JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
**48**, 3, pp. 645-658, Warsaw 2010

# OPTIMAL AXIAL TENSION AND INTERNAL PRESSURE STABILIZING POST-BUCKLING PATH FOR CYLINDRICAL SHELLS UNDER TORSION

Jacek Krużelecki
Daniel Trybuła

*Cracow University of Technology, Institute of Applied Mechanics, Kraków, Poland*

*e-mail: Jacek.Kruzelecki@pk.edu.pl; DTrybula@interia.pl*

The effect of stabilization of unstable post-buckling behavior of a structure usually is obtained by changing its geometry. In this paper, a possibility of stabilization of the initially unstable post-buckling path for a cylindrical shell under torsion by application of additional independent loadings acting on the structure without changing the shape and size of the shell is investigated. It occurred that axial tension improves the resistance against buckling for the cylindrical shell under torsion and can stabilize the unstable post-buckling path. On the other hand, internal pressure does not stabilize the post-buckling path but it improves the resistance of such a structure against instability.

*Key words:* post-buckling path, stabilization, cylindrical shell, torsion, tension, pressure

## 1. Introduction

Thin-walled shells under different states of loadings can be subjected to loss of stability, and their post-buckling behavior can be unstable. This means that the loss of stability of such structures is associated with a snap-through. It seems to be very dangerous, since it can lead to very large displacements and, finally, to destruction of the structure. The post-buckling path for a cylindrical shell under twisting moment is unstable. This type of behavior was discussed, for example, by Volmir (1967) and Yamaki (1984).

The effect of modification of the post-buckling behavior is in most cases obtained by changing size variables, which are usually dimensions of the designed elements. It requires certain modification of the standard structural optimization problem under stability constraints. Namely, post-buckling constraints of a special form added to the formulation of the optimization problem enable modification of the post-buckling path, and the stable post-buckling path can be created, even in the case of unstable behavior of the reference structure. This type of problems were considered, for example, by Perry and Gürdal (1995), Pietrzak (1996), Bochenek (1997, 2001), Cardoso *et al.* (1997), Mróz and Piekarski (1998), Suasa *et al.* (1999), Jasion (2009).

In some practical engineering applications, changes of shape and dimensions of a structure are undesirable or even impossible, for example, if the structure is already designed or it must have prescribed shape and dimensions. Then, a non-standard approach to such a problem is necessary.

In this paper, an alternative concept is applied, namely stabilization of the post-buckling path is obtained by application of additional loadings acting on the shell without changing geometry of the optimized structure. These loadings, in general, can be either active forces applied to the structure or passive ones (reactions of additional supports), or both active and passive forces acting simultaneously. The idea of increasing the buckling load for shells by application of initial pretensions, which lead to stiffening of structures, was presented, for example, in papers by Weingarten *et al.* (1965), Lofblad (1959) (axial compression and internal pressure), Haris *et al.* (1958) (axial compression and torsion, internal pressure), Berskowitz *et al.* (1967) (axial compression, torsion and external or internal pressure) in the late 1950's and 1960's. However, the influence of an initial pretension on the stabilization of an initially unstable post-buckling path has not been analysed at all. Such problems, for a finite-degree-of-freedom rod system that models behavior of a real shell structure under external pressure was considered by Bochenek and Krużelecki (2001). On the other hand, Krużelecki and Król (2006) and also Król *et al.* (2009) examined real cylindrical shells under external pressure, whereas Krużelecki and Trybuła (2007) investigated such shells under twisting moment. Some solutions were also presented by Bochenek and Krużelecki (2007). Those results showed that axial loadings can stabilize the initially unstable post-buckling path. Mathon and Limam (2006) experimentally examined the influence of internal pressure on buckling and the equilibrium path for a cylindrical shell under bending. They showed that the internal pressure and axial tension due to this pressure applied to the cylindrical shell can significantly increase the critical bending moment and even can stabilize of the post-critical path.

In the present paper, the stabilization of the post-buckling path for an elastic simply supported cylindrical shell under twisting moment is formulated as a certain modified non-standard problem of optimization. Two different types of stabilizing loadings are investigated, namely axial tension, internal pressure and both these loadings acting simultaneously. Calculations are performed using the ANSYS code for elastic deformations of shells of different lengths and thicknesses.

## 2. Formulation of the optimization problem

We consider an elastic simply supported at both ends cylindrical shell of the length $L$, radius $R$, constant thickness $h$, loaded by twisting moment $M$ applied to both ends of the structure. The problem of optimal design against instability is usually formulated as the maximisation of the minimal buckling load for the given total volume of the optimized structure. In general, it is a min-max problem under buckling constraints taken into account only in such a *standard* formulation. In this paper, an alternative approach is proposed, namely the results of post-buckling analysis and post-buckling constraints of a special form are implemented into the formulation of the design problem. Such an optimization problem with post-buckling and buckling constraints is called the *modified* problem. A general classification of such problems was given by Bochenek (2001).

In the present paper, it is assumed that stabilization of an unstable post-buckling path can be obtained by application of an additional axial load $L_{N_p} = N$ applied to the ends of the structure or by application of a uniformly distributed internal pressure $L_{N_p} = p$ or by both these loadings $L_{N_p} = (N,p)$ acting simultaneously (combined loadings), where $L_{N_p}$ denotes the loading parameter defining additional stabilizing loads. The problem of optimization can be stated as follows. The minimum value of the stabilizing loading parameter $L_{N_p}$ ($N$ or $p$ or $(N,p)$), which leads to stable behavior of the shell, is sought

$$\begin{aligned} &\text{Minimize} \quad L_{Np} \\ &\text{subject to} \quad \frac{\partial M}{\partial f}(f^*, L_{Np}) = \frac{\partial^2 M}{\partial f^2}(f^*, L_{Np}) = 0 \end{aligned} \tag{2.1}$$

where $f$ denotes, in general, characteristic displacement of the wall of the shell. The displacement $f^*$ in equality constraints $(2.1)_2$ refers to the horizontal inflexion point at the equilibrium path, Fig. 1. Conditions $(2.1)_2$ lead to elimination of the snap-through and, finally, one obtains a stable post-buckling path even if the original equilibrium path is unstable. It is shown in Fig. 1 where the thick line refers to the stable equilibrium path under the minimum stabilizing load $L_{N_p}$.

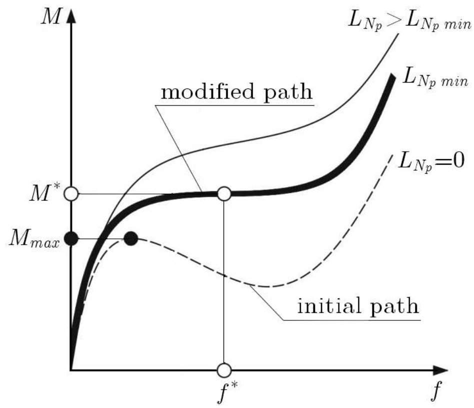


Fig. 1. Equilibrium paths for $L_{N_p} = 0$ and $L_{N_p} \neq 0$

This formulation of the optimization problem contains only one design variable $L_{N_p}$ (in three variants) and two constraints shown in equations $(2.1)_2$, which are imposed on the post-buckling state. They ensure stable behavior of the cylindrical shell under twisting moment. The condition of constant volume of the structure is automatically fulfilled because that formulation does not take into account the modification of the shell geometry.

The equilibrium path, which was obtained by numerical geometrically non-linear analysis, is given by discrete points. Therefore, constraints $(2.1)_2$ were verified numerically (for any $L_{N_p}$) using the finite differences method. To obtain the horizontal inflexion point at the equilibrium path, which refers to stabilization of the equilibrium path, one finds

$$M_{i-1} = M_i = M_{i+1} \tag{2.2}$$

where $M_j$ denote values of twisting moment in three neighboring points distant by $\Delta f$ from each other. That equality of moments was satisfied numerically with very good accuracy under additional conditions

$$M_{i-1} \leqslant M_i \leqslant M_{i+1} \qquad\qquad \frac{M_{i+1} - M_{i-1}}{M_i} \leqslant \varepsilon \tag{2.3}$$

where $\varepsilon$ is the assumed small tolerance.

The search for the minimum stabilizing load $L_{N_p}$, for which conditions (2.2) and (2.3) are satisfied and the horizontal inflexion point is obtained with the assumed accuracy, was performed using the algorithm presented in

Fig. 2. It starts from the assumed value of the stabilizing load $L_{N_{p1}}$ for which the equilibrium path is generated using the ANSYS code. If condition $(2.3)_1$ for the stabilizing load $L_{N_{pj}}$ is not satisfied in the vicinity of the expected horizontal inflexion point then its value is increased by using the relation $L_{N_{pj+1}} = \alpha L_{N_{pj}}$ $(\alpha > 1)$, and the new equilibrium path is generated for the new $L_{N_{pj+1}}$. Otherwise, condition $(2.3)_2$ is checked out. If that condition is satisfed, it means that the minimal stabilizing load $L_{N_{pj}} = L_{N_p}$ is found and the horizontal inflexion point is obtained with the assumed accuracy, otherwise $L_{N_{pj}}$ is decreased using the relation $L_{N_{pj+1}} = \beta L_{N_{pj}}$ $(\beta < 1)$, and the new equilibrium path is generated for the changed value of the stabilizing loading $L_{N_{pj+1}}$. Such a procedure is repeated until conditions (2.3) are satisfied or the final number of iterations $j_{max}$ is reached. During the iteration procedure, conditions (2.3) are verified not for the whole equilibrium path but only for that part of it which corresponds to sufficiently large displacements, $f \geqslant \widehat{f}$. The minimum displacement $\widehat{f}$ from which conditions (2.3) are checked out depends on $L_{N_{pj}}$ and $\widehat{f}$. It is apprioprately updated for each iteration. The displacemnts $\widehat{f} > f_1$, where $f_1$ corresponds to $M_{max}$ at the equilibrium path for $L_{N_p} = 0$, Fig. 2. The number of necassary iterations clearly depends on the starting point $L_{N_{p1}}$, but quite fast convergence was obtained assuming $\alpha = 1.05$, $\beta = 0.97$.

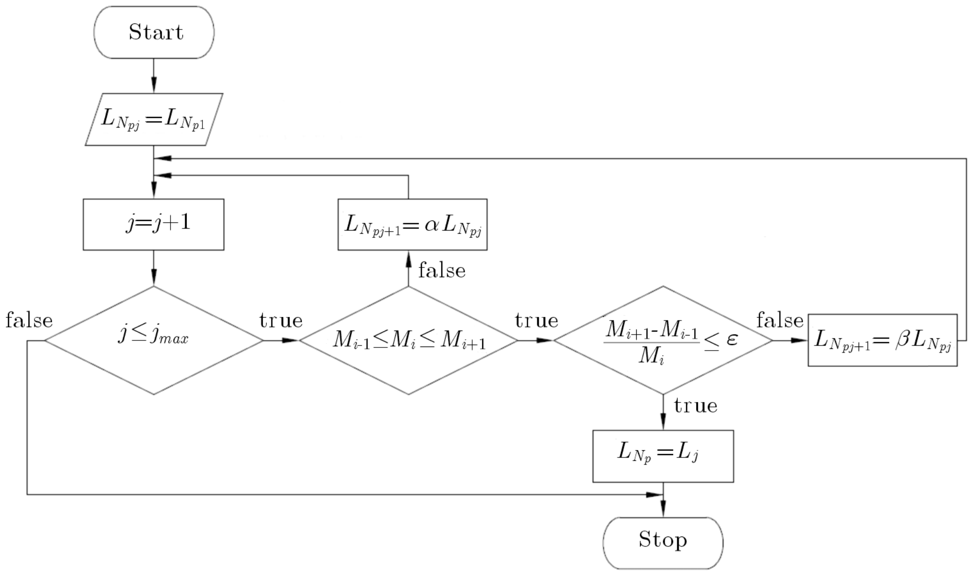


Fig. 2. Flowchart of the proposed concept of seeking for the minimum stabilizing load

## 3. Variations of stabilization

Stabilization of the post-buckling path for cylindrical shells can be achieved, at least, in three different ways, namely: by direct application of additional tensile loadings (active loadings), by imposing additional constraints connected with displacements (passive loadings) and by application of the 'mixed variant' consisting of both these types of loadings. These three variants of stabilization are discussed in details by Król *et al.* (2009) for cylindrical shells under external pressure. It occurred that the way connected with a direct application of additional loadings, called the 'active loadings', which are independent of deformations of a structure, is the most efficient one. This variant is applied in the present paper. In the case under consideration with two different independent stabilizing active loadings, namely an axial tensile force and internal pressure, there are three basic variants of application of additional loadings which can lead to stabilization of the post-buckling path:

(a) Application of the axial force $N$ only ($L_{N_p} = N$, pressure $p = 0$) to the ends of the shell before the twisting moment $M$ is applied. In this case, the minimum value of the axial force $N$ which stabilizes the post-buckling path is sought.

(b) Application of the internal pressure $p$ only ($L_{N_p} = p$, axial force $N = 0$) to the shell before the twisting moment $M$ is applied. In this case, the minimum value of the internal pressure $p$ which stabilizes the post-buckling path is sought.

(c) Simultaneous application of the axial force $N$ and internal pressure $p$ (combined loading $L_{N_p} = (N, p)$) to the shell before the twisting moment $M$ is applied. In this case, the minimum value of the loading parameter $L_{N_p}$ which stabilizes the post-buckling path is sought.

In each case (a), (b) and (c) discussed above, values of additional loadings ($N$, $p$, $(N, p)$) are independent of the twisting moment during deformations of the structure, and they are also independent of deformations themselves. These three variants of stabilization of the post-buckling path and the loading sequence are presented in Fig. 3.

## 4. Results of calculations

Calculations were performed for three variants of loadings discussed above, using the ANSYS code for elastic deformations of shells of different lengths

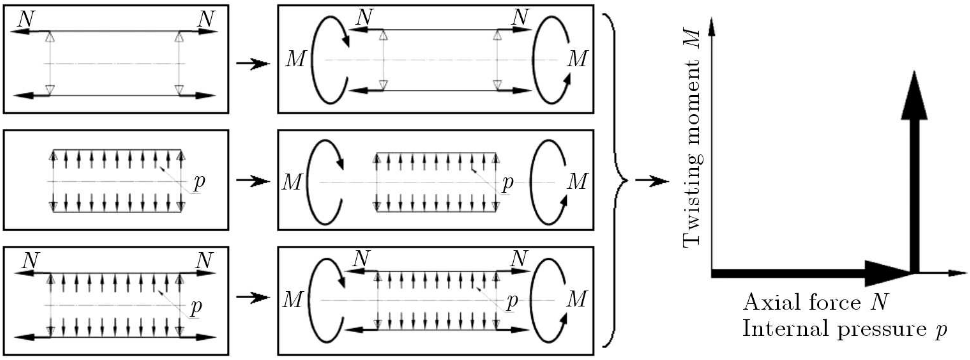


Fig. 3. Three variants of stabilization and loading sequence

and thicknesses. We present the results only for shells with the length parameters: $L/R = 1.0$, $2.0$ and the thickness parameters $h/R = 0.02$, $0.005$, assuming that the radius is constant, $R = 1\,\mathrm{m}$. The material is defined by the following material constants: Young modulus $E = 200\,\mathrm{GPa}$, Poisson's ratio $\nu = 0.3$. The reference stress $\sigma_0$ ($\sigma_0 = 225\,\mathrm{MPa}$) was applied to define the dimensionless axial stress $s = \sigma_z/\sigma_0 = N/(A\sigma_0)$ and dimensionless hoop stress $p^* = \sigma_\theta/\sigma_0 = pR/(h\sigma_0)$ used as a measure of the stabilizing axial force and internal pressure, respectively.

To obtain the equilibrium path using the ANSYS code, it was necessary to introduce small initial geometrical imperfections into a perfect structure. Then, the ANSYS standard 'Arc-Length' algorithm based on the Riks and the Newton-Raphson methods was used. The initial imperfections in form of the buckling mode connected with geometry of the considered structure and with type of loadings were applied. It was found that possibility of stabilization of the postbuckling path did not depend on the magnitude of a small imperfection $\varDelta$ but the introduction of geometrical imperfections was necessary to follow the postbuckling path using the ANSYS code. On the other hand, the stibilizing loads could depend on $\varDelta$ and thus the convergence of the applied procedure would strongly depend on $\varDelta$ as well. It was decided that in each considered case, the magnitude of the imperfection $\varDelta$ was chosen in such a way that the relative small decrease of the critical twisting moment, in comparison with the perfect structure, was held within the same assumed small tolerance. It gave a possibility to compare the results. It occurred that the value of $\varDelta$, which decreased the critical twisting moment by less than 5% – it means 5% tolerance – was a good compromise between the above mentioned conditions.

### 4.1. Shells subject to axial force only

Figure 4 presents the post-buckling paths in the dimensionless coordinates: $M/M_{cr}$ and $\varphi/\varphi_{cr}$, where $M_{cr}$ denotes the critical twisting moment (maximum moment at the equilibrium path for $N = 0$) and $\varphi$ stands for the angle of torsion of the shell, for two different shell geometrical parameters: $L/R = 1$, $h/R = 0.02$ and $L/R = 2$, $h/R = 0.005$.

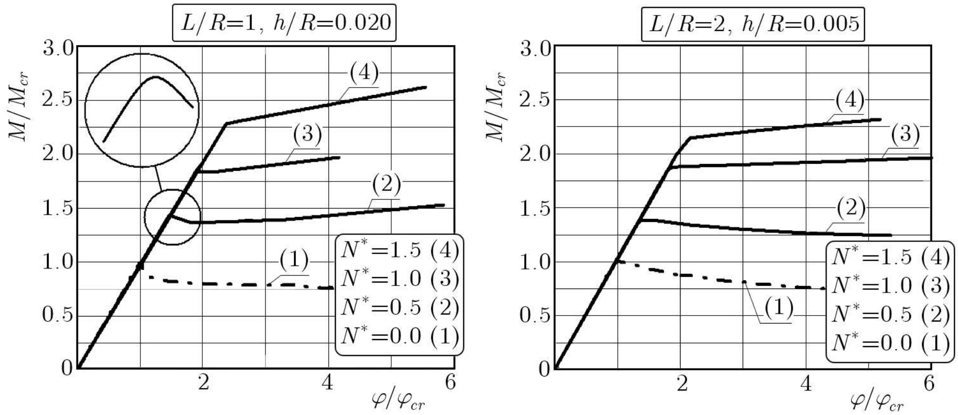


Fig. 4. Post-buckling paths for four different stabilizing forces $N$

As a measure of the applied active axial force, we took dimensionless quantity $N^* = s/s_{stab}$. The value of the minimum stabilizing force $N$ ($s_{stab}$) depends on geometry of the shell. The values of $s_{stab}$ are given in Table 1 for all considered cases. Broken lines (1) represent the equilibrium paths for shells under twisting moment only, it means the classical post-buckling paths for simply supported elastic cylindrical shells of constant thickness under torsion. Lines (2), which prescribe still unstable equilibrium paths, were obtained for axial forces smaller than the stabilizing ones, namely 50% of the stabilizing axial tension. On the other hand, lines (3) represent the stable post-buckling paths obtained for the minimal needed axial tension ($N^* = 1.0$). It means that those equilibrium paths satisfied conditions $(2.1)_2$ and the horizontal inflexion point occurred at those curves. Lines (4) were obtained for the axial loading 50% larger than the stabilizing one. It should be stressed that the stabilization of the post-buckling paths by axial tension was obtained for all considered shells.

In Fig. 5a, shape of the deformed shell ($L/R = 1$, $h/R = 0.005$) for the twisting moment and stabilizing tension $N^* = 1.0$, referring to torsion $\varphi/\varphi_{cr} \approx 2$ at the equilibrium path, is presented. Figure 5b shows distribution of the Mises equivalent stress, obtained for deformations presented in Fig. 5a.

**Table 1.** Values of $s_{stab}$ for all considered cases

| | $h/R = 0.005$ | $h/R = 0.02$ |
|---|---|---|
| $L/R = 1$ | 2.9 | 8.8 |
| $L/R = 2$ | 2.6 | 10.9 |

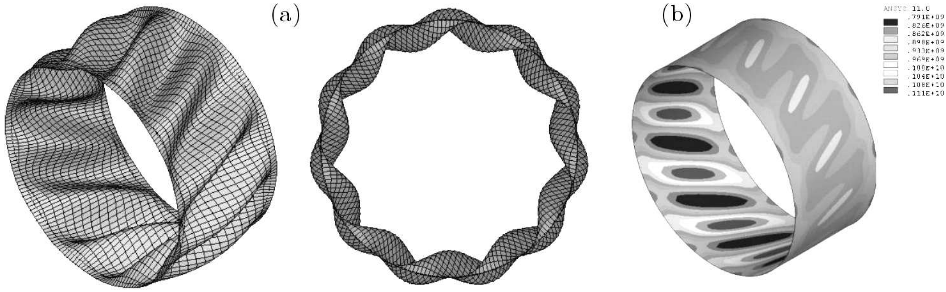


Fig. 5. Shape of the deformed shell (a) and distribution of the Mises equivalent stress (b)

### 4.2. Shells subject to internal pressure only

In Figure 6, the post-buckling paths for two different shells defined by: $L/R = 1$, $h/R = 0.005$ and $L/R = 2$, $h/R = 0.02$ are presented, where broken lines (1) refer to the classical post-buckling curves. For increasing values of the internal pressure, defined here by $p^*$, we obtained subsequent lines (2), (3), (4), whose shapes are similar to each other (for the same shell), namely the post-critical parts of these curves are practically parallel to each other. It means that the internal pressure has no influence on stabilization of the post-buckling path. Such a phenomenon was observed for any value of the internal pressure and any geometry of the considered shells.

The equilibrium paths obtained in geometrically non-linear analysis and presented for chosen geometry in Fig. 6, show higher values of the critical twisting moments at the presence of internal pressure in comparison with the structures without such an additional loading. It means that the internal pressure improves resistance against buckling for shells under torsion. In Fig. 7, influence of the internal pressure on the critical twisting moment is shown.

In Fig. 8a, shape of the deformed shell ($L/R = 1$, $h/R = 0.005$) for the twisting moment and internal pressure $p^* = 1.0$, referring to torsion $\varphi/\varphi_{cr} \approx 2$ at the equilibrium path, is presented. Figure 8b shows distribution of the Mises equivalent stress obtained for deformations presented in Fig. 8a.

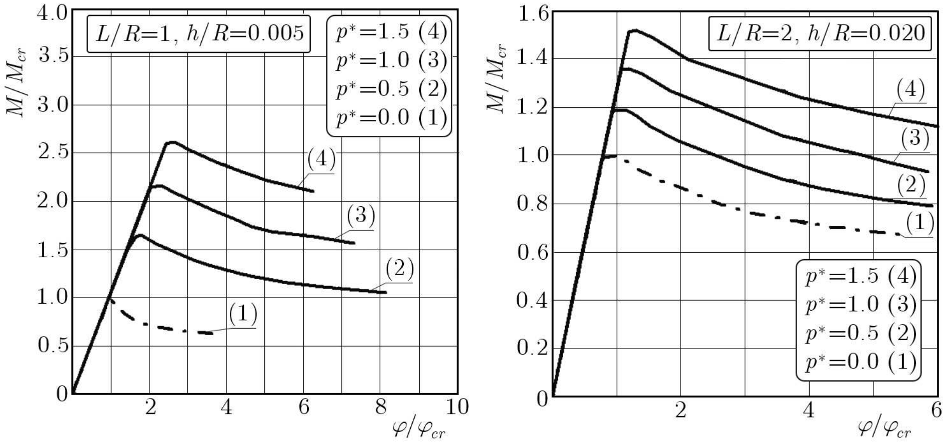


Fig. 6. Post-buckling paths for four different internal pressures $p$

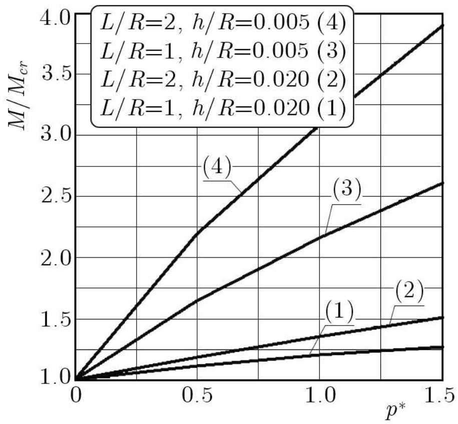


Fig. 7. Effect of internal pressure on the critical twisting moment

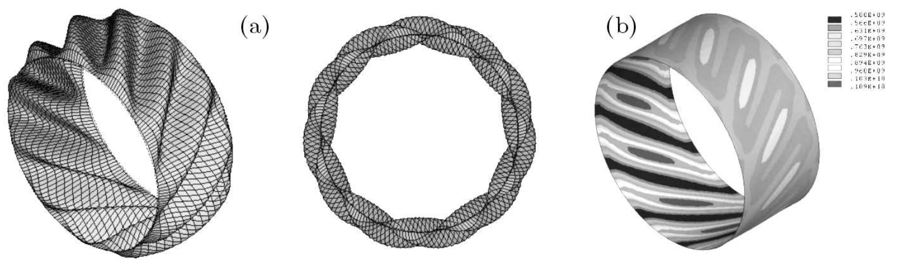


Fig. 8. Shape of the deformed shell (a) and distribution of the Mises equivalent stress (b)

### 4.3. Shells subject to both axial force and internal pressure

Figure 9 presents the post-buckling paths for the shell with $L/R = 1$ and $h/R = 0.005$ under combined stabilizing loadings $L_{N_p}$ with different contribution of $N^*$ and $p^*$. Broken line (1) represents the classical post-buckling path for the shell subject to the twisting moment only whereas line (2) refers to the stabilized equilibrium path only by axial tension with $N^* = 1.5$ $(p^* = 0)$. A small internal pressure, $p^* = 0.25$, increases the critical twisting moment and the post-buckling path (line (3)) remains stable. The same situation is observed for $p^* = 0.5$ and $p^* = 1.0$, namely post-buckling paths (4) and (5) remain stable as well. On the other hand, larger values of $p^*$ cause destabilization of the post-buckling paths (lines 6 and 7) but the applied $p^*$ increases the critical loads. It means that the internal pressure improves resistance against buckling of shells under twisting moment but it can destabilize previously the stabilized post-buckling path by the axial tensile force.

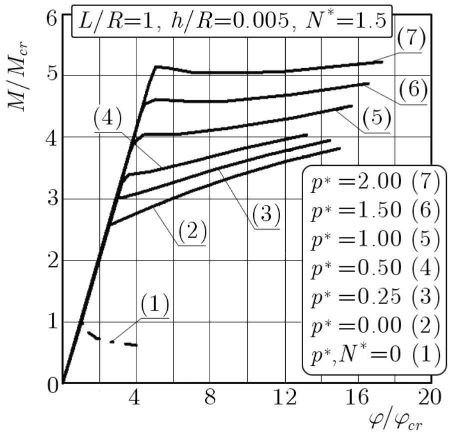


Fig. 9. Post-buckling paths for the axial force and internal pressure

Figure 10 presents relationships between the minimum stabilizing axial force and internal pressure for all considered shells. It shows that application of any internal pressure demands larger axial tension to stabilize the post-buckling path. On the other hand, these curves allow one to choose an appropriate couple of loadings $(N^*, p^*)$ which can lead to stabilization of the post-buckling path for the shell subject to twisting moment.

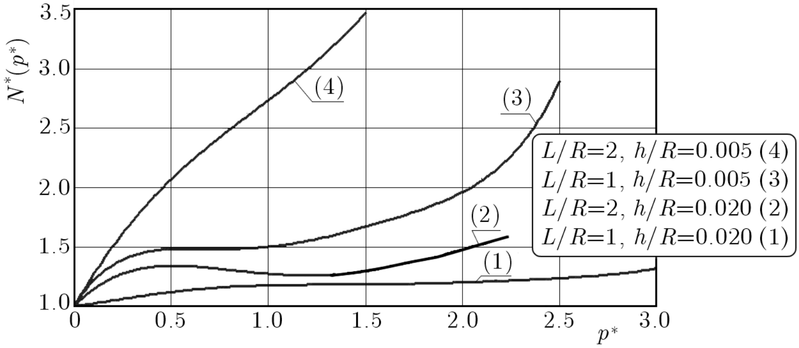


Fig. 10. Minimum axial force stabilizing the post-buckling path vs. internal pressure

## 5. Final remarks

Numerical analysis showed that axial tension can improve resistance against buckling of a cylindrical shell under twisting moment. It can also stabilize the post-buckling path without any modification to geometry of the structure. On the other hand, the internal pressure cannot stabilize of the post-buckling path but it can substantially increase the critical twisting moment in comparison with a structure loaded by torsion only. Simultaneous application of both considered additional loadings improves resistance of the shell under torsion against buckling, but stabilization the post-buckling path for the thus loaded structure requires a larger force in comparison with the structure stabilized by the axial force only. Because of a rather high level of stresses obtained in the elasticity analysis, the possibility of stabilization of the post-buckling path for elasto-plastic deformations of cylindrical shells subject to torsion should be verified.

## References

1. Berskovitz A., Singer J., Weller T., 1967, Buckling of unstiffened conical shells under combined loading, *Experimental Mechanics*, **11**, 458-467

2. Bochenek B., 1997, Optimisation of geometrically nonlinear structures with respect to buckling and postbuckling constraints, *Engineering Optimization*, **29**, 401-415

3. Bochenek B., 2001, Problems of optimal design of structures for post-buckling behaviour, *Zeszyty Naukowe Podstawowe Nauki Techniczne*, **22**, Politechnika Krakowska, Kraków [in Polish]

4. Bochenek B., Krużelecki J., 2001, A new concept of optimization for postbuckling behaviour, *Engineering Optimzation*, **33**, 503-522
5. Bochenek B., Krużelecki J., 2007, *Optimization of Stability of Structures. Modern Problems*, WPK, Kraków [in Polish]
6. Cardoso J.B., Sausa L.G., Castro J.A., Valido A.J., 1997, Optimal design of nonlinear structures and mechanical systems, *Engineering Optimzation*, **29**, 1/4, 277-291
7. Harris L.A., Suer H.S., Skene W.T., 1958, The effect of internal pressure on the buckling stress of thin-walled circular cylinders under combined axial compression and torsion, *J. Aeronaut. Sci.*, **25**, 2, 142-143
8. Jasion P., 2009, Stability analysis of shells of revolution under pressure conditions, *Thin-Walled Structures*, **47**, 311-317
9. Król M., Krużelecki J., Trybuła D., 2009, Optimal stabilization of the post-buckling path for cylindrical shells under external pressure, *Engineering Optimzation*, **41**, 1, 59-72
10. Krużelecki J., Król M., 2006, Optimization of post-buckling path for cylindrical shells under external pressure, *Proc. III European Conference on Computational Mechanics; Solids, Structures and Coupled Problems in Engineering* (CD-ROM only)
11. Krużelecki J., Trybuła D., 2007. Optimal stabilization of post-buckling path for cylindrical shells under torsion, *Proc. Seventh World Congress on Structural and Multidisciplinary Optimization*, Seoul: ISSMO, 1523-1533 (CD-ROM only)
12. Lofblad R.P., 1959, Elastic stability of thin-walled cylinders and cones with internal pressure under axial compression, Massachusetts Institute of Technology, Technical Report, 25-29
13. Mathon C., Limam A., 2006, Experimantal collapse of thin cylindrical shells submitted to internal pressure and pure bending, *Thin-Walled Structures*, **44**, 39-50
14. Mróz Z., Piekarski J., 1998, Sensitivity analysis and optimal design of nonlinear structures, *International Journal for Numerical Methods in Engineering*, **42**, 7, 1231-1262
15. Perry C.A., Gürdal Z., 1995, Design trends of minimum-weight blade-stiffened composite panel for postbuckling response, *Proc. First World Congress of Structural and Multidisciplinary Optimization*, Pergamon, 733-740
16. Pietrzak J., 1996, An alternative approach to optimization of structures prone to instability, *Structural Optimization*, **11**, 88-94

17. Suasa L.G., Valido A.J., Cardoso J.B., 1999, Optimal design of elastic-plastic structures with post-critical behaviour, *Structural Optimization*, **17**, 2, 147-154

18. Volmir A.S., 1967, *Stability of Deformable Systems*, Nauka, Moskva [in Russian]

19. Weingarten V.I., Morgan E.J., Seide P., 1965, Elastic stability of thin-walled cylindrical and conical shells under combined internal pressure and axial compression, *AIAA Journal*, **3**, 6, 1118-1125

20. Yamaki N., 1984, *Elastic Stability of Circular Cylindrical Shells*, North-Holland, Amsterdam-New York-Oxford

## Optymalne rozciąganie osiowe i ciśnienie wewnętrzne stabilizujące ścieżkę pokrytyczną dla skręcanych powłok cylindrycznych

Streszczenie

Efekt stabilizacji niestatecznego zachowania konstrukcji osiągany jest zazwyczaj przez modyfikację jej geometrii. W pracy tej badana jest możliwość stabilizacji pierwotnie niestatecznej ścieżki równowagi dla cylindrycznych powłok skręcanych bez zmiany jej geometrii i wymiarów, mianowicie przez zastosowanie niezależnych, dodatkowych obciążeń przyłożonych do konstrukcji. Okazało się, że rozciąganie osiowe poprawia odporność na wyboczenie powłoki skręcanej i może prowadzić do stabilizacji jej niestatecznej ścieżki równowagi. Natomiast dodatkowe ciśnienie wewnętrzne nie powoduje stabilizacji pokrytycznej ścieżki, ale podnosi odporność konstrukcji na utratę stateczności.

*Manuscript received December 3, 2009; accepted for print December 29, 2009*