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A NEW 2D SINGLE SERIES MODEL OF TRANSVERSE VIBRATION OF A MULTI-LAYERED SANDWICH BEAM WITH PERFECTLY CLAMPED EDGES

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A new two-dimmensional, single series local model of the transverse vibration of a multi-layer, one-span sandwich beam composed of isotropic layers with ideally (perfectly) clamped ends is proposed in the paper. The model is derived within the local theory of linear elastodynamics and it is composed of two two-dimmensional fields and of two approximations of three-dimmensional fields satisfying exactly the equations of motion as well as the Saint-Venant compatibility equations of the theory. All through-the-thickness boundary conditions of the local theory of elastodynamics as well as all local compatibility equations (for the displacements and stresses) between adjoining layers are fulfiled in the model. Both the cross-sectional warping and the transverse compliance(s) in each layer of the beam are taken into account, thus the model is applicable to the classical three-layer sandwich beam and to a multi-layer sandwich or laminated narrow structure.

Key words: sandwich beam, perfect clamping, transverse vibration, local model

Notations

E	–	Young's modulus
h_j	–	thickness of jth layer of beam
L	–	length of beam
$U_x, U_y, U_z, u_x, u_y, u_z$	–	displacements in directions x, y, z, respectively
$u_{x(j)}, u_{y(j)}, u_{z(j)}$	–	displacements within jth layer
t	–	time
x, y, z	–	space variables
$X^{(T)}$	–	trigonometric function of variable x
$X^{(H)}$	–	hyperbolic function of variable x

z_j	–	coordinate of one (upper) surface of jth layer
ε_{qr}	–	strain tensor
λ_L, μ_L	–	Lame's parameters
$\mu = \mu_L, \mu_{(j)}$	–	shear modulus and shear modulus of jth layer, respectively
ν	–	Poisson's ratio
$\rho, \rho_{(j)}$	–	density and density of jth layer, respectively
$\sigma_{zz}, \sigma_{zx}, \sigma_{zz(j)}, \sigma_{zx(j)}$	–	stresses and stresses in jth layer, respectively

1. Introduction

Many papers have been published lately on vibration analysis of sandwich structures and, in particular of sandwich beams. Unfortunately, most of them are devoted to presentation of general analytical models and are limited to numerical investigation of the simply supported structures – see e.g. Frostig and Baruch (1994), Cabańska-Płaczkiewicz (1999), Kapuria *et al.* (2004). It is noted that the paper by Lewiński (1991) contains theoretical considerations while papers by Lewiński (1991), Cupiał and Nizioł (1995), Szabelski and Kaźmir (1995) refer to rectangular plates. In papers by Chen and Sheu (1994), Fasana and Marchesiello (2001), Nilsson and Nilsson (2002), Backstom and Nilson (2006, 2007), some numerical results for clamped-clamped, clamped-free and free-free beams are also presented. In some of the papers, the numerical results are not tabulated and, therefore, are not useful for detail comparisons. Some of the above papers contain comparisons of numerical results for different theories, see Kapuria *et al.* (2004), Backstom and Nilson (2006), Hu *et al.* (2006, 2008), Wu and Chen (2008). In a paper by Backstom and Nilson (2006) the numerical results (amplitudes) are compared with measured values for the beam with both ends free.

Majority of analytical beam models were derived following the variational procedure and the same path as in the case of laminated composites – see e.g. Kapuria *et al.* (2004), Hu *et al.* (2008), Wu and Chen (2008). After looking through the analytical and numerical results for the simply supported beams, one may notice that the models of the eigenvalue problem of sandwich structures have got some deficiencies. Some of them are shown e.g. in a paper by Hu *et al.* (2006), where the evaluation of kinematic assumptions applied by different authors is proposed.

Some other deficiencies can be easily noticed. For example, in paper of Frostig and Baruch (1994) the in-plane normal stresses in the core are omited and the equilibrium equation instead of the equation of motion for the core is applied. Despite of the simplifications, the model is not compared (in Frostig and Baruch, 1994) with other models. In paper of Kapuria *et al.* (2004), high inaccuracy of eigenfrequencies of a sandwich beam predicted by the FSDT is shown. In Backstom and Nilson (2006, 2007), the compatibility equations of stresses between adjoining layers are not satisfied. In Wu and Chen (2009), high percentage differences between predictions of eigenfrequencies by different analytical models are given and commented.

Because of various assumptions and simplifications introduced into the models of vibration of sandwich structures, the comparisons limited to simply supported members can imply misunderstandings since the comparative results for any two models may be dependent on boundary conditions of the structure(s). To some extent, it is suggested e.g. in Fasana and Marchesiello (2001), where the percentage differences between the eigenfrequencies predicted by the two models are within the range (3.86-0.56) for the simply supported structure and within the range (5.85-4.22) for the free-free structure. Thus, instead of investigating the simply supported beams, a direct investigation of the clamped-clamped (C-C) sandwich structures is much more desired since it can be useful because of their practical importance.

There is much less papers devoted to vibration analysis of clamped-clamped unidirectional three-layer sandwich structures. Here, a few are collected (Nilsson and Nilsson, 2002; Raville *et al.*, 1961; Sakiyama *et al.*, 1996; Sokolinsky and Nutt, 2002; Howson and Zare, 2005). It is noticed that the experimental data given in Raville *et al.* (1961) are compared in Sakiyama *et al.* (1996), Sokolinsky and Nutt (2002), Howson and Zare (2005). Vibrational models presented in Nilsson and Nilsson (2002), Raville *et al.* (1961), Sokolinsky and Nutt (2002) were obtained according to the variational procedure. In Sakiyama *et al.* (1996), the Green functions approach is used, and in Howson and Zare (2005) a direct approach is employed to obtain the equations of motion.

It is an aim of the paper to present and discuss the new two-dimmensional (2D) model of transverse vibration of a C-C sandwich multi-layered beam with perfectly clamped edges, that is to show both its mathematical details and some comparison of numerical results. This model is a next result of investigations of sandwich structures by the present author within the local theory of linear elastodynamics. Several vibrational models for the unidirectional, both cantilever (Karczmarzyk, 1995, 1996) and clamped-clamped (C-C) (Karczmarzyk, 1999, 2005), sandwich structures have been elaborated within the approach. The former models and the new local model of the present author were obtained without the *a priori* expanding displacement and stress fields (within the structures) into series. However, the stress and displacment fields in the new model are finally expanded into the single series of the eigenfunctions of the classical Bernoulli-Euler theory of beams. All through-the-thickness boundary conditions and the compatibility equations of the local theory of linear elastodynamics as well as some specific edge boundary conditions have been satisfied in the former models (Karczmarzyk, 1995, 1996, 1999, 2005) and in the new model. However, as far as the present author knows, the perfect clamping edge boundary conditions for the sandwich beam are fulfiled for the first time within the local elastodynamic approach in the present paper.

The new model is directly applicable to the beams consisting of any number of layers. This is its important feature since the multi-layered sandwich structures are rarely investigated in the literature but they occur frequently in the modern composite constructions (see e.g. Wu *et al.*, 2003).

There are many formal differences between the new model and the models presented by other authors above mentioned in particular for the C-C sandwich beam. First, displacements and stresses within the new local model satisfy the well known differential equations of motion of the local theory of linear elastodynamics – expressed in stresses and displacements. The equations of motion of the other authors were derived for an assumed number of layers (usually equal to three) usually within the variational procedure or within the Bolle-Mindlin procedure. Thus, to apply (eventually) the variational theories, e.g. for a five-layer sandwich beam one needs to derive first new equations of motion. Secondly, the kinematic assumptions in the present new model and in the former models are quite different. The functions of space variable in the direction perpendicular to the interfaces appearing in the present model are unknown while their counterparts in the former models are assumed as known (linear or nonlinear) functions of the variable. On the other hand, the form of functions of space variable in the direction parallel to the interfaces (to length) of the beam is assumed in the present model whereas the functions are derived from the equations of motion in the former models. Thirdly, the final (computational) form of the problem within the present local model is derived after satisfying both the local edge boundary conditions and all through-the-thickness local boundary conditions and compatibility equations. In fact, the final form of the problem consist of two transcendental uncoupled equations. The computational form of the problem within the former models is derived by using only the edge boundary conditions since through-the-thickness conditions have been satisfied (more or less exactly) in the procedure(s) of deriving the equations of motion.

There are more formal distinctions which imply some merit differences, not discussed here, between the local new model and the former models however, dispite of them the numerical results show merit compatibility of the models. The main advantage of the new model, stressed here, is its direct applicability to the analysis of multi-layered (eg. five-layered) sandwich structures that is when the adjoining layers in such structures are of incomparable stiffnesses. It is also emphasized that the eigenfunctions for the C-C beam within the new local model are the same as in the classical Bernoulli-Euler beam theory.

The exemplary structures considered in the paper are shown in Fig. 1. They are composed of homogeneous, isotropic layers. The layers are perfectly bonded one to another. Each layer is perfectly clamped at the edges i.e., in Fig. 1a at $x = \pm L/2$. Any parameter of the structure(s) is not formally limited.

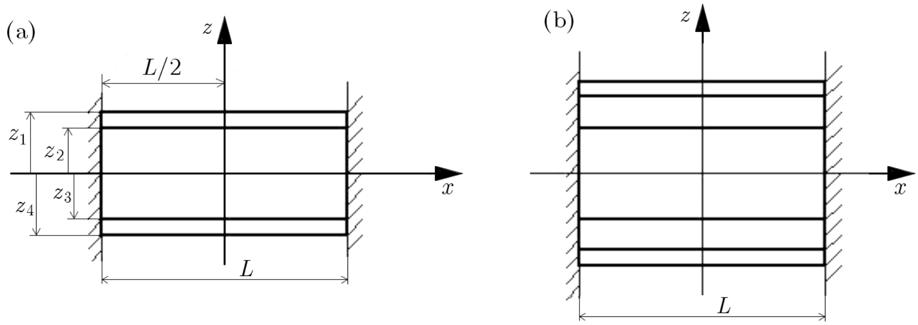

Fig. 1. Multi-layered sandwich C-C beams: (a) three-layer, (b) five-layer. Thickness of jth layer $h_j = z_j - z_{j+1}$

In the case of the beam symmetric about its middle plane (mid-plane) it is desired to impose the following assumptions on location of the origin of coordinate system. It is convenient to place it in the middle plane of the structure and in the middle of the span (mid-span) – as shown in Fig. 1a. Location of the origin in the mid-plane enables us to split the boundary problem in two subproblems – the transverse flexural problem and the transverse breath problem. Location of the origin in the mid-span enables decoupling of the symmetric and anti-symmetric modes of vibration.

The new model is presented in the further text as follows. All the equations and conditions of the local theory of linear elastodynamics, however without the well known Hooke law, are listed in the 2nd section. Two 2D solutions to the local 2D equations of motion of the theory of linear elastodynamics, derived here by the present author in an original way, are described in the 3rd section. Two 3D solutions to the local 3D equations of motion of the theory of linear elastodynamics are given in the 4th section. The 2D and 3D solutions are presented widely in order to facilitate understanding the content of the 5th section. The essential new ideas of deriving the new model (after combining the 2D and 3D fields) are presented in the 5th section. Exact formulas necessary to create the final numerical form of the boundary problem (i.e. to create the matrix of the problem) and some details on the final form are given in the 6th section. Numerical results and comparisons as well as some comments are given in the 7th section. Section 8th contains a few conclusions.

2. Statement of the problem

The boundary problem is formulated and solved entirely within the local linear theory of elastodynamics. The new solution (model) is composed of two 2D (plane) components and two 3D components.

The following 2D local equations of motion, containing the plane stress state components σ_{xx}, σ_{zz}, σ_{zx} and the corresponding displacements u_x, u_z, are satisfied by the 2D components of the model within each layer of the structure separately

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2} \qquad \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \tag{2.1}$$

Equations (2.1) can be expressed entirely in terms of the field u_x, u_z (Karczmarzyk, 1996) that is

$$\begin{aligned} \mu \nabla^2 u_x + (\lambda + \mu)\Big(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial x \partial z}\Big) &= \rho \frac{\partial^2 u_x}{\partial t^2} \\ \mu \nabla^2 u_z + (\lambda + \mu)\Big(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_z}{\partial z^2}\Big) &= \rho \frac{\partial^2 u_z}{\partial t^2} \end{aligned} \tag{2.2}$$

The parameters λ, μ in Eqs (2.2) are defined as follows

$$\lambda = \lambda_L \frac{1 - 2\nu}{1 - \nu} = 2\mu_L \frac{\nu}{1 - \nu} \qquad \lambda_L = 2\mu_L \frac{\nu}{1 - 2\nu} \qquad \mu = \mu_L \tag{2.3}$$

where λ_L, μ_L are the Lame material parameters and ν denotes the Poisson ratio of a particular homogeneous layer of the structure. Symbols ρ, t in (2.1) and (2.2) stand for the layer density and time, respectively.

The 3D components of the new solution satisfy the full 3D equations of motion of the local linear theory of elastodynamics, i.e.

$$\begin{aligned}
\mu\nabla^2 u_x + (\lambda_L + \mu)\Big(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial x} + \frac{\partial^2 u_z}{\partial z \partial x}\Big) &= \rho\frac{\partial^2 u_x}{\partial t^2} \\
\mu\nabla^2 u_y + (\lambda_L + \mu)\Big(\frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z \partial y}\Big) &= \rho\frac{\partial^2 u_y}{\partial t^2} \\
\mu\nabla^2 u_z + (\lambda_L + \mu)\Big(\frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial z^2}\Big) &= \rho\frac{\partial^2 u_z}{\partial t^2}
\end{aligned} \tag{2.4}$$

The 2D and 3D fields, satisfying the above equations of motion, fulfil the Saint-Venant compatibility equations expressed in terms of strains ε_{qr} in the following well known abbreviated form

$$\varepsilon_{kl,mn} + \varepsilon_{mn,kl} - \varepsilon_{km,\ln} - \varepsilon_{\ln,km} = 0 \tag{2.5}$$

The following through-the-thickness local boundary conditions, (2.6), and compatibility equations (2.7) for the whole structure are satisfied by the total stress and displacemnnt fields within the new model

$$\begin{aligned}
&\widetilde{\sigma}_{zz(1)}(x, z{=}z_1) = \widetilde{\sigma}_{zx(1)}(x, z{=}z_1) = \widetilde{\sigma}_{zz(p)}(x, z{=}z_{p+1}) = \\
&= \widetilde{\sigma}_{zx(p)}(x, z{=}z_{p+1}) = 0
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\widetilde{\sigma}_{zz(j)}(x, z = z_{j+1}) &= \widetilde{\sigma}_{zz(j+1)}(x, z = z_{j+1}) \\
\widetilde{\sigma}_{zx(j)}(x, z = z_{j+1}) &= \widetilde{\sigma}_{zx(j+1)}(x, z = z_{j+1}) \\
\widetilde{u}_{z(j)}(x, z = z_{j+1}) &= \widetilde{u}_{z(j+1)}(x, z = z_{j+1}) \\
\widetilde{u}_{x(j)}(x, z = z_{j+1}) &= \widetilde{u}_{x(j+1)}(x, z = z_{j+1})
\end{aligned} \tag{2.7}$$

where $j = 1, 2, \ldots, p-1$.

The symbols with the sign "$\sim$" denote the total stresses and displacements, the subscript p means the number of layers, subscripts, 1, j, $j+1$ identify the 1st, jth and $(j+1)$th layer, respectively. The coordinates z_1, z_j, etc. are explained in Fig. 1.

It is noticed that assuming in Eqs (2.6) the normal stresses as non-equal to zero, we have the boundary conditions for the forced vibration. It is explained that the stresses result from the Hooke law applied in the paper.

The following edge boundary conditions are satisfied within the new model $(j = 1,2,\dots,p)$

$$
\begin{aligned}
&\widetilde{u}_{x(j)}(x=\pm L/2,z)=0 \qquad\qquad \widetilde{u}_{z(j)}(x=\pm L/2,z)=0\\
&\frac{\partial \widetilde{u}_{z(j)}}{\partial x}\Big|_{(x=\pm L/2,z)}=0
\end{aligned}
\tag{2.8}
$$

As far as the present author knows, local edge boundary conditions (2.8) for the perfect clamping of the edges for all layers of the sandwich structure have been fulfiled for the first time within the local elastodynamic approach.

3. Solutions to the 2D (plane) local equations of motion of the linear elastodynamics

In order to derive 2D solutions for an isotropic continuous layer, the following kinematic assumptions are used

$$
\begin{aligned}
&u_x=-g(z)T(t)\frac{dX^{(T)}}{dx} \qquad\qquad u_z=f(z)X^{(T)}T(t)\\
&\frac{d^2X^{(T)}}{dx^2}=-\alpha^2X^{(T)} \qquad\qquad \alpha^2>0
\end{aligned}
\tag{3.1}
$$

The functions g, f of the space variable z are unknown, the function $X^{(T)}$ of the space variable x will be defined later. The function $T(t)=\exp(\mathrm{i}\omega t)$, where $\mathrm{i}^2=-1$ and ω, t are the vibration frequency and time, respectively. Due to (3.1), Eqs (2.2) can be transformed to the following form

$$
\begin{aligned}
&-\mu\frac{d^2g}{dz^2}+[(\lambda+2\mu)\alpha^2-\rho\omega^2]g+(\lambda+\mu)\frac{df}{dz}=0\\
&(\lambda+2\mu)\frac{d^2f}{dz^2}-(\mu\alpha^2-\rho\omega^2)f+(\lambda+\mu)\alpha^2\frac{dg}{dz}=0
\end{aligned}
\tag{3.2}
$$

Equations (3.2) can be solved in many ways, and one of them, which is very convenient, is shown below. It is noticed that Eqs (3.2) may be rearranged as follows

$$
\begin{aligned}
&-\mu\frac{d^2g}{dz^2}+(\mu\alpha^2-\rho\omega^2)g+\underline{(\lambda+\mu)\Big(\alpha^2g+\frac{df}{dz}\Big)}=0\\
&\mu\frac{d^2f}{dz^2}-(\mu\alpha^2-\rho\omega^2)f+\underline{(\lambda+\mu)\Big(\alpha^2\frac{dg}{dz}+\frac{d^2f}{dz^2}\Big)}=0
\end{aligned}
\tag{3.3}
$$

The underlined term occurs in each of Eqs (3.3). It is seen form Eqs (3.3) that,

$$\begin{aligned} &\frac{d^2g}{dz^2} = \Big(\alpha^2 - \frac{\rho\omega^2}{\mu}\Big)g \equiv \beta_1^2 g \qquad && \beta_1^2 = \alpha^2 - \frac{\rho\omega^2}{\mu} \\ &\frac{d^2f}{dz^2} = \Big(\alpha^2 - \frac{\rho\omega^2}{\mu}\Big)f \equiv \beta_1^2 f \qquad && g = -\frac{1}{\alpha^2}\frac{df}{dz} \end{aligned} \tag{3.4}$$

Thus, the first rearrangement of Eqs (3.2) leads to the first solution, expressed in the following matrix form:
— for $\beta_1^2 > 0$

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} \cosh(\beta_1 z) & \sinh(\beta_1 z) \\ -\dfrac{\beta_1}{\alpha^2}\sinh(\beta_1 z) & -\dfrac{\beta_1}{\alpha^2}\cosh(\beta_1 z) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{3.5}$$

— for $\beta_1^2 < 0$, $\overline{\beta}_1^2 = -\beta_1^2$

$$\begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} \cos(\overline{\beta}_1 z) & \sin(\overline{\beta}_1 z) \\ \dfrac{\overline{\beta}_1}{\alpha^2}\sin(\overline{\beta}_1 z) & -\dfrac{\overline{\beta}_1}{\alpha^2}\cos(\overline{\beta}_1 z) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{3.6}$$

Equations (3.2) can be also rearranged in a second manner

$$\begin{aligned} &-(\lambda+2\mu)\frac{d^2g}{dz^2} + [(\lambda+2\mu)\alpha^2 - \rho\omega^2]g + (\lambda+\mu)\underline{\Big(\frac{d^2g}{dz^2} + \frac{df}{dz}\Big)} = 0 \\ &(\lambda+2\mu)\frac{d^2f}{dz^2} - [(\lambda+2\mu)\alpha^2 - \rho\omega^2]f + (\lambda+\mu)\alpha^2\underline{\Big(\frac{dg}{dz} + f\Big)} = 0 \end{aligned} \tag{3.7}$$

Again, there is a term (underlined) occuring in both Eqs (3.7). It is seen directly from Eqs (3.7) that the functions g,f are now defined as follows

$$\begin{aligned} &\frac{d^2g}{dz^2} = \Big(\alpha^2 - \frac{\rho\omega^2}{\lambda+2\mu}\Big)g \equiv \beta_2^2 g \qquad && \beta_2^2 = \alpha^2 - \frac{\rho\omega^2}{\lambda+2\mu} \\ &\frac{d^2f}{dz^2} = \Big(\alpha^2 - \frac{\rho\omega^2}{\lambda+2\mu}\Big)f \equiv \beta_2^2 f \qquad && f = -\frac{dg}{dz} \end{aligned} \tag{3.8}$$

Thus, the second rearrangement of Eqs (3.2) leads to the second solution, expressed in the following matrix form: — for $\beta_2^2 > 0$

$$\begin{bmatrix} f_2 \\ g_2 \end{bmatrix} = \begin{bmatrix} \cosh(\beta_2 z) & \sinh(\beta_2 z) \\ -\dfrac{1}{\beta_2}\sinh(\beta_2 z) & -\dfrac{1}{\beta_2}\cosh(\beta_2 z) \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} \tag{3.9}$$

— for $\beta_2^2 < 0$, $\overline{\beta}_2^2 = -\beta_2^2$

$$\begin{bmatrix} f_2 \\ g_2 \end{bmatrix} = \begin{bmatrix} \cos(\overline{\beta}_2 z) & \sin(\overline{\beta}_2 z) \\ -\dfrac{1}{\overline{\beta}_2}\sin(\overline{\beta}_2 z) & \dfrac{1}{\overline{\beta}_2}\cos(\overline{\beta}_2 z) \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} \tag{3.10}$$

The constants C_l, $l = 1,2,3,4$, in (3.5), (3.6) and (3.9), (3.10) are unknown.

4. Solutions to the 3D (plate) local equations of motion of the linear elastodynamics

In order to derive 3D solutions for an isotropic continuous layer, the following kinematic assumptions are used

$$\begin{aligned} & U_x = -G(z)Y(y)\frac{dX^{(H)}}{dx}T(t) && U_y = -G(z)\frac{dY}{dy}X^{(H)}T(t) \\ & U_z = F(z)Y(y)X^{(H)}T(t) && \frac{d^2X^{(H)}}{dx^2} = \alpha^2 X^{(H)} \qquad \alpha^2 > 0 \\ & \frac{d^2Y}{dy^2} = -\beta^2 Y && \beta^2 > 0 \end{aligned} \tag{4.1}$$

U_x, U_y and U_z are displacements dependent on three space variables x, y, z. The functions G, F of the variable z are unknown. The function $X^{(H)}$ of the variable x as well as $Y(y)$ will be defined later. $T(t)$ is the same function of time which appears in (3.1). Now Eqs (2.4) can be transformed to the following (two) ordinary differential equations

$$\begin{aligned} & -\mu\frac{d^2G}{dz^2} - [(\lambda_L + 2\mu)(\alpha^2 - \beta^2) + \rho\omega^2]G + (\lambda_L + \mu)\frac{dF}{dz} = 0 \\ & (\lambda_L + 2\mu)\frac{d^2F}{dz^2} + [\mu(\alpha^2 - \beta^2) + \rho\omega^2]F - (\lambda_L + \mu)(\alpha^2 - \beta^2)\frac{dG}{dz} = 0 \end{aligned} \tag{4.2}$$

It is noticed that a first rearrangement of Eqs (4.2) is as follows

$$\begin{aligned} & -\mu\frac{d^2G}{dz^2} - [\mu(\alpha^2 - \beta^2) + \rho\omega^2]G - \underline{(\lambda_L + \mu)\Big[(\alpha^2 - \beta^2)G - \frac{dF}{dz}\Big]} = 0 \\ & \mu\frac{d^2F}{dz^2} + [\mu(\alpha^2 - \beta^2) + \rho\omega^2]F - \underline{(\lambda_L + \mu)\Big[(\alpha^2 - \beta^2)\frac{dG}{dz} - \frac{d^2F}{dz^2}\Big]} = 0 \end{aligned} \tag{4.3}$$

The underlined term occurs in each of Eqs (4.3). It is seen form Eqs (4.3) that

$$\begin{aligned} \frac{d^2G}{dz^2} &= -\Big(\alpha^2-\beta^2+\frac{\rho\omega^2}{\mu}\Big)G \equiv -R_1^2 G \qquad && R_1^2=\alpha^2-\beta^2+\frac{\rho\omega^2}{\mu} \\ \frac{d^2F}{dz^2} &= -\Big(\alpha^2-\beta^2+\frac{\rho\omega^2}{\mu}\Big)F \equiv -R_1^2 F \qquad && G=\frac{1}{\alpha^2-\beta^2}\frac{dF}{dz} \end{aligned} \tag{4.4}$$

Now it is seen that the first solution to Eqs (4.3) can be expressed in the matrix form:
— for $R_1^2 < 0$, $\overline{R}_1^2 = -R_1^2$

$$\begin{bmatrix} F_1 \\ G_1 \end{bmatrix} = \begin{bmatrix} \cosh(\overline{R}_1 z) & \sinh(\overline{R}_1 z) \\ \dfrac{\overline{R}_1}{\alpha^2-\beta^2}\sinh(\overline{R}_1 z) & \dfrac{\overline{R}_1}{\alpha^2-\beta^2}\cosh(\overline{R}_1 z) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \tag{4.5}$$

— for $R_1^2 > 0$

$$\begin{bmatrix} F_1 \\ G_1 \end{bmatrix} = \begin{bmatrix} \cos(R_1 z) & \sin(R_1 z) \\ -\dfrac{R_1}{\alpha^2-\beta^2}\sin(R_1 z) & \dfrac{R_1}{\alpha^2-\beta^2}\cos(R_1 z) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \tag{4.6}$$

The second rearrangement of Eqs (4.2) is as follows

$$\begin{aligned} &-(\lambda_L+2\mu)\frac{d^2G}{dz^2} - [(\lambda_L+2\mu)(\alpha^2-\beta^2)+\rho\omega^2]G + (\lambda_L+\mu)\underline{\Big(\frac{d^2G}{dz^2}+\frac{dF}{dz}\Big)} = 0 \\ &(\lambda_L+2\mu)\frac{d^2F}{dz^2} + [(\lambda_L+2\mu)(\alpha^2-\beta^2)+\rho\omega^2]F + \\ &\qquad -(\lambda_L+\mu)(\alpha^2-\beta^2)\underline{\Big(\frac{dG}{dz}+F\Big)} = 0 \end{aligned} \tag{4.7}$$

Directly from Eqs (4.7), one obtains

$$\begin{aligned} &\frac{d^2G}{dz^2} = -\Big(\alpha^2-\beta^2+\frac{\rho\omega^2}{\lambda_L+2\mu}\Big)G \equiv -R_2^2 G \\ &R_2^2 = \alpha^2-\beta^2+\frac{\rho\omega^2}{\lambda_L+2\mu} \\ &\frac{d^2F}{dz^2} = -\Big(\alpha^2-\beta^2+\frac{\rho\omega^2}{\lambda_L+2\mu}\Big)F \equiv -R_2^2 F \qquad F=-\frac{dG}{dz} \end{aligned} \tag{4.8}$$

Finally, it is seen that the second solution to Eqs (4.2) can be expressed in the following matrix form:

— for $R_2^2 < 0$, $\overline{R}_2^2 = -R_2^2$

$$\begin{bmatrix} F_2 \\ G_2 \end{bmatrix} = \begin{bmatrix} \cosh(\overline{R}_2 z) & \sinh(\overline{R}_2 z) \\ -\dfrac{1}{\overline{R}_2}\sinh(\overline{R}_2 z) & -\dfrac{1}{\overline{R}_2}\cosh(\overline{R}_2 z) \end{bmatrix} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} \tag{4.9}$$

— for $R_2^2 > 0$

$$\begin{bmatrix} F_2 \\ G_2 \end{bmatrix} = \begin{bmatrix} \cos(R_2 z) & \sin(R_2 z) \\ -\dfrac{1}{R_2}\sin(R_2 z) & \dfrac{1}{R_2}\cos(R_2 z) \end{bmatrix} \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} \tag{4.10}$$

The constants D_l, $l = 1,2,3,4$, in (4.5), (4.6) and (4.9), (4.10) are unknown.

5. Idea of the new solution to the boundary problem – combination of the 2D and 3D fields

Let us assume the following relationships

$$\begin{aligned} \frac{d^2 g_i}{dz^2} &= \frac{d^2 G_i}{dz^2} \Leftrightarrow \beta_i^2 g_i = -R_i^2 G_i \\ \frac{d^2 f_i}{dz^2} &= \frac{d^2 F_i}{dz^2} \Leftrightarrow \beta_i^2 f_i = -R_i^2 F_i \qquad i = 1,2 \end{aligned} \tag{5.1}$$

The above assumption is one of new ideas in the paper. Equations (5.1) will be satisfied if the following equalities are valid

$$\begin{aligned} &\beta_i^2 = -R_i^2 \quad \wedge \quad g_i \equiv G_i \quad \wedge \quad f_i \equiv F_i \quad \Rightarrow \\ &\Rightarrow \quad \frac{d^2 g_i}{dz^2} = \frac{d^2 G_i}{dz^2} \quad \wedge \quad \frac{d^2 f_i}{dz^2} = \frac{d^2 F_i}{dz^2} \qquad i = 1,2 \end{aligned} \tag{5.2}$$

It is evident that Eqs (5.1) will be satified if β^2 is defined as follows

$$\begin{aligned} \beta_1^2 = -R_1^2 \quad &\Leftrightarrow \quad \beta^2 = 2\alpha^2 \\ \beta_2^2 = -R_2^2 \quad &\Leftrightarrow \quad \beta^2 = 2\alpha^2 + \rho\omega^2\Big(\frac{1}{\lambda_L + 2\mu} - \frac{1}{\lambda + 2\mu}\Big) \end{aligned} \tag{5.3}$$

If $\lambda_L \approx \lambda$, as assumed in the further text, we can write down

$$\beta_1^2 = -R_1^2 \quad \wedge \quad \beta_2^2 = -R_2^2 \quad \Leftrightarrow \quad \beta^2 = 2\alpha^2 \tag{5.4}$$

It is noted however that the final form of the solution proposed here is the same irrespective of applying or omiting the assumption $\lambda_L \cong \lambda$.

Due to Eqs (5.1)-(5.4), one can write displacements (4.1) by replacing $G_i(z)$ with $g_i(z)$ and $F_i(z)$ with $f_i(z)$, for $i = 1,2$, respectively

$$\begin{aligned} &U_{xi} = -g_i(z)Y(y)\frac{dX^{(H)}}{dx}T(t) && U_{yi} = -g_i(z)\frac{dY}{dy}X^{(H)}T(t) \\ &U_{zi} = f_i(z)Y(y)X^{(H)}T(t) && i = 1,2 \\ &\frac{d^2X^{(H)}}{dx^2} = \alpha^2 X^{(H)} && \alpha^2 > 0 \qquad \frac{d^2Y}{dy^2} = -2\alpha^2 Y \end{aligned} \tag{5.5}$$

It is explained here that the function Y is assumed to be even, i.e.

$$Y(\sqrt{2}\alpha y) = Y(-\sqrt{2}\alpha y) \qquad\qquad Y = \cos(\sqrt{2}\alpha y) = \cos[\alpha L(\sqrt{2}y/L)] \tag{5.6}$$

It is obvious that for a sufficiently small y, the following approximations are valid

$$Y \cong 1 \qquad\qquad \frac{dY}{dy} \cong 0 \tag{5.7}$$

Approximations (5.7) imply a limitation of the model proposed here to a narrow structure. After taking into account (5.7), one obtains approximations of diplacements (5.5)

$$\begin{aligned} &U_{xi} \cong -g_i(z)\frac{dX^{(H)}}{dx}T(t) && U_{yi} \cong 0 && U_{zi} \cong f_i(z)X^{(H)}T(t) \\ &i = 1,2 && \frac{d^2X^{(H)}}{dx^2} = \alpha^2 X^{(H)} && \alpha^2 > 0 \end{aligned} \tag{5.8}$$

Let us assume the following (total) displacement field within the isotropic layer

$$\begin{aligned} &\widetilde{u}_x = \sum_i \widetilde{u}_{xi} = \sum_i (u_{xi} - U_{xi}) \cong -\sum_i g_i(z)\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t) && \widetilde{u}_{yi} \cong 0 \\ &\widetilde{u}_z = \sum_i \widetilde{u}_{zi} = \sum_i (u_{zi} - U_{zi}) \cong \sum_i f_i(z)(X^{(T)} - X^{(H)})T(t) && i = 1,2 \end{aligned} \tag{5.9}$$

Assumption (5.9) is the next new idea of the solution presented here. It is stated that the 3D components of the solutions (of the equations of motion) derived in Section 4 do not occur in the final, total displacement field (5.9). They have "disapeared" due to assumptions (5.1) and (5.7). The only trace of including the 3D components into field (5.9) is the function $X^{(H)}$ and its derivative. We further assume, as in Karczmarzyk (1999), that the functions of the variable x for the symmetric (about the mid-span) vibration are as follows

$$X^{(T)} = \frac{\cos(\alpha x)}{\cos\frac{\alpha L}{2}} \qquad\qquad X^{(H)} = \frac{\cosh(\alpha x)}{\cosh\frac{\alpha L}{2}} \tag{5.10}$$

When the anti-symmetric (about the mid-span) vibrations are considered, the functions of the variable x are defined as follows

$$X^{(T)} = \frac{\sin(\alpha x)}{\sin\frac{\alpha L}{2}} \qquad\qquad X^{(H)} = \frac{\sinh(\alpha x)}{\sinh\frac{\alpha L}{2}} \tag{5.11}$$

It is noted that functions (5.10), (5.11) are the eigenfunctions within the classical Bernoulli-Euler theory of beam. It is seen that irrespective of the type of vibration, the following equalities are satisfied, for $x = \pm L/2$

$$X^{(T)}(x = \pm L/2) - X^{(H)}(x = \pm L/2) = 0 \quad \Leftrightarrow \quad \widetilde{u}_{zi}(x = \pm L/2) = 0 \tag{5.12}$$

The right-hand side of Eq. (5.12) is one of the edge(s) boundary conditions for the perfect clamping of the edge(s). It means that irrespective of value of the variable z, the transversal (out-of-plane) vibrational displacement at the edges $x = \pm L/2$ is equal to zero. It is noticed that the derivative of the transverse displacement equals to zero at $x = \pm L/2$ for any value of z.

The second edge boundary condition for the perfect clamping solution is as follows

$$\widetilde{u}_{xi}(x = \pm L/2) = 0 \quad \Leftrightarrow \quad \left.\frac{dX^{(T)}}{dx}\right|_{x=\pm L/2} - \left.\frac{dX^{(H)}}{dx}\right|_{x=\pm L/2} = 0 \tag{5.13}$$

After substituting functions (5.10) into the right-hand side Eq. (5.13), one obtains the following transcendental equation for the symmetric modes of vibration enabling us to calculate α

$$\frac{\sin\frac{\alpha L}{2}}{\cos\frac{\alpha L}{2}} + \frac{\sinh\frac{\alpha L}{2}}{\cosh\frac{\alpha L}{2}} = 0 \tag{5.14}$$

When functions (5.11) are used, the right-hand side Eq. (5.13) is transformed to the form

$$\frac{\cos\frac{\alpha L}{2}}{\sin\frac{\alpha L}{2}} - \frac{\cosh\frac{\alpha L}{2}}{\sinh\frac{\alpha L}{2}} = 0 \tag{5.15}$$

In the literature, there are the following approximate values of α satisfying Eqs (5.14) and (5.15), i.e.:

— for symmetric modes

$$\frac{\alpha_1 L}{2} \cong 2.365 \qquad \frac{\alpha_k L}{2} \cong \frac{(4k-1)\pi}{4} \qquad k = 2,3,4,\dots$$
$$\frac{\alpha_1 L}{2} \cong 2.365 \qquad \frac{\alpha_m L}{2} \cong \frac{(2m+1)\pi}{4} \qquad m = 3,5,7,\dots \tag{5.16}$$

— for anti-symmetric modes

$$\frac{\alpha_1 L}{2} \cong 3.927 \qquad \frac{\alpha_l L}{2} \cong \frac{(4l+1)\pi}{4} \qquad l = 2,3,4,\dots$$
$$\frac{\alpha_2 L}{2} \cong 3.927 \qquad \frac{\alpha_m L}{2} \cong \frac{(2m+1)\pi}{4} \qquad m = 4,6,8,\dots \tag{5.17}$$

6. Through-the-thickness boundary and compatibility equations and a numerical form of the boundary value problem

Upon the basis of the displacement field, defined by (5.9), (3.5), (3.6), (3.9) and (3.10), we are able to derive the total strain field and, after its substitution to the Hooke law, we obtain the following expressions for the total stresses within the layer $(i = 1, 2)$

$$\begin{aligned}
\widetilde{\sigma}_{zx} &= \sum_i \widetilde{\sigma}_{zxi} = \sum_i \mu\Big(f_i - \frac{dg_i}{dz}\Big)\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t) = \\
&= \sum_i S_{zxi}\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t) \\
\widetilde{\sigma}_{zz} &= \sum_i \widetilde{\sigma}_{zzi} = \sum_i \Big(\lambda\alpha^2 g_i + (\lambda+2\mu)\frac{df_i}{dz}\Big)(X^{(T)} - X^{(H)})T(t) = \\
&= \sum_i S_{zzi}(X^{(T)} - X^{(H)})T(t)
\end{aligned} \tag{6.1}$$

It is seen from Eqs (6.1) that irrespective of value of the variable z (included in the functions g_i, f_i), the total shear stresses and normal (out-of-plane) stresses are equal to zero at the edges $x = \pm L/2$. In order to write the explicit expressions for the stresses, we use the following relationships for the stress components, i.e.

$$\begin{aligned}
\widetilde{\sigma}_{zx1} &= \mu\Big(1 + \frac{\beta_1^2}{\alpha^2}\Big)f_1\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t) \\
\widetilde{\sigma}_{zz1} &= 2\mu\frac{df_1}{dz}(X^{(T)} - X^{(H)})T(t)
\end{aligned}$$

$$
\begin{aligned}
&\widetilde{\sigma}_{zx2} = 2\mu f_2\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t)\\
&\widetilde{\sigma}_{zz2} = \Big[2\mu + \lambda\Big(1 - \frac{\alpha^2}{\beta_2^2}\Big)\Big]\frac{df_2}{dz}(X^{(T)} - X^{(H)})T(t) \qquad (6.2)\\
&\widetilde{\sigma}_{zx1} = S_{zx1}\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t) \qquad S_{zx1} = \mu\Big(1 + \frac{\beta_1^2}{\alpha^2}\Big)f_1\\
&\widetilde{\sigma}_{zz1} = S_{zz1}(X^{(T)} - X^{(H)})T(t) \qquad S_{zz1} = 2\mu\frac{df_1}{dz}\\
&\widetilde{\sigma}_{zx2} = S_{zx2}\Big(\frac{dX^{(T)}}{dx} - \frac{dX^{(H)}}{dx}\Big)T(t) \qquad S_{zx2} = 2\mu f_2\\
&\widetilde{\sigma}_{zz2} = S_{zz2}(X^{(T)} - X^{(H)})T(t)\\
&S_{zz2} = \Big[2\mu + \lambda\Big(1 - \frac{\alpha^2}{\beta_2^2}\Big)\Big]\frac{df_2}{dz}
\end{aligned}
$$

If we use (3.5), (3.6), (3.9), (3.10) and (6.2), we obtain $S_{zx} = S_{zx1} + S_{zx2}$ and $S_{zz} = S_{zz1} + S_{zz2}$ in the explicit form:

— for $\beta_1^2 > 0$, $\beta_2^2 > 0$

$$
\begin{bmatrix} S_{zz} \\ S_{zx} \end{bmatrix} = \begin{bmatrix} 2\mu\beta_1\sinh(\beta_1 z) & 2\mu\beta_1\cosh(\beta_1 z) & \mathcal{A}\sinh(\beta_2 z) & \mathcal{A}\cosh(\beta_2 z) \\ \mathcal{B}\cosh(\beta_1 z) & \mathcal{B}\sinh(\beta_1 z) & 2\mu\cosh(\beta_2 z) & 2\mu\sinh(\beta_2 z) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{6.3}
$$

where

$$
\mathcal{A} = \Big[2\mu + \lambda\Big(1 - \frac{\alpha^2}{\beta_2^2}\Big)\Big]\beta_2 \qquad \mathcal{B} = \mu\Big(1 + \frac{\beta_1^2}{\alpha^2}\Big)
$$

— for $\overline{\beta}_1^2 = -\beta_1^2 > 0$, $\overline{\beta}_2^2 = -\beta_2^2 > 0$

$$
\begin{bmatrix} S_{zz} \\ S_{zx} \end{bmatrix} = \begin{bmatrix} -2\mu\overline{\beta}_1\sin(\overline{\beta}_1 z) & 2\mu\overline{\beta}_1\cos(\overline{\beta}_1 z) & -\overline{\mathcal{A}}\sin(\overline{\beta}_2 z) & \overline{\mathcal{A}}\cos(\overline{\beta}_2 z) \\ \overline{\mathcal{B}}\cos(\overline{\beta}_1 z) & \overline{\mathcal{B}}\sin(\overline{\beta}_1 z) & 2\mu\cos(\overline{\beta}_2 z) & 2\mu\sin(\overline{\beta}_2 z) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{6.4}
$$

where

$$
\overline{\mathcal{A}} = \Big[2\mu + \lambda\Big(1 + \frac{\alpha^2}{\overline{\beta}_2^2}\Big)\Big]\overline{\beta}_2 \qquad \overline{\mathcal{B}} = \mu\Big(1 - \frac{\overline{\beta}_1^2}{\alpha^2}\Big)
$$

Analogously, $g = g_1 + g_2$, $f = f_1 + f_2$ in the explicit form are as follows:

— for $\beta_1^2 > 0$, $\beta_2^2 > 0$

$$\begin{bmatrix} -g \\ f \end{bmatrix} = \begin{bmatrix} \dfrac{\beta_1}{\alpha^2}\sinh(\beta_1 z) & \dfrac{\beta_1}{\alpha^2}\cosh(\beta_1 z) & \dfrac{1}{\beta_2}\sinh(\beta_2 z) & \dfrac{1}{\beta_2}\cosh(\beta_2 z) \\ \cosh(\beta_1 z) & \sinh(\beta_1 z) & \cosh(\beta_2 z) & \sinh(\beta_2 z) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{6.5}$$

— for $\overline{\beta}_1^2 = -\beta_1^2 > 0$, $\overline{\beta}_2^2 = -\beta_2^2 > 0$

$$\begin{bmatrix} -g \\ f \end{bmatrix} = \begin{bmatrix} -\dfrac{\overline{\beta}_1}{\alpha^2}\sin(\overline{\beta}_1 z) & \frac{\overline{\beta}_1}{\alpha^2}\cos(\overline{\beta}_1 z) & \dfrac{1}{\overline{\beta}_2}\sin(\overline{\beta}_2 z) & -\dfrac{1}{\overline{\beta}_2}\cos(\overline{\beta}_2 z) \\ \cos(\overline{\beta}_1 z) & \sin(\overline{\beta}_1 z) & \cos(\overline{\beta}_2 z) & \sin(\overline{\beta}_2 z) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} \tag{6.6}$$

For a particular mode of free vibration, the symbol α in expressions (6.3)-(6.6) must be replaced with α_m, $m = 1, 2, 3$ – see $(5.16)_2$, while the symbol ω (frequency) appearing in β_1, β_2 must be replaced with ω_m (mth eigenfrequency).

It is noticed that any limitation on parameters of the beam (such as thicknesses, densities etc.) as well as any restriction on the ratios $h_{(j)}/h_{(j+1)}$, $\rho_{(j)}/\rho_{(j+1)}$, $\mu_{(j)}/\mu_{(j+1)}$, L/h_t, $h_t = h_1 + h_2 + \ldots + h_p$, etc., have not been introduced into the model. Therefore, it can be applied for the vibration analysis of both the multi-layered, slender and thickset, sandwich beams and the classical laminated beams consisting of stiffness-comparable layers. Obviously, this statement is true provided that the edge fixing of the structure assures perfect edge clamping boundary conditions (2.8).

It is noted (repeated) that the stress and displacement fields derived in Sections 3-6 occur in an isotropic, homogeneous (let us say in jth) layer of the multi-layered structure. If we want to have such fields for any $(j+k)$th layer, we have to substitute the material parameters ρ, λ and μ for this particular layer into the all above expressions – in particular into formulas (6.3)-(6.6). The distinction between the stress and displacement fields for two different layers, let us say jth and $(j+k)$th, is seen in the following exemplary expressions

$$\begin{aligned} &\frac{d^2 g_{(j)}}{dz^2} = \Big(\alpha^2 - \frac{\rho_{(j)}\omega^2}{\mu_{(j)}}\Big) g_{(j)} \equiv \beta_{1(j)}^2 g_{(j)} \qquad \beta_{1(j)}^2 = \alpha^2 - \frac{\rho_{(j)}\omega^2}{\mu_{(j)}} \\ &\frac{d^2 g_{(j+k)}}{dz^2} = \Big(\alpha^2 - \frac{\rho_{(j+k)}\omega^2}{\mu_{(j+k)}}\Big) g_{(j+k)} \equiv \beta_{1(j+k)}^2 g_{(j+k)} \\ &\beta_{1(j+k)}^2 = \alpha^2 - \frac{\rho_{(j+k)}\omega^2}{\mu_{(j+k)}} \end{aligned} \tag{6.7}$$

In order to obtain a numerical form of the boundary problem, we have to use the derived expressions for the displacements and stresses and substitute it into through-the-thickness boundary conditions (2.6) and compatibility equations (2.7). The creation and the structure of the resulting matrix of the eigenvalue problem for a three-layered sandwich beam is illustrated in the following formal expression (scheme)

$$\begin{bmatrix} z=z_1 & \widetilde{\sigma}_{zz(1)}=0 \\ & \widetilde{\sigma}_{zx(1)}=0 \\ z=z_2 & \widetilde{\sigma}_{zz(1)}=\widetilde{\sigma}_{zz(2)} \\ & \widetilde{\sigma}_{zx(1)}=\widetilde{\sigma}_{zx(2)} \\ & \widetilde{u}_{z(1)}=\widetilde{u}_{z(2)} \\ & \widetilde{u}_{x(1)}=\widetilde{u}_{x(2)} \\ z=z_3 & \widetilde{\sigma}_{zz(2)}=\widetilde{\sigma}_{zz(3)} \\ & \widetilde{\sigma}_{zx(2)}=\widetilde{\sigma}_{zx(3)} \\ & \widetilde{u}_{z(2)}=\widetilde{u}_{z(3)} \\ & \widetilde{u}_{x(2)}=\widetilde{u}_{x(3)} \\ z=z_4 & \widetilde{\sigma}_{zz(3)}=0 \\ & \widetilde{\sigma}_{zx(3)}=0 \end{bmatrix} \equiv \begin{bmatrix} ++++ & & \\ ++++ & & \\ ++++ & ++++ & \\ ++++ & ++++ & \\ ++++ & ++++ & \\ ++++ & ++++ & \\ & ++++ & ++++ \\ & ++++ & ++++ \\ & ++++ & ++++ \\ & ++++ & ++++ \\ & & ++++ \\ & & ++++ \end{bmatrix} \begin{bmatrix} C_{1(1)} \\ C_{2(1)} \\ C_{3(1)} \\ C_{4(1)} \\ C_{1(2)} \\ C_{2(2)} \\ C_{3(2)} \\ C_{4(2)} \\ C_{1(3)} \\ C_{2(3)} \\ C_{3(3)} \\ C_{4(3)} \end{bmatrix} = \mathbf{0} \equiv$$

$$\equiv \mathbf{A}\boldsymbol{C} = \mathbf{0} \tag{6.8}$$

The pluses in the matrix $\mathbf{A}$ denote, in a general case, the non-zero elements of the matrix. After solving the equation $\det \mathbf{A} = 0$, one obtains the eigenfrequencies ω_m. There are many ways for numerical solving of the eigenvalue equation. One of them is using a standard software module for evaluation the determinants. The other way may be transformation of the matrix to the smallest dimension and then obtaining a computational code. It is noted that the whole eigenvalue problem is expressed by one of Eqs (5.14), (5.15) and Eq. (6.8).

If the matrix $\mathbf{0}$ in Eq. (6.8) is replaced with a non-zero matrix containing components of sinusoidally varying loads of the structure, Eq. (6.8) together with one of Eqs (5.14) and (5.15) will be the final, matrix form of the boundary problem of the forced vibration (in this case, the loads will be expanded into series (5.14) and (5.15)). Anyway, the boundary problem in its final form consists of two uncoupled Eqs: (5.14) or (5.15) and (6.8).

The dimension of the square matrix $\mathbf{A}$ for the structure consisting of p layers is equal to $4p \times 4p$. It is easy to show that for the structure symmetric about the middle plane, the matrix dimension can be decreased two times. For the symmetric structure, the boundary problem (6.8) splits into two subproblems: one for the transverse flexural vibration and the other for the transverse breath problem. Thus, for the classical sandwich beam symmetric about the mid-plane (as shown in Fig. 1a), whose outer layers are of the same thickness and the same materials, the matrix $\mathbf{A}$ dimension is equal to $6 \times 6 \equiv 2p \times 2p$, $p = 3$.

Let us finally note that after substituting into (3.5), (3.6), (3.9), (3.10), (6.3)-(6.6) and (6.8) $\alpha_m = m\pi/L$ for $m = 1, 2, 3, \ldots$, after replacing the function $X^{(T)} - X^{(H)}$ in (5.9) by the sinus Fourier series function $X(x) = \sin(m\pi x/L)$ and the function $d(X^{(T)} - X^{(H)})/dx$ in (5.9) by the function $\alpha_m \cos(m\pi x/L)$, we obtain a local 2D solution to the sinusoidal vibration problem of the simply supported multi-layer sandwich beam (Karczmarzyk, 1999). This advantageous property of the model proposed here shows its efficiency and its (limited) similarity to the classical Bernoulli-Euler theory of homogeneous beam based on the assumption of plane cross-sections.

The opposite idea of replacing the sinus Fourier series functions with the Bernoull-Euler eigenfunctions was first proposed and numerically verified by the present author in Karczmarzyk (2005). It was only an intuitive proposition. In the present paper, the idea checked in Karczmarzyk (2005) has been justified for the first time mathematically. Due to the present paper we know, among other things, that the model is exact and accurate only for sufficiently narrow sandwich structures (beams) and not for wide rectangular plates with two parallel edges clamped and the other edges free.

7. Numerical results and comparisons

In order to check the new model, some computations have been made for the input data given in Table 1. The results are listed and compared in Table 2. Eight eigenfrequencies ω_{SK} obtained after numerical solving of eigenvalue problem (5.14), (5.15), (6.8) for the C-C beam, for the input data given in Raville *et al.* (1961), Sakiyama *et al.* (1996), Sokolinsky and Nutt (2002), Howson and Zare (2005), are presented. Apart from the new results, the reader will find in Table 2 the eigenfrequencies presented (for the structure) in the literature, i.e., ω_{ExExp} – obtained experimentally (Raville *et al.*, 1961) and listed in Sakiyama *et al.* (1996), Sokolinsky and Nutt (2002), Howson and Zare (2005) and ω_{RAV}, ω_{SAK}, ω_{VSS}, ω_{HZ} computed according to the models by Raville *et al.* (1961), Sakiyama *et al.* (1996), Sokolinsky and Nutt (2002), Howson and Zare (2005), respectively. The percentage differences between the results predicted by the different models are shown in Fig. 2.

Table 1. Parameters of the classical three-layered sandwich beam of length $L = 1.21872\,\mathrm{m}$

Parameter	h [mm]	E [Pa]	ν [–]	ρ [kg·m^{-3}]	μ [Pa]	λ [Pa]
Layer 1	0.40624	$0.6890 \cdot 10^{10}$	0.33	2687.3	$0.2590 \cdot 10^{10}$	$0.2551 \cdot 10^{10}$
Layer 2	6.34750	$0.1833 \cdot 10^{9}$	0.33	119.69	$0.6891 \cdot 10^{8}$	$0.6788 \cdot 10^{8}$
Layer 3	0.40624	$0.6890 \cdot 10^{10}$	0.33	2687.3	$0.2590 \cdot 10^{10}$	$0.2551 \cdot 10^{10}$

Table 2. Flexural eigenfrequencies of the sandwich C-C beam according to different models

Vibr. [rad·s^{-1}]	Mode (m)							
	1(s)	2(a)	3(s)	4(a)	5(s)	6(a)	7(s)	8(a)
ω_{SK}	220.50	597.91	1144.0	1834.7	2645.1	3551.4	4537.4	5568.3
ω_{ExExp}	–	–	1165.5	1761.2	2509.5	3362.8	4277.0	5448.8
ω_{RAV}	229.88	617.81	1173.8	1872.0	2685.6	3596.0	4575.3	5618.7
ω_{VSS}	217.40	584.96	1113.4	1776.9	2552.9	3419.9	4358.6	5353.3
ω_{SAK}	210.88	567.77	1081.2	1727.3	2484.5	3332.2	4252.7	5230.3
ω_{HZ}	217.38	584.96	1113.1	1776.8	2553.0	3420.1	4359.2	5354.2

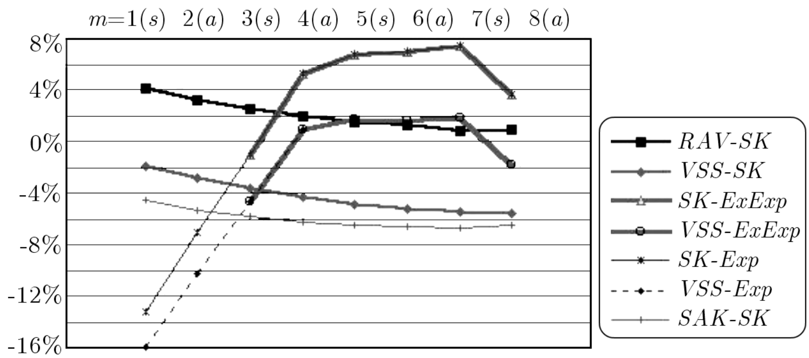

Fig. 2. Percentage differences between eigenfrequencies listed in Table 2

It is explained that the notation $m = [1(s), 3(s), 5(s), 7(s)]$ is used in Fig. 2 to denote vibration symmetric about the middle of the beam span (mid-span), while the notation $m = [2(a), 4(a), 6(a), 6(a)]$ refers to vibration anti-symmetric about the mid-span. The abbreviations used in Fig. 2 are defined as follows: $RAV - SK = 100(\omega_{RAV} - \omega_{SK})/\omega_{RAV}$, $VSS - SK = 100(\omega_{VSS} - \omega_{SK})/\omega_{VSS}$, $SAK - SK = 100(\omega_{SAK} - \omega_{SK})/\omega_{SAK}$, $SK - ExExp = 100(\omega_{SK} - \omega_{ExExp})/\omega_{ExExp}$, $VSS - ExExp = 100(\omega_{VSS} - \omega_{ExExp})/\omega_{ExExp}$.

The abbreviations $SK - Exp$ and $VSS - Exp$ do not stand for definitions analogous to the above outlined, but they are approximate, potential elongations of the curves $SK - ExExp$ and $VSS - ExExp$, respectively. Unfortunately, the eigenfrequencies of the first symmetric mode and first unsymmetric mode of vibration are not explicitely given (tabulated) in the literature (Raville *et al.*, 1961; Sakiyama *et al.*, 1996; Sokolinsky and Nutt, 2002; Howson and Zare, 2005) and, therefore, the present author was not able to calculate $SK - Exp = 100(\omega_{SK} - \omega_{ExExp})/\omega_{ExExp}$, $VSS - Exp = 100(\omega_{VSS} - \omega_{ExExp})/\omega_{ExExp}$ for the two lower modes.

It is seen from Table 2 and in Fig. 2 that the following relationships, concerning the computational eigenfrequencies, are observed, $\omega_{RAV} > \omega_{SK} > \omega_{VSS} > \omega_{SAK}$. It is noted that the eigenfrequencies ω_{VSS} are almost equal to the eigenfrequencies ω_{HZ}. This means that models presented in Sokolinsky and Nutt (2002), Howson and Zare (2005) are compatible. The results predicted by the new model and models by Raville *et al.* (1961), Sokolinsky and Nutt (2002), Howson and Zare (2005) are close. The curves $RAV - SK$ and $VSS - SK$ are parallel and distant approximately by 6%. The model by Sakiyama *et al.* (1996) gives lower eigenfrequencies than the new model and the models by Sokolinsky and Nutt (2002), Howson and Zare (2005).

However, the comparisons of the computational results and the existing experimental results (Raville *et al.*, 1961), see the curves $SK - ExExp$ and $VSS - ExExp$, suggest high inaccuracy of all the models (see Raville *et al.*, 1961; Sakiyama *et al.*, 1996; Sokolinsky and Nutt, 2002; Howson and Zare, 2005; and the new one) for the lower modes of vibration. This is suggested by the elongations $SK - Exp$ and $VSS - Exp$. The computed eigenfrequencies for the lower modes of vibration are probably much lower than the corresponding measured values. The first mode eigenfrequency according to the model by Sokolinsky and Nutt (2002) seems to be some 16% lower than the expected experimental value. It is difficult to explain this phenomenon exactly, but one of potential explanations is suggested here. Most probably, the existing experimental eigenfrequencies ω_{ExExp}, presented in Raville *et al.* (1961) and listed in Sakiyama *et al.* (1996), Sokolinsky and Nutt (2002), Howson and Zare (2005), were measured for the vibrating sandwich beam with fixed (or free – see e.g. Nilsson and Nilsson, 2002) edges (ends), which in any case were not perfectly clamped – see definition of boundary conditions (2.8).

8. Conclusions

A new two-dimmensional, single series local model of transverse vibration of a multi-layered one-span sandwich beam with perfectly clamped edges has been presented in the present paper. It is derived in the local theory of linear elastodynamics after satisying all the rigorous requirements of the theory.

The eigenfunctions for the C-C sandwich multi-layered beam within the new model are the same as in the classical theory of homogeneous beam, based on the assumption of plane cross-sections.

The model is applicable to beams composed of any number of layers irrespective of their parameters. It is applicable to structures with both edges clamped or simply supported, after replacing (if it is necessary) the Bernoulli-Euler eigenfunctions with the sinus Fourier series functions.

In the case of a beam symmetric about its middle plane, the model splits into two submodels: one for the transverse flexural anti-symmetric vibration and the second for the transverse symmetric (breathing) vibration.

The model predicts the eigenfrequencies close to the counterparts predicted by different former models published by other authors.

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Nowy dwuwymiarowy pojedyńczo szeregowy model drgań poprzecznych wielowarstwowej belki sandwiczowej z idealnie utwierdzonymi krawędziami

Streszczenie

W tej pracy jest przedstawiony nowy dwuwymiarowy, pojedyńczo szeregowy, lokalny model drgań poprzecznych wielowarstwowej, jednoprzęsłowej belki sandwiczowej, złożonej z warstw izotropowch, z idealnie utwierdzonymi końcami. Model ten, otrzymany w ramach lokalnej teorii liniowej elastodynamiki, składa się z dwóch pól dwuwymiarowych i dwóch aproksymacji pól trójwymiarowych spełniających ściśle równania ruchu oraz warunki zgodności Saint-Venanta. W modelu zostały spełnione wszystkie warunki brzegowe po grubości, jak również lokalne warunki ciągłości (przemieszczeń i naprężeń) między przylegającymi warstwami. Uwzględniono deplanacje przekrojowe, jak też poprzeczne podatności każdej warstwy i dlatego model ten jest stosowalny zarówno do klasycznej trójwarstwowej belki sandwiczowej, jak i do wielowarstwowej struktury sandwiczowej czy laminatowej.

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