3 and 4) [3]

$$c_2^x(\tau_1) = m_2^x(\tau_1) \tag{2}$$

$$c_3^x(\tau_1, \tau_2) = m_3^x(\tau_1, \tau_2) \tag{3}$$

$$\begin{aligned} c_4^x(\tau_1, \tau_2, \tau_3) = {} & m_4^x(\tau_1, \tau_2, \tau_3) - m_2^x(\tau_1) \cdot m_2^x(\tau_3 - \tau_2) \\ & - m_2^x(\tau_2) \cdot m_2^x(\tau_3 - \tau_1) \\ & - m_2^x(\tau_3) \cdot m_2^x(\tau_2 - \tau_1) \end{aligned} \tag{4}$$

Notice, if $x(t)$ is zero-mean, the 2nd- and 3rd-order cumulants are equal to the 2nd- and 3rd-order moments, respectively but to create 4th-order cumulant it is necessary to know the 4th-

and 2nd-order moments. By putting zero lags, i.e. $\tau_i = \tau_2 = \tau_3 = 0$ we obtain well-known statistical relationships

$$\begin{aligned} c_2^x(0) &= \gamma_2^x = E\left[x^2(t)\right] && \text{(variance)} \\ c_3^x(0,0) &= \gamma_3^x = E\left[x^3(t)\right] && \text{(skewnees)} \\ c_4^x(0,0,0) &= \gamma_4^x = E\left[x^4(t)\right] - 3\left[\gamma_2^x\right]^2 && \text{(kurtosis)} \end{aligned} \tag{5}$$

In the frequency domain the higher order spectra are defined as the multidimensional Fourier transforms of cumulant functions (and of moments, what was mentioned above), beginning from the power spectrum as follows

$$C_2^x(\omega) = \sum_{\tau=-\infty}^{+\infty} c_2^x(\tau) \cdot \exp\left[-j(\omega\tau)\right] \tag{6}$$

where $|\omega| \le \pi$. In the similar way, we can write the relationships of bispectrum

$$C_3^x(\omega_1,\omega_2) = \sum_{\tau_1=-\infty}^{+\infty} \sum_{\tau_2=-\infty}^{+\infty} c_3^x(\tau_1,\tau_2) \cdot \exp\left[-j(\omega_1\tau_1 + \omega_2\tau_2)\right] \tag{7}$$

where $|\omega_1| \le \pi$, $|\omega_2| \le \pi$, $|\omega_1 + \omega_2| \le \pi$, and trispectrum:

$$C_4^x(\omega_1,\omega_2,\omega_3) = \sum_{\tau_1=-\infty}^{+\infty} \sum_{\tau_2=-\infty}^{+\infty} \sum_{\tau_3=-\infty}^{+\infty} c_4^x(\tau_1,\tau_2,\tau_3) \cdot \exp\left[-j(\omega_1\tau_1 + \omega_2\tau_2 + \omega_3\tau_3)\right] \tag{8}$$

where $|\omega_1| \le \pi$, $|\omega_2| \le \pi$, $|\omega_3| \le \pi$, $|\omega_1 + \omega_2 + \omega_3| \le \pi$.

The cumulant spectra are more useful in the processing of random signals than are moment spectra for several important reasons [4]:

- if an analysed signal is Gaussian one than cumulant spectra of order $n \rangle 2$ are zero and thus from non-zero spectra we can determine an extent of non-Gaussianity,
- cumulants provide a suitable measure of the extent of the statistical dependence in time series,
- the cumulant of two statistically independent random processes equal the sum of the individual random processes, whereas the same is not true for higher order moments.

In particular, this last property enables us to use the cumulants as an operator, what is more practical and much easier.

2. QUADRATIC PHASE COUPLING

In many papers we can find a qualification that power spectrum and autocorrelation function are phase blind. It comes from the fact that phase relationships of signals are lost in the power spectrum and in the autocorrelation function. The higher order spectra make possible detection and quantitative description of nonlinearities in signals (not only stochastic). Such signals arise, when they are passed through the systems with nonlinear characteristic. In practice, we meet situation very often, in which two harmonic signals are passed by the nonlinear system. Consider the signal

$$x(t) = A_1 \cos(\omega_1 t + \varphi_1) + A_2 \cos(\omega_2 t + \varphi_2) \tag{9}$$

which is passed trough the simple quadratic nonlinear system

$$h(t) = a\, x^2(t) \tag{10}$$

where a is non-zero constant. On the output of the system, the signal $x(t)$ will include the harmonic components: $(2\omega_1, 2\varphi_1)$, $(2\omega_2, 2\varphi_2)$, $(\omega_1+\omega_2, \varphi_1+\varphi_2)$ and $(\omega_1-\omega_2, \varphi_1-\varphi_2)$. Such phenomenon, which produces a formation of these phase relations, is called quadratic phase coupling (sometimes it is met a definition – phase coupling of the second order). Note that these phase relations are identical as the frequency relations. The power spectrum always is to show the peaks on the same positions (related to the frequencies) irrespective of the phases of sinusoids – the information about phase relations is lost. Thus, in order to obtain the information we should use the higher order spectra. Moreover, if we have exact information about the input and output signals of the system, than we can use the higher order spectra to the identification of nonlinear systems. Sometimes we deal with situations that a nonlinear system is excited by the independent sinusoidal sources and it is to generate a harmonic signal with quadratically coupled frequency pairs. Hence identification of these pairs makes possible us to identify some characteristics about the system and also the number of the independent sources.

In general, if we have a signal that is composed of three sinusoids with frequencies and phases (ω_1, φ_1), (ω_2, φ_2) and (ω_3, φ_3) respectively, than sinusoids 1 and 2 are said to be quadratically phase coupled (QPC) if and only if

$$\begin{aligned} &\omega_1+\omega_2=\omega_3 \\ \text{and} \quad &\varphi_1+\varphi_2=\varphi_3 \end{aligned} \tag{11}$$

Note the pairs (ω_1, ω_2) and (ω_2, ω_1) are unordered QPC pair. In general the harmonic signal $x(t)$ may be composed of k (let assume complex) sinusoids

$$x(t)=\sum_{i=1}^{k} A_i \exp\left(j(\omega_i t+\varphi_i)\right) \tag{12}$$

Because one sinusoid can be coupled to itself, so there is the interesting question: how many different phase coupled pairs can appear in such signal? Assume that p is the number of sinusoids coupled to itself and s will be the total number of unordered pairs of sinusoids in the signal. Then the total number of the ordered pairs is $2s$, whereas the total number of different ordered pairs are $2s–p$. Hence the different ordered QPC pairs we can simply obtain as

$$\begin{aligned} &\omega_{1,i}+\omega_{2,i}=\omega_{3,i} \\ \text{and} \quad &\varphi_{1,i}+\varphi_{2,i}=\varphi_{3,i} \end{aligned} \tag{13}$$

where $1 \le i \le 2s-p$. In the case of the harmonic signals, it is a summation of sinusoids with zero mean that is why the third order cumulants of these signals are equal to the third order moments (3). Since the integration sinusoid from 0 to 2π is zero, the third order cumulant of signal $x(t)$ we can write as

$$c_3^x(\tau_1,\tau_2)=\sum_{i=1}^{q} \alpha_i \exp\left(j(\omega_{1,i}\tau_1+\omega_{2,i}\tau_2\right) \tag{14}$$

where q is the number of different ordered coupled sinusoids in $x(t)$. It is important to observe that in eq. (14), only the phase coupled components appear. Hence the bispectrum function is the useful tool for detection of the quadratic phase coupling. In many publications we can find the terms: a normalized higher order spectrum or nth order coherency index. Both the terms determine functions that combine the nth order cumulant spectrum with the power spectrum. The coherency indexes of 3rd- and 4th-order are adequately defined as

$$P_3^x(\omega_1,\omega_2) = \frac{C_3^x(\omega_1,\omega_2)}{\sqrt{C_2^x(\omega_1)C_2^x(\omega_2)C_2^x(\omega_1+\omega_2)}} \quad \text{(bicoherency)} \tag{15}$$

$$P_4^x(\omega_1,\omega_2,\omega_3) = \frac{C_3^x(\omega_1,\omega_2,\omega_3)}{\sqrt{C_2^x(\omega_1)C_2^x(\omega_2)C_2^x(\omega_3)C_2^x(\omega_1+\omega_2+\omega_3)}} \quad \text{(tricoherency)} \tag{16}$$

and they are very useful in the detection and characterization of nonlinearities in time series and in discriminating linear processes from nonlinear ones [4]. Moreover we can use the 3rd order coherency index for detecting quadratic phase coupling phenomena and therefore the suitable estimation of the bispectrum function is very important. There are several techniques of bispectrum estimation, which can be used for analysis quadratic phase coupling. But the conventional techniques on the basis of the Fourier transform and parametric methods (AR and ARMA) are used most often. Both the methods have some advantages and a number of limitations. In general, the conventional techniques are better suitable as quantifiers of the quadratic phase coupling and the parametric methods offer the promise of high resolution and are used more often as detectors. More information about the bispectrum estimation and methods of QPC estimation are available in [4], [7], [8] and [9].

3. NUMERICAL RESULTS

The Matlab is an interactive open computational environment, which integrate a numerical analysis, matrix operations and signal processing. Therefore it can be a very good tool in the analysis of QPC phenomena, especially if we use the HOSA toolbox.

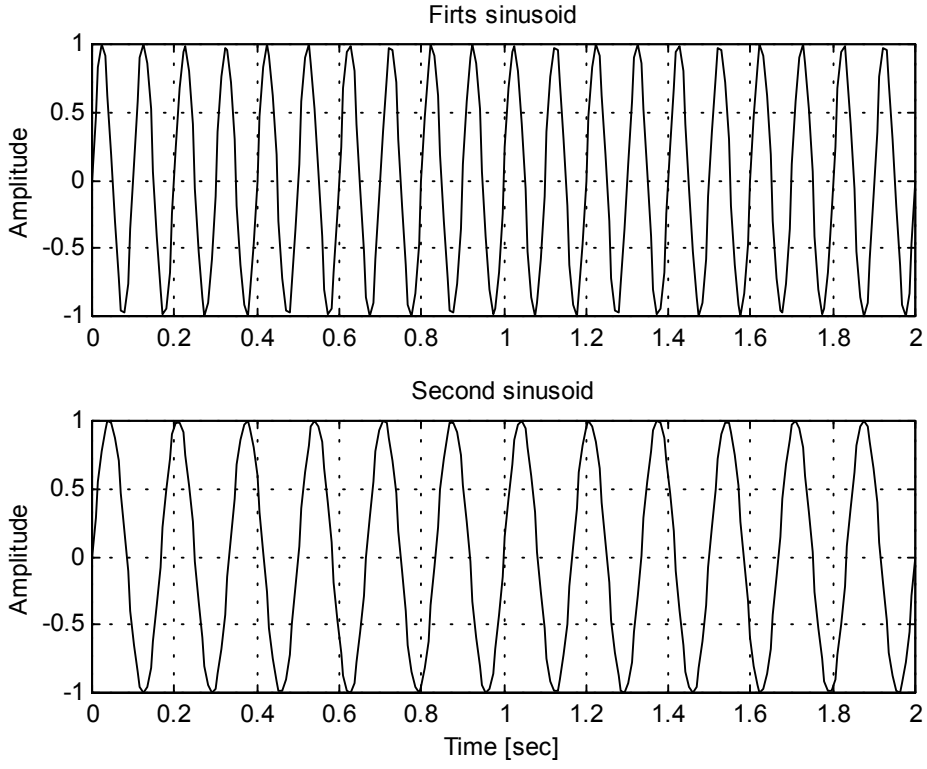

Fig. 1. An example of two sinusoids: the first – 10 Hz; the second – 6 Hz

Thus program in Matlab language was written and it was used for the QPC analysis. The program carries out the following operations:

- generates two sinusoids;
- adds these two sinusoids (9);
- squares the sum of the sinusoids (10);
- computes the Discrete Fourier Transform (FFT function) of such signal;
- using the standard matlab function – qpctor, detects the quadratic phase coupling and computes the value of the bispectrum function for the phase coupled components.

The first sinusoid on frequency 10 Hz is constant, whereas the frequency of the second sinusoid can be changed from 1 to 20 Hz with step 1 Hz. Both sinusoids are sampled with frequency equal to 128 Hz and the time of the observation is 2 seconds. The sampled frequency was chosen for the sake of the better estimation of FFT (128 is a power of two) and the time of the observation was chosen for the sake of the frequency 1 Hz (we have two full periods in 2 seconds). Figure 1 illustrates two sinusoids with frequency 10 Hz and 6 Hz respectively. Figure 2 depicts sum of these sinusoids and they square. To eliminate the constant component, the square of the sum of the sinusoids was reduced about the mean.

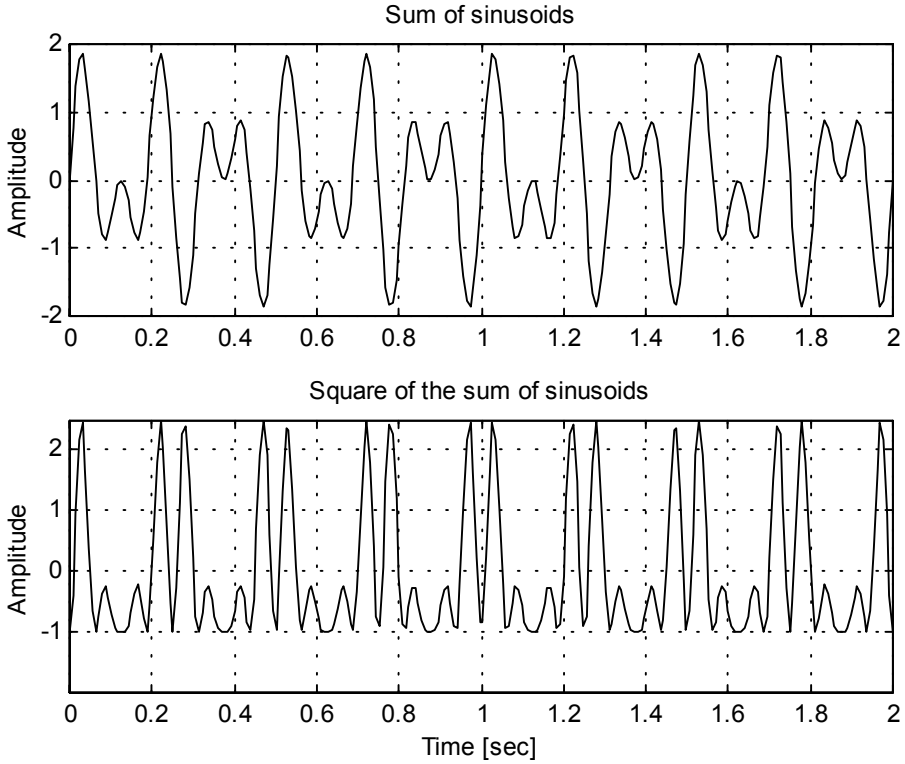

Fig. 2. The sum of two sinusoids and they square

As was mentioned above, the standard matlab function – qpctor was used for the detection of QPC. This function detects quadratically phase coupled harmonics using the TOR (Third Order Recursion) method and computes the bispectrum value for the quadratically coupled components. The syntax of qpctor is:

[*arvec, bspec*] = qpctor (*y, maxlag, ar_order, nfft, samp_seg, overlap, flag*)

where:

y – the data matrix; each column of y is assumed to correspond to a different realization.

maxlag – specifies the maximum number of third-order cumulant lags, $c_3^x(\tau,\tau)$, to be used.
ar_order – specifies the AR order to use.
nfft – specifies the FFT length; its default value is 64.
samp_seg – specifies the number of samples per segment; the default value is the length of the time series, or the row dimension if y is a matrix.
overlap – specifies the percentage overlap between segments; maximum allowed value is 99; default value is 0; the parameter is ignored if y is a matrix.
If *flag* is biased, then biased sample estimates of cumulants are computed (default); if the first letter is not 'b', unbiased estimates are computed.
arvec – the vector of estimated AR parameters.
bspec – the estimated parametric bispectrum. It is an nfft/2-by-nfft array whose upper-left hand corner corresponds to the origin in the bispectral plane.

The original qpctor function was modified in a not large degree and it was used with the following parameters:

qpctor (*y, 72, 48, 128*)

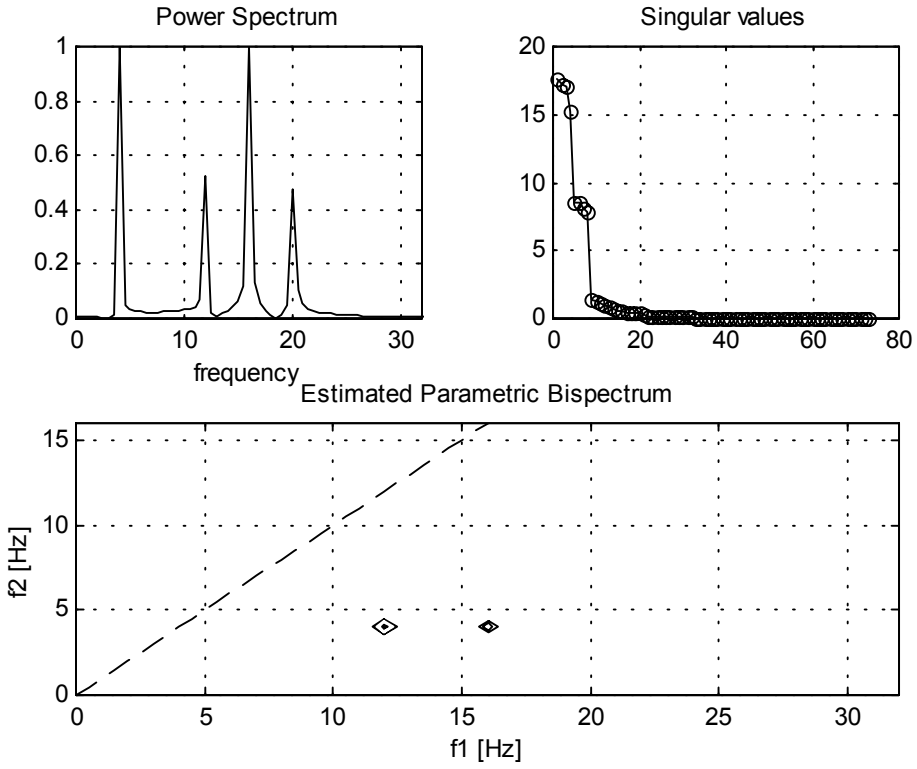

Fig. 3. Three plots coresponding to the input frequencies: 10 and 6 Hz

Figure 3 illustrates three plots corresponding to our input frequencies. Notice that four peaks (corresponding to the frequencies 4, 12, 16 and 20 Hz) occur in the power spectrum plot and frequencies 4, 12, 16 and 4, 16, 20 form two quadratically coupled triplets (Estimated Parametric Bispectrum plot). The singular values plot shows two groups of significant singular values (four in each) corresponding to these two triplets. The values of the parameters in the qpctor function were selected for the sake of the range of the second frequency changes (1 to 19 Hz) and to get high resolution. Moreover the bigger values of the *maxlag* and *ar_order* give better results hence it was used the values 72 and 48 adequately.

The test was performed in two stages:

1. For the constant frequency 10 Hz of the first sinusoid, the second was changed from 1 to 19 Hz without 10 Hz with step 1 Hz; the amplitude of the both sinusoids was 1,
2. For the frequencies 10 and 6 Hz of the sinusoids, the amplitude of the second was changed from 0,1 to 1; the amplitude of the first sinusoid was 1.

In the both cases, the initial phase was zero. The experiment was aimed at analysing the quadratic phase coupling and the bispectrum value depending on the frequency changes and amplitude changes. The results are shown in Table.1 and Table.2.

Tab. 1. Bispectrum value for QPC triplets depending on frequency f_2 (f_1 constant – 10 Hz)

f_2 [Hz]	Power Spectrum Components [Hz]				QPC triplets [Hz]			Bispectrum Value $\cdot 10^6$
	$2 \cdot f_1$	$2 \cdot f_2$	$\lvert f_1 - f_2 \rvert$	$f_1 + f_2$	f_{1j}	f_{2i}	f_{3I}	
1	20	2	9	11	9	11	20	10,8391
					2	9	11	6,1696
2	20	4	8	12	4	8	12	1,9600
					8	12	20	1,1880
3	20	6	7	13	6	7	13	2,9302
					7	13	20	2,0745
4	20	8	6	14	6	8	14	4,1609
					6	14	20	3,8215
5	20	10	5	15	5	15	20	0,4124
					5	10	15	0,3364
6	20	12	4	16	4	12	16	1,8349
					4	16	20	0,6424
7	20	14	3	17	3	14	17	0,9160
					3	17	20	0,5071
8	20	16	2	18	2	16	18	0,3260
					2	18	20	0,1354
9	20	18	1	19	1	19	20	0,1129
					1	18	19	0,1101
11	20	22	1	21	1	20	21	0,0611
					1	21	22	0,0610
12	20	24	2	22	2	20	22	0,1287
					2	22	24	0,0416
13	20	26	3	23	3	20	23	0,1969
					3	23	26	0,0934
14	20	28	4	24	4	20	24	0,1642
					4	24	28	0,0592
15	20	30	5	25	5	20	25	0,1145
					5	25	30	0,0571
16	20	32	6	26	6	20	26	0,1123
					6	26	32	0,0620
17	20	34	7	27	7	20	27	0,0786
					7	27	34	0,0507
18	20	36	8	28	8	20	28	0,1240
					8	28	36	0,0534
19	20	38	9	29	9	20	29	0,1460
					9	29	38	0,0582

Tab. 2. Bispectrum value for QPC triplets depending on the mplitude of f_2 (f_1 = 10 Hz, f_2 = 6 Hz, amplitude of f_1 is 1; power spectrum components 4, 12, 16, 20 Hz)

Amplitude A_{f2}	QPC triplets [Hz]			Bispectrum Value $\cdot 10^{6}$
	$f_{1\,i}$	$f_{2\,i}$	$f_{3\,i}$	
0,1	4	12	20	-
	4	16	20	0,0223
0,2	4	12	20	0,0190
	4	16	20	0,1044
0,3	4	12	20	0,0509
	4	16	20	0,1941
0,4	4	12	20	0,1211
	4	16	20	0,2833
0,5	4	12	20	0,2862
	4	16	20	0,4047
0,6	4	12	20	0,5657
	4	16	20	0,5201
0,7	4	12	20	0,9088
	4	16	20	0,5973
0,8	4	12	20	1,2740
	4	16	20	0,6412
0,9	4	12	20	1,6118
	4	16	20	0,6581
1,0	4	12	20	1,8349
	4	16	20	0,6424

Analysis of the problem stated above and the results shown in the Tab.1 may be discussed as follow:

- the quadratic phase coupling was occurred in all cases and two quadratic coupled triplets were observed for each realization (for each pair of input frequencies),
- the bispectrum value is so much the bigger if the two first harmonics of the QPC triplets ($f_{1\,i}$ and $f_{2\,i}$) are closer to each other,
- the bispectrum value is bigger if the two first harmonics of the QPC triplets ($f_{1\,i}$ and $f_{2\,i}$) are sum and difference of the input frequencies.

Hence the biggest value of the bispectrum for f_2 = 1Hz and QPC triplet 9, 11, 20 Hz. The frequencies 9 and 11 Hz are difference and sum of the input frequencies 10 an 1 Hz respectively. You may notice that we have one exception to these rules – for f_2 = 5 Hz. It is an error, which results from the application of numerical computation and from value of sampling frequency (number of samples).

It was tested the dependence of the bispectrum value on the amplitude changes of the second sinusoid in the second stage (Tab.2). It was chosen two frequencies 10 and 6 Hz because the difference and sum of these frequencies are 4 and 16 Hz adequately and they are power of two. So it was expected to receive the exacter results. We may observe that the bispectrum value rises with the increase of the amplitude f_2. For A_{f2} = 0,1 only one peak of QPC triplet (4, 16 and 20 Hz) occurs – 4 and 16 Hz are the difference and sum of the input frequencies. But for A_{f2} = 0,2 the second peak (4, 12 and 20 Hz) appears and the bispectrum value rises faster than the value of the first, for A_{f2} = 0,6 it is bigger. It follows from the fact that the frequencies 4 and 12 Hz are closer to each other.

4. CONCLUSIONS

The higher order statistics, i.e. third-order cumulants and bispectrum are extensively developed in the literature. In particular they are insensitive to any symmetrically distributed noise and also exhibit the capability of better characterising non-Gaussian signals. By exploiting these HOS properties, it is possible to devise a robust method for identifying and classifying signals affected by noise with different distributions and even with very low signal-to-noise ratios.

All real signals that are normally called „periodic" have some amplitude and phase variation from period to period. The evaluation and statistical description of the amplitude and phase variation of the hydroacoustic signals using the quadratic phase coupling model is very promising approach in signal processing for object recognition in underwater environment.

REFERENCES

[1] Brocket P. L., Hinich M., Wilson G. R., Nonlinear and non-Gaussian Ocean Noise, JASA, vol 82, no 4, pp. 1386-1394, 1987.

[2] Kiciński W., Analiza danych w μC systemach pomiarowych z zastosowaniem funkcji bispectrum, XII Conference Application Microprocessors in Automatic Control and Measurement 9-10.10.2000, Wyd. PIE, pp. 147-154, Warszawa 2000.

[3] Murino V., Ottonello C., Pagman S., Noisy Texture Classification: A Higher – Order Statistics Approach, Pattern Recognition, vol. 31, no 4, pp. 383-393, 1998.

[4] Nikias C.L., Mendel J.M., Signal Processing with Higher-Order Spectra, IEEE Signal Processing Magazine, no 7, pp. 10-37, 1993.

[5] Oppenheim A.V., Schafer R.W., Discrete-Time Signal Processing, Prentice Hall, Englewood Cliffs, NJ 1989.

[6] Rabiner L.R, Schafer R.W., Digital Processing of Speech Signals, Prentice Hall, Englewood Cliffs, NJ 1978.

[7] Raghuveer M.R., Nikias C.L., Bispectrum Estimation via AR Modelling, Signal Processing, vol. 10, pp. 35-48, 1986.

[8] Raghuveer M.R., Nikias C.L., Bispectrum Estimation: A digital signal processing framework." Proc. of IEEE vol. 75, no 7, pp. 869 – 889, 1987.

[9] Raghuveer M.R., Time Domain Approaches to QPC Estimation", IEEE Trans. On Automatic Control, vol. 35, no 1, pp. 48 – 56, 1990.

[10] Regazzoni C., Tacconi G., Tesel A., An Application to Local Optimal Detection of Higher – Order Spectral Analysis, Proc. of the Conference of Underwater Defence Technology, pp. 249-293, 1995.

[11] www.ant.uni-bremen.de/HOSHOME/