Scientific Journals
Maritime University of Szczecin

Zeszyty Naukowe
Akademia Morska w Szczecinie

2012, 30(102) pp. 5–11

2012, 30(102) s. 5–11

Kinematic decomposition of a ship nondimensional trajectory in turning circle test

Analiza kinematyczna bezwymiarowej trajektorii statku podczas próby cyrkulacji

Jarosław Artyszuk

Maritime University of Szczecin, Faculty of Navigation, Institute of Marine Traffic Engineering
Akademia Morska w Szczecinie, Wydział Nawigacyjny, Instytut Inżynierii Ruchu Morskiego
70-500 Szczecin, ul. Wały Chrobrego 1–2, e-mail: j.artyszuk@am.szczecin.pl

Key words: ship manoeuvring, ship turning, trajectory, drift angle, yaw velocity (nondimensional), motion kinematics, curvilinear motion

Abstract

The paper presents basic principles of a ship's nondimensional trajectory definition (for arbitrary manoeuvres) by means of drift angle and nondimensional yaw velocity. This enables faster and better tuning of mathematical model of ship manoeuvring motions against the provided full scale sea maneouvring trials (usually in a form of position trajectories) – since it is known the governing, basic factors behind the trajectory. A special focus has been applied to the standard turning circle test.

Słowa kluczowe: manewrowanie statku, cyrkulacja, trajektoria, kąt dryfu, prędkość kątowa (bezwymiarowa), kinematyka ruchu, ruch krzywoliniowy

Abstrakt

W pracy dokonano nowatorskiego i kompleksowego ujęcia kinematyki ruchu krzywoliniowego, szczególnie w odniesieniu do bezwymiarowej (mierzonej jednostkami długości statku) trajektorii statku podczas manewru cyrkulacji. Sformułowano analitycznie kształt trajektorii oraz jej ewoluty (położenia środków krzywizny). Ponadto przedstawiono praktyczne znaczenie i możliwości kształtowania trajektorii poprzez dobór parametrów zmian kąta dryfu i bezwymiarowej prędkości kątowej. Pozwala to zredukować problem kształtu trajektorii do dwóch parametrów o bardzo ważnej interpretacji hydrodynamicznej, co ma olbrzymie znaczenie w procesie identyfikacji i/lub walidacji matematycznego modelu manewrowania statku.

Introduction

The turning trajectory of a ship is apparently rather complex manoeuvring response. Due to a navigation importance and availability of full-scale data (easy measurement by GPS techniques), it is often used to validate various ship manoeuvring prediction codes on the uppermost level – i.e. with regard to the combined output of final motion and without going into details (components). This way, it is possible to ensure the same (or quite similar) turning trajectory, in the tuning circle test for instance, by significantly different means. The adequacy and accuracy of particular regions in the manoeuvring mathematical models can then be questioned, which actually strongly limits the validity / application range of the model. This study is aimed to provide a solution to such problems in that the nondimensional trajectory (in ship's length units) is used, which is further decomposed into the drift angle and nondimensional yaw velocity. Hence, the trajectory-based convergence / accuracy criteria (quality indices) for the mathematical models can be replaced with those comprising only the basic motions – drift and/or yaw response. The both latter items are much easier to be handled in the development process of manoeuvring models than the complex trajectory.

Fundamental terms in ship curvilinear motion kinematics

Let us define: a) a moving ship-related, cartesian, right-hand, horizontal plane reference system Mxy, b) a similar but stationary earth-related coordinate system Ox_Oy_O, c) a moving trajectory / curve-related $M\tau n$ axes of the same orientation as Mxy – refer to figure 1.

Vertical z-axes, from the reader's eye to the figure's plane, are also denoted for purpose of further conversion to 3D case. The moving body system origin is fixed to a point M that represents a geometric middle of a ship (lying precisely at the intersection of the centre plane and the midship section). A ship (its origin) is assumed to follow a given trajectory l with linear velocity v_{xy} and rotate with a yaw velocity ω_z. The linear velocity (also referred to as the total / resulting velocity) is always tangential to l and can be decomposed to its components in ship body axes – the longitudinal (surge) velocity v_x and transverse (sway) velocity v_y. At the current (instant) position on the trajectory, defined by the pair of coordinates $[x_O, y_O]$ (the symbol M is omitted), a curvature circle (as tangential to the trajectory) can be specified, the radius ρ of which is dependent on geometric properties of the curve l.

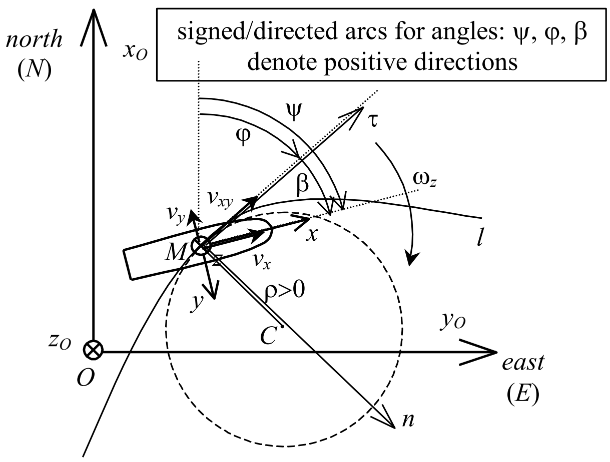

Fig. 1. Definition of ship kinematics
Rys. 1. Kinematyka ruchu statku po trajektorii – oznaczenia

The radius is essentially a signed radius – if the circle centre (depicting a direction of convexity) lies to the right of the movement direction (τ direction), it takes a positive value. It means the curvature radius coincides with the positive part of the normal n. In the case of a hypothetical, reverse movement of a ship in figure 1, a negative radius is obtained. The radius can easily be derived from the trajectory equations, it is always (through the inherent geometric properties) perpendicular to the trajectory. It also instantly changes along the trajectory together with the center of curvature circle (as lying on this perpendicular direction at distance ρ from the ship's origin M), which describes a curve called the evolute – in such a case the input trajectory takes the name of involute.

The trajectory definition can be given in a parametric form (more flexible) – an elapsed time t or travelled distance s is often used to parametrise the position coordinates $[x_O, y_O]$:

$$\begin{cases} x_O = x_O(t) \\ y_O = y_O(t) \end{cases}, \quad \begin{cases} x_O = x_O(s) \\ y_O = y_O(s) \end{cases} \tag{1a}$$

where

$$s = \int_0^t v_{xy}(t)\,\mathrm{d}t \quad \text{or} \quad \mathrm{d}s = v_{xy}(t)\cdot \mathrm{d}t = \mathrm{d}x_O + \mathrm{d}y_O \tag{1b}$$

or in a direct form:

$$y_O = y_O(x_O) \tag{1c}$$

Concerning the parametric form and a turning circle manoeuvre, it is obvious that the discrete position points at constant parameter intervals will not be equally spaced along the trajectory only in the case of time as parameter. A ship is normally reducing her linear velocity while turning, the velocity is even minimum during the steady phase of turning – i.e. when the circular trajectory is assumed. So, initially the locations are spaced farther from each other due to the higher speed and these mutual distances vary with the varying velocity v_{xy}.

Introducing a trajectory definition in a dimensionless form (by dividing all linear dimensions: x_O, y_O, and s by a ship's length L, which is usually the length between perpendiculars) as the most convenient to compare ships of different size and type from the viewpoint of similarity in ship hydrodynamics, one gets:

$$\begin{cases} x'_O = x'_O(s') \\ y'_O = y'_O(s') \end{cases} \tag{2a}$$

where: $$x'_O = \frac{x_O}{L},\ y'_O = \frac{y_O}{L},\ s' = \frac{s}{L}$$

$$y'_O = y'_O(x'_O) \tag{2b}$$

The nondimensional trajectory for a ship turning and stopping (in the latter aspect, also a nondimensional distance as the length of the trajectory is often additionally introduced) is roughly independent from a ship's size (represented e.g. by a ship's length L). For a ship turning due to the stern rudder action also the independence from the initial linear velocity can be proved (assuming of course much the same propeller thrust loading, represented in case of a fixed-pitch propeller by the ratio of advance velocity and propeller rpm). Considering different types of conventional ships and propulsion systems, however, the possible differences in non-dimensional trajectory patterns are not too significant for lots of standard manoeuvring regimes.

For heading angle ψ, trajectory angle φ (known as the course over ground – COG, or the course made good – CMG) related to the body system origin M, and hull drift angle β, shown in figure 1, both conventions of defining the possible range for angles are supported in the formulas derived hereafter. It does not matter whether an angle comes from the range $\langle 0°, 360°)$ ($\langle 0, 2\pi)$ in radians, in which there is also no limit on the upper bound – the angle may continue up to infinity, and assume negative values as well, if necessary) or is contained within the region $(-180°, 180°\rangle$ ($(-\pi, \pi\rangle$ in radians, with the same level of flexibility as in the former case). The equivalency of both approaches comes from the inherent nature of the trigonometric functions involved (sine and cosine) – specifically from their symmetries and periodicity. The motion case in figure 1 is usual for the stern rudder application, a ship always inclines (with her heading) to the trajectory interior thus "developing" a certain drift angle β to the outer side. The drift angle is usually positive for starboard turning, as also followed by the positive yaw velocity ω_z – the portside turning involves that the both variables simultaneously change in sign.

In all situations, the following rule applies (taking signed values into account):

$$\psi = \varphi + \beta \tag{3}$$

The trajectory angle is essentially the angle of its tangent in a particular point:

$$\cos\varphi = \frac{\mathrm{d}x_O}{\mathrm{d}s},\ \sin\varphi = \frac{\mathrm{d}y_O}{\mathrm{d}s},\ \tan\varphi = \frac{\mathrm{d}y_O}{\mathrm{d}x_O} \tag{4}$$

According to the commonly known kinematic principles, well covered in mechanical engineering science (as actually / really adopted from the differential geometry of curves – see e.g. [1]) it can write with the full support of sign:

$$\rho = \frac{\mathrm{d}s}{\mathrm{d}\varphi} \tag{5a}$$

where the sign of ρ depends on the sign of $\mathrm{d}\varphi$, i.e. whether φ is increasing (positive ρ), or decreasing (negative ρ),

$$\rho(x_O) = \frac{\left[1+\left(\frac{\mathrm{d}y_O}{\mathrm{d}x_O}\right)^2\right]^{3/2}}{\frac{\mathrm{d}^2 y_O}{\mathrm{d}x_O^2}} \tag{5b}$$

or

$$\rho(t) = \frac{\left[\left(\frac{\mathrm{d}x_O}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y_O}{\mathrm{d}t}\right)^2\right]^{3/2}}{\frac{\mathrm{d}x_O}{\mathrm{d}t}\cdot\frac{\mathrm{d}^2 y_O}{\mathrm{d}t^2}-\frac{\mathrm{d}^2 x_O}{\mathrm{d}t^2}\cdot\frac{\mathrm{d}y_O}{\mathrm{d}t}} \tag{5b}$$

or using the well-known ship-related terms (of linear or angular velocity nature):

$$\rho(t) = \frac{\mathrm{d}s}{\mathrm{d}\varphi}\cdot\frac{\mathrm{d}t}{\mathrm{d}t} = \frac{\frac{\mathrm{d}s}{\mathrm{d}t}}{\frac{\mathrm{d}\varphi}{\mathrm{d}t}} = \frac{v_{xy}(t)}{\omega_z(t)-\frac{\mathrm{d}\beta}{\mathrm{d}t}} \tag{5c}$$

$$\rho(s) = \frac{1}{\frac{\omega_z(s)}{v_{xy}(s)}-\frac{1}{v_{xy}}\cdot\frac{\mathrm{d}\beta}{\mathrm{d}t}} = \frac{1}{\frac{\omega_z(s)}{v_{xy}(s)}-\frac{\mathrm{d}\beta}{\mathrm{d}s}} \tag{5d}$$

where the following relationships have been used (see also Eq. 3):

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\psi}{\mathrm{d}t} - \frac{\mathrm{d}\beta}{\mathrm{d}t}$$

where by definition:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v_{xy} \quad \text{and} \quad \frac{\mathrm{d}\psi}{\mathrm{d}t} = \omega_z \tag{6}$$

But we also can write (see Eq. 1b):

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}s} - \frac{\mathrm{d}\beta}{\mathrm{d}s}$$

or more conveniently:

$$\boxed{\frac{\mathrm{d}\varphi}{\mathrm{d}s'} = \frac{\mathrm{d}\psi}{\mathrm{d}s'} - \frac{\mathrm{d}\beta}{\mathrm{d}s'}} \tag{7}$$

where $\mathrm{d}\varphi / \mathrm{d}s$, as the inverse of curvature radius ρ, is just called "a curvature" and usually marked by k, and finally get:

$$\boxed{\frac{\mathrm{d}\varphi}{\mathrm{d}s'} = \omega'_z - \frac{\mathrm{d}\beta}{\mathrm{d}s'}},\quad \rho'(s) = \frac{1}{\omega'_z - \frac{\mathrm{d}\beta}{\mathrm{d}s}} \tag{8}$$

where ω'_z is the so-called nondimensional yaw velocity, a very fundamental quantity in determining hull manoeuvring forces, which can also be written in a more canonical form:

$$\omega'_z = \frac{\mathrm{d}\psi}{\frac{v_{xy}\,\mathrm{d}t}{L}} = \frac{\omega_z L}{v_{xy}} \tag{9}$$

The nondimensional trajectory (see Eqs. 2a and 4) is thus characterised differentially by:

$$\begin{cases} \mathrm{d}\,x'_O = \cos\varphi \cdot \mathrm{d}\,s' \\ \mathrm{d}\,y'_O = \sin\varphi \cdot \mathrm{d}\,s' \end{cases} \tag{10}$$

or in an integral form according to:

$$\begin{cases} x'_O(s') = \int_0^{s'} \cos\varphi \cdot \mathrm{d}\,s' \\ y'_O(s') = \int_0^{s'} \sin\varphi \cdot \mathrm{d}\,s' \end{cases}, \quad \varphi(s') = \left(\int_0^{s'} \omega'_z \,\mathrm{d}\,s' \right) - \beta(s') \tag{11}$$

where the drift angle β and nondimensional yaw velocity ω'_z (of a direct hydrodynamic interpretation) are the only two contributing factors. They are to be derived from the differential equations of ship manoeuvring motions. The usual differential equations, written for derivatives of v_x, v_y, ω_z (versus time t and/or nondimensional distance s', the latter is equivalent to the so-called nondimensional time) on the left-hand side, can be rearranged as to directly provide derivatives of desired quantities: β, ω'_z, and v_{xy}.

The differential equation for v_{xy} is needed only in the case of coupling / influence to β, ω'_z – in other words, it is necessary when we are not able to write all involved hydrodynamic forces (e.g. of hull, propeller, rudder, etc.) as proportional to the linear velocity square v^2_{xy}. Such an adverse situation is surprisingly a frequent event in the classical ship manoeuvring with propeller and rudder – despite of relatively low Froude number (that is typical in ship manoeuvring and thus the velocity square law for hull forces is valid), it is very hard to keep the constant propeller thrust loading coefficient c_{Th} (as function of the advance coefficient J) through all phases of manoeuvring, refer e.g. to [2]. While a ship turning – and thus the natural forward speed reduction – the thrust loading increases under constant propeller / engine rpm or pitch. Hence finally, one can observe that both the propeller thrust and the part of rudder steering force / moment as originating from the propeller jet are kept almost constant – although the linear velocity square decreases, the propeller thrust loading coefficient at the same time increases to the same degree.

The second part of the rudder excitation comes from a ship's wake flow, the velocity of which is practically proportional to the ship's forward velocity, and the force is dependent to the square of velocity – so, with the rudder and idling propeller (the so-called coasting turn) there is no a ship's linear velocity influence on β and ω'_z.

Of course, under certain circumstances one can attempt to introduce some valid approximations for the propeller and rudder manoeuvres and thus make the variation of β and ω'_z be independent from the linear velocity. Let us notice for example that the rudder lateral force and moment (as governing the initial but not the final drift angle and nondimensional yaw velocity, correspondingly) are relatively low in comparison to the hull generalized forces – so the rudder forces can then be disregarded and the velocity square-related hull forces are solely used.

According to the observations in full-scale and computer simulations with sophisticated manoeuvring mathematical models, it can establish the following relationship for a ship's standard (constant rpm and/or pirch) turning manoeuvre (even with large rudder angles) – also refer to [3]:

A) The steady values of β and ω'_z are correlated according to the hull hydrodynamic characteristics only.
B) The steady value of β depends both on the rudder angle δ and the propeller loading coefficient c_{Th} during the steady phase of turning (a ship's steady linear velocity has to be known in this case).
C) The transient of both the drift angle and nondimensional yaw velocity, as far as we are concerned with the nondimensional distance s' as the independent variable, is practically not affected by the initial ship speed / engine throttle. A major question is here arising with reference to the response "time constants", e.g. in the first- or second order linear approximation (linearisation) of the differential equations. The detailed (comprising some more or less minor nonlinear effects) chart of β and ω'_z versus s' seems to be of less importance at the basic stage of research as performed in the present paper. In general, the "time" constants – the speed of increase – may be different for β and ω'_z. This is particularly important to remember, since we make a linearisation of somehow nonlinear ship manoeuvring at large rudder angle (even close to maximum one).

Trajectory performance with first order linear models for drift and yaw

For very small rudder (and thus resulting rather low drift angle and nondimensional yaw velocitiy) the well known, mutually coupled, linear differential equations (of constant coefficients) can be considered for β (in radians) and ω'_z – the rudder angle δ is here an input control:

$$\begin{cases} \dfrac{\mathrm{d}\beta}{\mathrm{d}s'} = a_1\beta + b_1\omega'_z + c_1\delta \\ \dfrac{\mathrm{d}\omega'_z}{\mathrm{d}s'} = a_2\beta + b_2\omega'_z + c_2\delta \end{cases} \tag{12}$$

where a_1, b_1, c_1 and a_2, b_2, c_2 are constants. Except for c_1 and c_2, as connected solely with the rudder hydrodynamic force, all other constants combine the effects from both a ship's hull and rudder. b_1 additionally has a very important contribution from the centrifugal force involved in the development of drift angle. All the constants can be determined from the detailed description of hydrodynamic forces.

This set of two first-order linear equations can easily be decoupled / resolved to the second-order linear equations for both nondimensional yaw velocity (more investigated in the literature, often called the Nomoto second-order equation) and drift angle (less interest given in the literature) as sole functions of the nondimensional distance:

$$T_1T_2\frac{\mathrm{d}^2\omega'_z}{\mathrm{d}s'^2} + \left(T_1+T_2\right)\frac{\mathrm{d}\omega'_z}{\mathrm{d}s'} + \omega'_z = K\left(\delta + T_3\frac{\mathrm{d}\delta}{\mathrm{d}s'}\right) \tag{13a}$$

$$T_1T_2\frac{\mathrm{d}^2\beta}{\mathrm{d}s'^2} + \left(T_1+T_2\right)\frac{\mathrm{d}\beta}{\mathrm{d}s'} + \beta = K_\beta\left(\delta + T_{3\beta}\frac{\mathrm{d}\delta}{\mathrm{d}s'}\right) \tag{13b}$$

where all T-class constants, interpreted as nondimensional time constants of the response are certain functions of the background constants in (12), as well as the the K-class constants, which are referred to as the amplification coefficients. Note that before and now there are still six parameters. Moreover, it can identify the hydrodynamic equations (12) based on the constants of the kinematic equations (13), which can be in turn determined / fitted from some full-scale manoeuvring behaviour. It also should be clearly emphasized that the most significant time constants on the left-hand side of (13) are equal / identical both for ω'_z and β – this effect can be explained through the inherent nature of the background set of linear coupled equations (12). Some other interpretation problems also arise here with regard to setting double initial conditions – in terms of $\mathrm{d}\omega'_z / \mathrm{d}s'$ and ω'_z at $s' = 0$ (normally both equal to zero) – for eqotion (13a), as usual in the only yaw-oriented application studies. Then, the initial value of β in eqotion (12) can not be zero that is against our frequent expectations.

The equations (13), after adopting some approximation to the response (different criteria are often being defined here), can be reduced to the first-order linear equations (more extensively used due to their simplicity and still good power):

$$T\frac{\mathrm{d}\omega'_z}{\mathrm{d}s'} + \omega'_z = K\delta \tag{14a}$$

$$T\frac{\mathrm{d}\beta}{\mathrm{d}s'} + \beta = K_\beta\delta \tag{14b}$$

The first equation (14a) is sometimes known as Nomoto (or K–T) first-order linear equation. The above processing – the conversion of second- to first-order – in the case of a ship manoeuvring shall be treated as an empirical fitting of higher order response to the first-order equation – see e.g. [4] – since it is very hard to state a firm hydrodynamic theory for this. Formally, the first-order linear equations can be derived from (12), but under the assumption of no coupling between ω'_z and β, i.e. when $b_1 = a_2 = 0$ (or $T_1 = T_2$, or either T_1 or T_2 is diminishing to zero), that is really physically doubtful. It is worthwhile to mention here for further reference that the linear first- and second-order response is well covered in the control engineering theory.

For the purpose of present study – an investigation into the turning trajectory at large helm – it can suppose the following (the order of equations is set back to the original one):

$$T_b\frac{\mathrm{d}\beta}{\mathrm{d}s'} + \beta = K_b\delta \tag{15a}$$

$$T_w\frac{\mathrm{d}\omega'_z}{\mathrm{d}s'} + \omega'_z = K_w\delta \tag{15b}$$

where there are two different pairs of constants – T_b, K_b, T_w, K_w (in general being functions of δ, thus "making" the model really nonlinear), and which can be transformed to:

$$T_b\frac{\mathrm{d}\beta}{\mathrm{d}s'} + \beta = \beta_0 \tag{16a}$$

$$T_w\frac{\mathrm{d}\omega'_z}{\mathrm{d}s'} + \omega'_z = \omega'_{z0} \tag{16b}$$

where β_0 and ω'_{z0} denote the steady (asymptotic) values of the corresponding parameters.

The response of β (in radians) and ω'_z for the most interesting constant rudder (δ = const) and zero initial conditions is thus given by:

$$\beta(s') = \beta_0\left(1 - e^{-s'/T_b}\right) \tag{17a}$$

$$\omega'_z(s') = \omega'_{z0}\left(1 - e^{-s'/T_w}\right) \tag{17b}$$

Therefore, substituting (17) to (8) and (11) one gets:

$$\begin{aligned} &\varphi(s') = \underline{\omega'_{z0}\cdot s'} + \omega'_{z0}T_w e^{-s'/T_w} - \beta_0\left(1-e^{-s'/T_b}\right) \\ &\varphi(0) = 0 \end{aligned} \tag{18}$$

where the first underlined term is responsible for the steady phase of turning (i.e. the circular trajectory), the other two components govern the unsteady or transient phase / leg of trajectory (sometimes called the initial spiral curve),

$$\begin{cases} x'_O(s') = \\ = \displaystyle\int_0^{s'} \cos\left(\omega'_{z0}s' + \omega'_{z0}T_w e^{-s'/T_w} - \beta_0\left(1-e^{-s'/T_b}\right)\right)\mathrm{d}s' \\ y'_O(s') = \\ = \displaystyle\int_0^{s'} \sin\left(\omega'_{z0}s' + \omega'_{z0}T_w e^{-s'/T_w} - \beta_0\left(1-e^{-s'/T_b}\right)\right)\mathrm{d}s' \end{cases} \tag{19}$$

The following figures 2, 3 and 4 show the effect of T_b and T_w on modelling the trajectory – s' is from the range $\langle 0, 20\rangle$, the steady values assumed for β and ω'_z are 25° and 0.6 correspondingly, the value 0^+ for T_b and/or T_w means an asymptotic approach to zero.

Since the drift angle plays a somehow different role than the yaw velocity in the trajectory performance, it has been possible to include the whole possible range of T_b i.e. from nearly zero up to infinity. Figure 2 therefore demonstrates the limits of trajectory change by means of T_b. Moreover, for a relatively low T_b (as compared to T_w) the so-called initial negative transfer (position shift) is experienced – i.e. to portside for the starboard rudder turning.

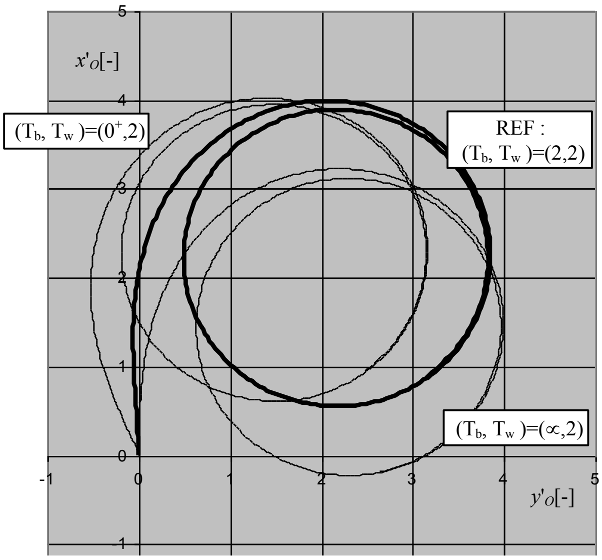

Fig. 2. The effect of drift time constant T_b on trajectory
Rys. 2. Wpływ stałej czasowej kąta dryfu na trajektorię

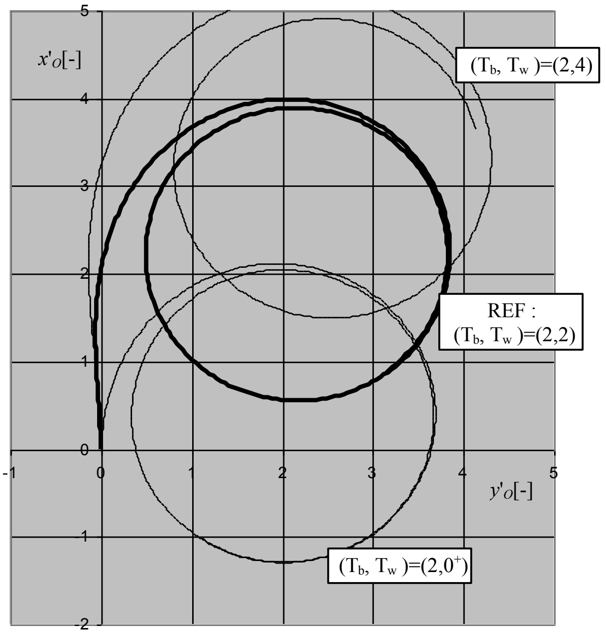

Fig. 3. The effect of yaw time constant T_w on trajectory
Rys. 3. Wpływ stałej czasowej prędkości kątowej na trajektorię

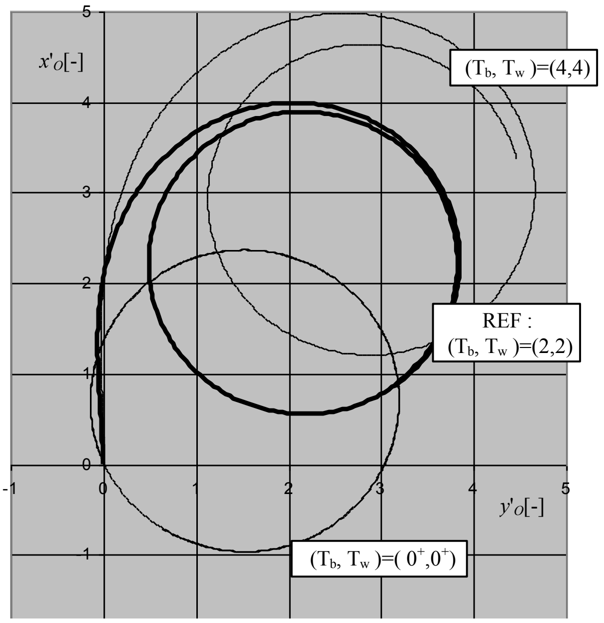

Fig. 4. The effect of equal yaw and drift time constants ($T_b = T_w$) on trajectory
Rys. 4. Wpływ równych stałych czasowych dryfu i prędkości kątowej na trajektorię

The instantaneous centre of curvature

The position of the curvature circle centre on the earth – coordinates denoted by x_{OC}, y_{OC} is defined by:

$$\begin{aligned} \begin{bmatrix} x_{OC} \\ y_{OC} \end{bmatrix} &= \begin{bmatrix} x_O \\ y_O \end{bmatrix} + \rho \begin{bmatrix} \cos(90° + \varphi) \\ \sin(90° + \varphi) \end{bmatrix} = \\ &= \begin{bmatrix} x_O \\ y_O \end{bmatrix} + \rho \begin{bmatrix} \cos(90° + \psi - \beta) \\ \sin(90° + \psi - \beta) \end{bmatrix} \end{aligned} \tag{20a}$$

while in a ship moving reference system, the position of curvature centre (i.e. relative to a ship) reads:

$$\begin{bmatrix} x_C \\ y_C \end{bmatrix} = \rho \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \cdot \begin{bmatrix} \cos(90^\circ + \varphi) \\ \sin(90^\circ + \varphi) \end{bmatrix} = \\ = \rho \begin{bmatrix} \sin\beta \\ \cos\beta \end{bmatrix} \quad !!! \tag{20b}$$

which finally and nondimensionally looks like:

$$x'_C = \frac{\sin\beta}{\omega'_z - \dfrac{\mathrm{d}\beta}{\mathrm{d}s'}}, \quad y'_C = \frac{\cos\beta}{\omega'_z - \dfrac{\mathrm{d}\beta}{\mathrm{d}s'}} \tag{20c}$$

Just for reference and further comparison, the following formula, taken from [5], presents the so-called instant pivot point of a ship dealt with as a rigid body:

$$x'_{PP} = \frac{\sin\beta}{\omega'_z}, \quad y'_{PP} = \frac{\cos\beta}{\omega'_z} \tag{21}$$

The term instant pivot point essentially differs from the curvature centre in that the former relates to a ship – a plane figure – subject to a combined translation and rotation about z axis, and thus produced local velocities over the figure's area, while the latter corresponds to a single point movement – for example a ship's origin – along a certain trajectory.

Based on (18), we havethere is in a nondimensional form:

$$\begin{bmatrix} x'_{OC} \\ y'_{OC} \end{bmatrix} = \begin{bmatrix} x'_O \\ y'_O \end{bmatrix} + \frac{1}{\omega'_{z0}\left(1 - e^{-s'/T_w}\right) - \dfrac{\beta_0}{T_b} e^{-s'/T_b}} \cdot \\ \cdot \begin{bmatrix} -\sin\left(\omega'_{z0} \cdot s' + \omega'_{z0} T_w e^{-s'/T_w} - \beta_0\left(1 - e^{-s'/T_b}\right)\right) \\ \cos\left(\omega'_{z0} \cdot s' + \omega'_{z0} T_w e^{-s'/T_w} - \beta_0\left(1 - e^{-s'/T_b}\right)\right) \end{bmatrix} \tag{22}$$

which converges during the steady phase ($s' \to \infty$) to

$$\begin{bmatrix} x'_{OC-\text{steady}} \\ y'_{OC-\text{steady}} \end{bmatrix} = \lim_{s' \to \infty} \left(\begin{bmatrix} x'_O \\ y'_O \end{bmatrix} + \frac{1}{\omega'_{z0}} \cdot \begin{bmatrix} -\sin(\omega'_{z0} \cdot s') \\ \cos(\omega'_{z0} \cdot s') \end{bmatrix} \right) \tag{23}$$

Conclusions

Though lots of efforts in this study have been connected with the simplest (1st order inertia) models of drift angle and yaw nondimensional velocity development – giving very good qualitative indications – other more sophisticated and thus more accurate description can be specified and investigated.

Further research shall also go on with regard to the coordinates of a transition point on a ship's trajectory where the transient (unsteady) curve changes into the circular (steady) curve. It is worthwhile to know whether the steady phase of turning is assumed faster or slower. From each of these behaviours one can deduce the parameters of the background change of drift angle and yaw nondimensional velocity and set a proper, additional hydrodynamics in the manoeuvring mathematical model as to realise the observed effect on the trajectory. Determination of the aforementioned coordinates shall be treated and defined of course in practical aspects – e.g. introducing an approximate definition of convergence criterion – since theoretically (asymptotically) there is no such transition.

References

1. FICHTENHOLZ G.M.: Differential and Integral Calculus, vol. 1. Ed. 8, PWN, Warszawa 1980 (in Polish).
2. ARTYSZUK J.: Ship Manoeuvring Auxiliary Differential Equations for Optimal Sea Trials Programme Design. Scientific Bulletin, 6(78) "Marine Traffic Engineering", Maritime University, Szczecin, 2005.
3. ARTYSZUK J.: Analysis of Steady State Turning Ability in View of Ship Manoeuvring Model Optimisation. Archives of Transport, vol. 17, 1, 2005.
4. ARTYSZUK J.: Data Smoothing Application to the Ship Motion Mathematical Model Identification. Annual of Navigation, 2/00, 2000.
5. ARTYSZUK J.: Pivot Point in Ship Manoeuvring. Scientific Journals, 20(92), Maritime University, Szczecin 2010, 13–24.