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# A SEQUENTIAL AND GLOBAL METHOD OF SOLVING AN INVERSE PROBLEM OF HEAT CONDUCTION EQUATION

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The paper presents a solution to an inverse problem based on the analytical form of the direct problem solution in the convolutional form. The analytical form $T(r,t)$ is a surface that is fitted to temperature patterns measured (and charged with errors) at the internal points. In the case of quickly-varying patterns, the solution to the inverse problem is highly sensitive to measurement errors (short sampling times). In order to obtain reliable results, the method of sequential (step by step) and global solving of the inverse problem was used together with smoothing the measurement results with the help of hyperbolic spline functions. The numerical results confirm effectiveness of the methods presented in the paper.

*Key words:* inverse problem, heat conduction, sequential method, global method

## 1. Introduction

In most heat conduction problems, the governing partial differential equation has to be solved with appropriate initial and boundary conditions. These problems are called direct problems. The problems with incomplete boundary conditions, when the solution is prescribed at some internal points in the domain, are called inverse problems.

Solving the inverse problems is not easy because of their ill-posedness in the Hadamard sense, i.e. the existence, uniqueness and stability of their solutions are not always guaranteed. There have appeared various approaches to the inverse heat conduction problem. They are widely reported in Xianwu and Atluri (2006). Some of the approaches were the following: the use of

Laplace transform, Ciałkowski and Grysa (1980), Grysa *et al.* (1981), the use of Helmholtz equation, Grysa (1989), the finite difference method Hensel and Hills (1984), the finite element approaches, Hore *et al.* (1977), Bass (1980), Tikhonov regularization method, Tikhonov and Arsenin (1977), Alifanov iterative regularization, Alifanov (1994), approaches based on Trefftz functions, Jirousek (1978), Qing-Hua Qin (2000), Ciałkowski (2006, 2007), Grysa and Leśniewska (2009).

In the case of transient heat conduction problems with unknown boundary condition on a part of the boundary, the unknown boundary temperature can be determined after each time step (sequential method) based on measured temperatures at the internal points of the solution domain. However, the measured temperatures are charged with errors what usually results in inaccurate solution to the inverse problem, in particular for small time steps. In the paper by Ciałkowski (2007), splines are used to smooth noisy results of measurements. Smoothing with the use of splines decreases essentially oscillations of the inverse problem solution.

In the paper by Grysa and Leśniewska (2009), smoothing of an inaccurate temperature internal response with the use of T-functions appeared to be a very efficient method ensuring stable solution of the considered IHCP.

Here we use the Laplace transform technique to find the best approximate solution for the case of inaccurate input data. The analytical form of the solution, $T(r,t)$, can be presented as a surface in the $r,t,T$ system of coordinates. When all results of the measurement are known, one can fit it into all the measured results simultaneously (global method).

In transient problems, discretization of the temperature time derivative with the back finite difference may cause instability of the solution for a very small time step. Therefore, a solution to an inverse problem is based here on an analytical form of the direct problem solution. Such an approach is possible only for the simple geometries and for linear heat conduction equation. Here, the problem of determination of the boundary temperature on the inner surface of the cylindrical layer is considered.

## 2. The direct problem solution

The linear heat conduction equation for a cylindrical layer can be written as

$$\rho c\frac{\partial T}{\partial t} = \lambda\Big(\frac{\partial^2 T}{\partial x^2} + \frac{1}{x}\frac{\partial T}{\partial x}\Big) \qquad x \in (R_1, R_2), \quad t > 0 \tag{2.1}$$

In technical applications (cylindrical sleeves, turbine tubes, combustion chambers and others) the inner surface, $x = R_1$, is a lot more loaded termically than the outer one, and in the case of thermal shock on the inner surface the change of heat flux on the outer surface, $x = R_2$, is negligible in the initial period of time. Hence, for the heat flux on the outer surface we assume the following condition

$$-\lambda \frac{\partial T}{\partial x}\Big|_{x=R_2} = q(t) \approx 0 \qquad t > 0 \tag{2.2}$$

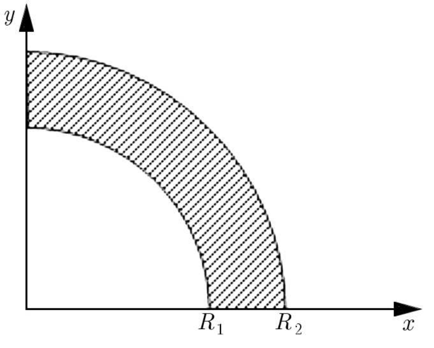


Fig. 1. Cross-section of the cylindrical layer

In the case of heat turbine, condition (2.2) is satisfied thanks to isolation of the turbine tube. However, in general $q(t) \neq 0$.

Assume the initial temperature is a constant, i.e.

$$T(x, t = 0) = T_0 = \text{ const} \qquad x \in \langle R_1, R_2 \rangle \tag{2.3}$$

and the inner surface temperature is a function of time

$$T(x = R_1, t_1) = T_f(t) \qquad t > 0 \tag{2.4}$$

If cooling is considered, then $T_0 \geqslant T_f$. We define dimensionless coordinates

$$\xi = \frac{x}{R_2} \qquad \tau = \frac{\lambda}{\rho c}\frac{t}{R_2^2} \qquad \vartheta = \frac{T}{T_0}$$
$$\vartheta_f = \frac{T_f}{T_0} \qquad g = \frac{R_2}{\lambda T_0} q$$

Here $\tau$ (the dimensionless time) is the Fourier number. Equation (2.1) with conditions (2.2) to (2.4) take the form as follows

— the heat conduction equation

$$\frac{\partial\vartheta}{\partial\tau} = \frac{\partial^2\vartheta}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\vartheta}{\partial\xi} \qquad \xi \in (\xi_1, \xi_2), \quad \tau > 0 \tag{2.5}$$

— the initial condition

$$\vartheta(\xi, 0) = \vartheta_0 = \text{const} \tag{2.6}$$

— the boundary condition on the inner surface, $\xi = \xi_1$

$$\vartheta(\xi = \xi_1, \tau) = \vartheta_f(\tau) \qquad \tau > 0 \tag{2.7}$$

— the boundary condition on the outer surface, $\xi = \xi_2$

$$\frac{-\partial\vartheta(\xi = \xi_2, \tau)}{\partial\xi} = g(\tau) \tag{2.8}$$

To solve equation (2.5) with conditions (2.6) to (2.8), the Laplace transform is applied (Doetsch, 1964)

$$\overline{\vartheta}(\xi, s) = \int\limits_0^\infty \vartheta(\xi, \tau)\mathrm{e}^{-s\tau}\, d\tau \tag{2.9}$$

As a result, we arrive to the following convolutional form of the considered direct problem solution

$$\begin{aligned} \vartheta(\xi, \tau) = [\vartheta_f'(\tau) + (\vartheta_f(0) - \vartheta_0)\delta(\tau)] * F(\xi, \tau) + \\ -[g'(\tau) + g(0)\delta(\tau)] * H(\xi, \tau) + \eta(\tau)\vartheta_0 \end{aligned} \tag{2.10}$$

with $\delta(t)$ standing for the Dirac function, $\eta(t)$ being the Heaviside function and

$$F(\xi, \tau) = \sum_{n=1}^{\infty} res_{s=s_n} \frac{1}{s} \frac{I_1(\sqrt{s})K_0(\sqrt{s}\xi) - K_1(\sqrt{s})I_0(\sqrt{s}\xi)}{I_0(\sqrt{s}\xi_1)K_1(\sqrt{s}) + K_0(\sqrt{s}\xi_1)I_1(\sqrt{s})}\mathrm{e}^{st}$$

$$H(\xi, \tau) = \sum_{n=1}^{\infty} res_{s=s_n} \frac{1}{s\sqrt{s}} \frac{K_0(\sqrt{s}\xi_1)I_0(\sqrt{s}\xi) - I_0(\sqrt{s}\xi_1)K_0(\sqrt{s}\xi)}{I_0(\sqrt{s}\xi_1)K_1(\sqrt{s}) + K_0(\sqrt{s}\xi_1)I_1(\sqrt{s})}\mathrm{e}^{st}$$

and $s_n$, $n = 0, 1, 2, \ldots$, being roots of the transcendental equation

$$s\Big(I_0(\sqrt{s}\xi_1)K_1(\sqrt{s}) + K_0(\sqrt{s}\xi_1)I_1(\sqrt{s})\Big) = 0$$

For $\sqrt{s} = \mathrm{i}\mu$, we obtain $\mu_0 = 0$ and

$$J_0(\mu_n \xi_1) Y_1(\mu_n) = Y_0(\mu_n \xi_1) J_1(\mu_n) \qquad n = 1, 2, \ldots \tag{2.11}$$

For $\mu \xi_1 \gg 1$ we arrive at the asymptotic form of equation (2.11), McLachlan (1964)

$$\cos[\mu(1 - \xi_1)] = 0$$

with the roots

$$\mu_n = \frac{\frac{\pi}{2} + (n-1)\pi}{1 - \xi_1} \qquad n = 1, 2, \ldots \tag{2.12}$$

Roots of the equation $\cos[\mu(1 - \xi_1)] = 0$, $\mu_n$ stand for the first approximation for the roots of equation (2.11) when the Newton method is used to calculate them with the required accuracy.

In particular, we find

$$\begin{aligned} F(\xi, \tau) &= 1 + 2 \sum_{n=1}^{\infty} f_n(\xi) \frac{\mathrm{e}^{-\mu_n^2 \tau}}{\mu_n} \\ H(\xi, \tau) &= \ln \frac{\xi}{\xi_1} + 2 \sum_{n=1}^{\infty} h_n(\xi) \frac{\mathrm{e}^{-\mu_n^2 \tau}}{\mu_n^2} \end{aligned} \tag{2.13}$$

with

$$\begin{aligned} f_n(\xi) &= \frac{1}{B} [J_1(\mu_n) Y_0(\mu_n \xi) - Y_1(\mu_n) J_0(\mu_n \xi)] \\ h_n(\xi) &= \frac{1}{B} [J_0(\mu_n \xi_1) Y_0(\mu_n \xi) - Y_0(\mu_n \xi_1) J_0(\mu_n \xi)] \end{aligned} \tag{2.14}$$

and

$$B = \xi_1 [J_1(\mu_n \xi_1) Y_1(\mu_n) - J_1(\mu_n) Y_1(\mu_n \xi_1)] - J_0(\mu_n \xi_1) Y_0(\mu_n) + Y_0(\mu_n \xi_1) J_0(\mu_n)$$

With regard to the convolutional form of the solution to problems (2.1) to (2.4), the functions $\vartheta_f$ and $g$ that appear in formula (2.10) may be replaced by their approximation with linear-wise functions in the successive time intervals (comp. Fig. 2, where $\Theta' = \vartheta_f$ or $\Theta' = g$). Consider the solution $\vartheta(\xi, \tau)$ for a certain $\tau = \tau_M < 1$, $M$ being a natural number. Because in (2.13) the function $\exp(-\mu_n^2 \tau)$ appears, we first calculate the convolution of the functions $\vartheta_f$ and $g$ with $\exp(-\mu_n^2 \tau)$. Using the average value theorem and dividing the time interval $\langle 0, \tau_M \rangle$ into small subintervals, $\langle 0, \tau_M \rangle = \bigcup\limits_{k=1}^{M} \langle \tau_{k-1}, \tau_k \rangle$, we obtain

$$
\begin{aligned}
I_1 &= \Theta'(\tau) * \mathrm{e}^{-\mu_n^2 \tau}\Big|_{\tau=\tau_M} = \int_0^{\tau_M} \Theta'(\eta)\mathrm{e}^{-\mu_n^2(\tau_M-\eta)}\, d\eta = \mathrm{e}^{-\mu_n^2\tau_M} \int_0^{\tau_M} \Theta'(\eta)\mathrm{e}^{\mu_n^2\eta}\, d\eta \approx \\
&\approx \mathrm{e}^{-\mu_n^2\tau_M} \sum_{k=1}^{M} \int_{\tau_{k-1}}^{\tau_k} \frac{\Theta_k-\Theta_{k-1}}{\tau_k-\tau_{k-1}} \mathrm{e}^{\mu_n^2\eta}\, d\eta = \mathrm{e}^{-\mu_n^2\tau_M} \sum_{k=1}^{M} \frac{\Theta_k-\Theta_{k-1}}{\tau_k-\tau_{k-1}} \int_{\tau_{k-1}}^{\tau_k} \mathrm{e}^{\mu_n^2\eta}\, d\eta \approx \\
&\approx \mathrm{e}^{-\mu_n^2\tau_M} \sum_{k=1}^{M} (\Theta_k-\Theta_{k-1})\mathrm{e}^{\mu_n^2[\tau_{k-1}+\gamma_k(\tau_k-\tau_{k-1})]} = \sum_{k=0}^{M} \Theta_k \Phi_{kn}(\tau_M)
\end{aligned}
\tag{2.15}
$$

where $\gamma_k \in (\tau_{k-1}, \tau_k)$, $\Theta_k \equiv \Theta(\tau_k)$ and $\Theta = \vartheta_f$ or $\Theta = g$. Moreover

$$
\Phi_{kn}(\tau_M) = \begin{cases} -\mathrm{e}^{-\mu_n^2(\tau_M-\gamma_1\tau)} & \text{for} \quad k=0 \\ \mathcal{B} & \text{for} \quad 0<k<M \\ \mathrm{e}^{-\mu_n^2[(1-\gamma_M)(\tau_M-\tau_{M-1})]} & \text{for} \quad k=M \end{cases}
\tag{2.16}
$$

where

$$
\mathcal{B} = \mathrm{e}^{-\mu_n^2[\tau_M-\gamma_k\tau_k-(1-\gamma_k)\tau_{k-1}]} - \mathrm{e}^{-\mu_n^2[\tau_M-\gamma_{k+1}\tau_{k+1}-(1-\gamma_{k+1})\tau_k]}
$$

For $\Theta_0 = \Theta_1 = \ldots = \Theta_M = \text{const}$, $\sum_{k=0}^{M} \Phi_{kn}(\tau_M) = 0$.

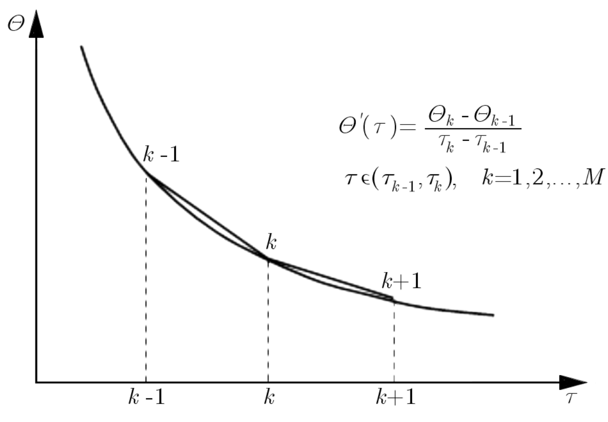


Fig. 2. Linear-wise approximation of the function $\Theta(\tau)$

The use the average value theorem is more advantageous than calculating the accurate value of the expression

$$
\mathrm{e}^{-\mu_n^2\tau_M} \int_{\tau_{k-1}}^{\tau_k} \mathrm{e}^{\mu_n^2\eta}\, d\eta \qquad n=1,2,\ldots
$$

which for growing $n$ becomes a product of two values, a very big one and a very small one, and may be troublesome due to numerical problems.

Another convolution that appears in (2.10) can be expressed as follows

$$I_2 = \Theta'(\tau) * \eta(\tau) = \int_0^\tau \Theta'(\eta)\eta(\tau-\eta)\, d\eta = \int_0^\tau \Theta'(\eta)\, d\eta = \Theta(\tau) - \Theta(0) \tag{2.17}$$

Here again $\Theta = \vartheta_f$ or $\Theta = g$. Hence, assuming that $\vartheta_f(0) = \vartheta_0$ and $g(0) = 0$, for $I_1 = \sum_{k=0}^{M} \Theta_k \Phi_{kn}(\tau_M)$ and $I_2 = \Theta(\tau) - \Theta(0)$ we obtain the following form of the function $\vartheta(\xi,\tau)$ for $\tau = \tau_M > 0$ and $\xi \in \langle \xi_1, 1\rangle$

$$\begin{aligned} \vartheta(\xi,\tau_M) = \vartheta_f(\tau_M) + \sum_{k=0}^{M} \vartheta_{f_k} \sum_{n=1}^{\infty} \frac{f_n(\xi)}{\mu_n} \Phi_{kn}(\tau_M) + \\ - \sum_{k=0}^{M} g_k \sum_{n=1}^{\infty} \frac{h_n(\xi)}{\mu_n^2} \Phi_{kn}(\tau_M) - g(\tau_M) \ln \frac{\xi}{\xi_1} \end{aligned} \tag{2.18}$$

or, in a more compact form,

$$\vartheta(\xi,\tau_M) = \{\chi(\xi,\tau_M)\}^\top \{\vartheta_f\} - V\{\xi,\tau_M\}^\top \{g\} \tag{2.19}$$

where

$$\begin{aligned} \chi_k(\xi,\tau_M) &= \sum_{n=1}^{\infty} \frac{f_n(\xi)}{\mu_n} \Phi_{kn}(\tau_M) \qquad && k = 1,2,\dots,M-1 \\ \chi_M(\xi,\tau_M) &= \sum_{n=1}^{\infty} \frac{f_n(\xi)}{\mu_n} \Phi_{Mn}(\tau_M) + 1 \\ V_k(\xi,\tau_M) &= \sum_{n=1}^{\infty} \frac{h_n(\xi)}{\mu_n^2} \Phi_{kn}(\tau_M) \qquad && k = 1,2,\dots,M-1 \\ V_M(\xi,\tau_M) &= \sum_{n=1}^{\infty} \frac{h_n(\xi)}{\mu_n^2} \Phi_{kn}(\tau_M) + \ln \frac{\xi}{\xi_1} \\ \vartheta_{f_M} &\equiv \vartheta_f(\tau_M) \end{aligned}$$

Solution (2.10) in form (2.19) is convenient to solve an inverse problem.

When calculating numerical sum of an infinite series, the summation is confined to such a number of its elements that leads to inaccuracy of the result not exceeding the assumed value.

Let us consider the functions $f_n(\xi)$ and $g_n(\xi)$, formulas (2.14). For sufficiently big values of the roots $\mu_n$, the functions can be expressed in asymptotic forms

$$f_n(\xi) \approx -\sqrt{\frac{\xi_1}{\xi}} \frac{\cos[\mu_n(1-\xi)]}{(1-\xi_1)\sin[\mu_n(1-\xi_1)]}$$
$$g_n(\xi) \approx -\frac{1}{\sqrt{\xi}} \frac{\sin[\mu_n(1-\xi)]}{(1-\xi_1)\sin[\mu_n(1-\xi_1)]}$$

By virtue of (2.12) we obtain $\sin[\mu_n(1-\xi_1)] = (-1)^{n-1}$. Hence

$$|f_n(\xi)| \leqslant \frac{\sqrt{\xi_1}}{\xi} \frac{1}{1-\xi_1} \qquad |g_n(\xi)| \leqslant \frac{1}{\sqrt{\xi}} \frac{1}{1-\xi_1}$$

For convergence of the series in the formula (2.18) the expression $\Phi_{kn}(\tau_M)/\mu_n$ is crucial. By virtue of (2.16), we find

$$\max_{k\leqslant M} \frac{1}{\mu_n} \Phi_{kn}(\tau_M) = \frac{1}{\mu_n} \mathrm{e}^{-\mu_n^2(1-\gamma_M)(\tau_M-\tau_{M-1})} \qquad 0 < \gamma_M < \tau_M < 1$$

Thus, for any positive $\varepsilon$ and sufficiently big $\mu_n$, by virtue of the formula (2.12), the following inequality holds

$$\frac{1}{\mu_n} \mathrm{e}^{-\mu_n^2(1-\gamma_M)(\tau_M-\tau_{M-1})} < \varepsilon$$

Hence, we can find such a value of the number $n$ for the first series in formula (2.18) that leads to its sufficiently accurate sum. The second series in (2.18) converges as well thanks to the expression $\mu_n^2$ in the denominator.

## 3. The inverse problem

Owing to the practical application, let us a consider an inverse problem for the cylindrical layer. The unknown temperature, $\vartheta_f(\tau_k)$, on the inner surface, for $\xi = \xi_1$, is to be found based on the initial temperature of the layer, the temperature on its outer surface, $\xi = 1$, and known values of temperature, $\vartheta_{meas}(\xi_i^*, \tau_k)$, at some inner points, $\xi_i^*$, $i = 1, \ldots, M_{meas}$ (see Fig. 3). The measured values, $\vartheta_{meas}(\xi_i^*, \tau_k)$ is charged with a noise.

In order to find the unknown inner boundary temperature, $\vartheta_f(\tau_k)$, a distance between the surface $\vartheta(\xi, \tau_k)$ and the given values $\vartheta_{meas}(\xi_i^*, \tau_k)$ is minimized. The functional that describes the distance (the mean-square error) reads

$$I_M = I(\tau_M) = \sum_{i=1}^{M_{meas}} \|\vartheta(\xi_i^*, \tau_M) - \vartheta_{meas}(\xi_i^*, \tau_M)\|^2 \qquad M = 1, 2, \ldots \tag{3.1}$$

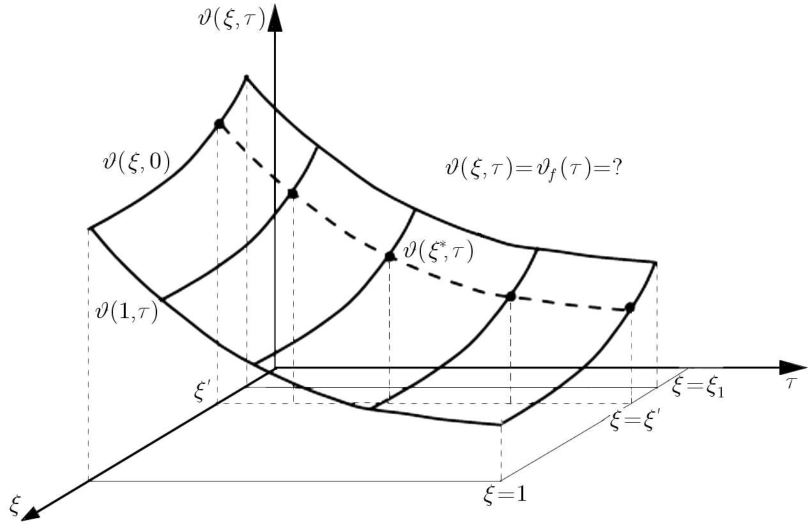


Fig. 3. Graphic form of the solution $\vartheta(\xi,\tau)$

By virtue of the formula (2.19) the functional $I_M$ takes the form

$$I_M = \sum_{i=1}^{M_{meas}} \Big(\sum_{k=0}^{M} \vartheta_{f_k}\chi_k(\xi_i^*,\tau_M) - \sum_{k=0}^{M} g_k V_k(\xi_i^*,\tau_M) - \vartheta_{meas}(\xi_i^*,\tau_M)\Big)^2 \tag{3.2}$$

In the functional $I_M$, the inner boundary temperature, $\vartheta_{f_M}$, is an unknown. Hence, from the necessary condition of the functional $I_M$ minimum, i.e. from the equation $\partial I_M/\partial\vartheta_{f_M} = 0$, we find

$$\vartheta_{f_M} = \sum_{k=0}^{M-1} \vartheta_{f_k} a_k + \sum_{i=1}^{M_{meas}} b_i\Big(\sum_{k=0}^{M} g_k V_k(\xi_i^*,\tau_M) + \vartheta_{meas}(\xi_i^*,\tau_M)\Big) \tag{3.3}$$

with

$$a_k = -\frac{\sum_{i=1}^{M_{meas}} \chi_M(\xi_i^*,\tau_M)\chi_k(\xi_i^*,\tau_M)}{\sum_{i=1}^{M_{meas}} \chi_M^2(\xi_i^*,\tau_M)} \qquad\qquad b_i = \frac{\chi_M(\xi_i^*,\tau_M)}{\sum_{i=1}^{M_{meas}} \chi_M^2(\xi_i^*,\tau_M)}$$

With $\vartheta_{f_M}$ known from formula (3.3) one can find successive values of the inner boundary temperature for $M = 1,2,\dots$ based on the measured inner point temperature values, $\vartheta_{meas}(\xi_i^*,\tau_k)$. In this case, the inner boundary temperature is calculated sequentially.

The sequential method is sensitive to noise of measured temperature values, in particular for initial moments of time, Grysa (1988). The noise present in the measured data can cause instabilities in the estimated heat fluxes. These instability problems demand proper numerical treatment.

The global method seems to be the proper one. In this case, in order to diminish the measurement noise impact on the searched values of the inner boundary temperature, $\vartheta_{f_1}, \vartheta_{f_2}, \ldots, \vartheta_{f_M}$, we minimize the functional describing the mean-value distance between the surface $\vartheta(\xi, \tau_k)$ and the given values $\vartheta_{meas}(\xi_i^*, \tau_k)$ in the whole time interval $\langle 0, \tau_M \rangle$

$$I = \sum_{k=1}^{M} I_k = \sum_{k=1}^{M} \sum_{i=1}^{M_{meas}} \|\vartheta(\xi_i^*, \tau_k) - \vartheta_{meas}(\xi_i^*, \tau_k)\|^2 \tag{3.4}$$

Using form (3.1) of the function $\vartheta(\xi, \tau)$ for $\tau = \tau_M$, we obtain

$$I = \{\vartheta_f\}^\top [\chi\chi]\{\vartheta_f\} - 2\{\vartheta_f\}^\top ([V\chi]\{g\} + \{\vartheta\chi\}) + C \tag{3.5}$$

Here

$$\begin{aligned}
[\chi\chi] &= \sum_{k=1}^{M} \sum_{i=1}^{M_{meas}} \{\chi(\xi_i^*, \tau_k)\}\{\chi(\xi_i^*, \tau_k)\}^\top \\
[V\chi] &= \sum_{k=1}^{M} \sum_{i=1}^{M_{meas}} \{V(\xi_i^*, \tau_k)\}\{\chi(\xi_i^*, \tau_k)\}^\top \\
\{\vartheta\chi\} &= \sum_{k=1}^{M} \sum_{i=1}^{M_{meas}} \vartheta_{meas}(\xi_i^*, \tau_k)\{\chi(\xi_i^*, \tau_k)\}^\top \\
C &= \sum_{k=1}^{M} \sum_{i=1}^{M_{meas}} [\{V(\xi_i^*, \tau_k)\}^\top \{g\} + \vartheta_{meas}(\xi_i^*, \tau_k)]^2
\end{aligned}$$

The functional $I$ minimized with respect to the vector $\{\vartheta_f\}$ leads to a system of algebraic equations

$$[\chi\chi]\{\vartheta_f\} = [V\chi]\{g\} + \{\vartheta\chi\} \tag{3.6}$$

The solution reads

$$\{\vartheta_f\} = [\chi\chi]^{-1}[V\chi]\{g\} + [\chi\chi]^{-1}\{\vartheta\chi\} \tag{3.7}$$

In the global method, solving an inverse problem leads to a system of algebraic equations.

## 4. Numerical example

In order to show the difference between the sequential and the global method, a solution of the inverse problem of cooling the inner surface of a cylindrical layer is presented. The inner radius of the layer, $R_1 = 0.9\,\mathrm{m}$, and outer one, $R_2 = 1.0\,\mathrm{m}$ (its thickness is equal to $0.10\,\mathrm{m}$). Thermophysical properties of the layer are as follows: $\lambda = 30\,\mathrm{W/(mK)}$, $\rho = 7800\,\mathrm{kg/m^3}$, $c = 500\,\mathrm{J/(kgK)}$.

The following location of two thermal probes are considered:

a) $s_1 = 2.0\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the inner surface,

b) $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the inner surface.

The dimensionless initial temperature is assumed as $\vartheta(\xi, 0) = 1$. Hence, for $T_0 = 1000°\mathrm{C}$ we have $T(\xi, 0) = 1000°\mathrm{C}$.

The dimensionless inner surface temperature is assumed in the following form

$$\vartheta(\xi_1, \tau) = \vartheta_f(\tau) = \vartheta_{max}\Big[1 - \Big(1 - \frac{\vartheta_{min}}{\vartheta_{max}}\Big) z\mathrm{e}^{1-z}\Big] \qquad z = \frac{\tau}{\tau_{min}}$$

For $\tau = \tau_{min}$ we have $\vartheta_f(\tau_{min}) = \vartheta_{min}$. For $\tau \in (0, \tau_{min})$ cooling takes place; for $\tau > \tau_{min}$ the temperature $\vartheta(\xi_1, \tau)$ increases. Intensity of cooling may be controlled with the minimal temperature value, $\tau_{min}$, and with the moment of time $\tau_{min}$.

As the measured data for the inverse problem, numerical data obtained by solving the direct problem can be used. For the direct problem, values of the inner surface temperature, $\vartheta(\xi_1, \tau) = \vartheta_f(\tau)$, and the heat flux $q(\tau)$ on the outer surface, for $\xi_2 = 1$, are prescribed. We add some random noise on the values of $\vartheta_{meas}(\xi_i^*, \tau_k)$ whose level is at most $5°\mathrm{C}$. For this purpose, the noise with normal distribution $N(0, 0.005)$ generated numerically in Fortran has been used.

The inverse problem has been solved:

a) for noise present in the measured data,

b) for measured data smoothed with hyperbolic splines.

The time step was $\Delta t = 0.2\,\mathrm{s}$.

In the case of sequential method, the solution sensitivity for initial moments of time is a result of solution $\vartheta(\xi, \tau)$ "response" for the input measured data with noise.

Changes in time of the noise present in the data measured in the first point (the distance from the boundary $s_1 = 2\,\mathrm{mm}$) and in the second one

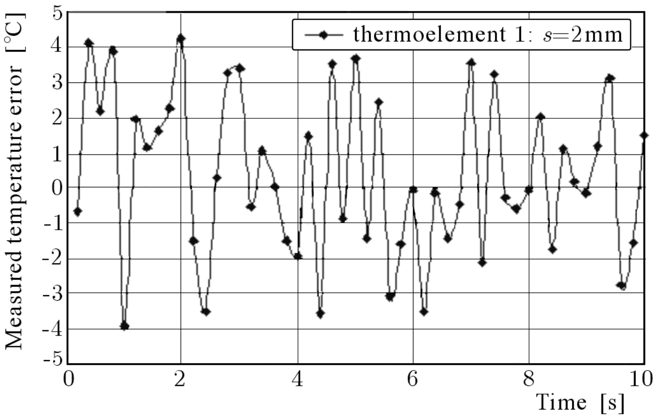


Fig. 4. Distribution of the random error of measured temperature in the first point for $\Delta t = 0.2\,\mathrm{s}$

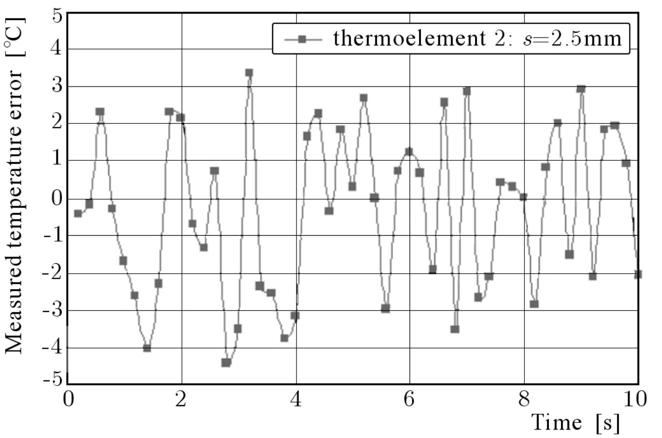


Fig. 5. Distribution of the random error of measured temperature in the second point for $\Delta t = 0.2\,\mathrm{s}$

($s_2 = 2.5\,\mathrm{mm}$) are presented in Fig. 4 and Fig. 5. In Fig. 6, the change of temperature in time at the two points is shown.

The global solution to the inverse problem for disturbed and undisturbed temperature measurements (here the measurements are simulated numerically) is presented in Fig. 7 (the inner boundary temperature) and Fig. 8 (heat flux on the inner boundary). The heat flux determination strongly depends on the temperature measurement error. For some moments of time, the heat flux error exceeds 50%. In order to reduce that disadvantageous situation, the measurements are smoothed with hyperbolic splines (Kosma, 1999). The idea of using the hyperbolic splines follows the character of parabolic equation so-

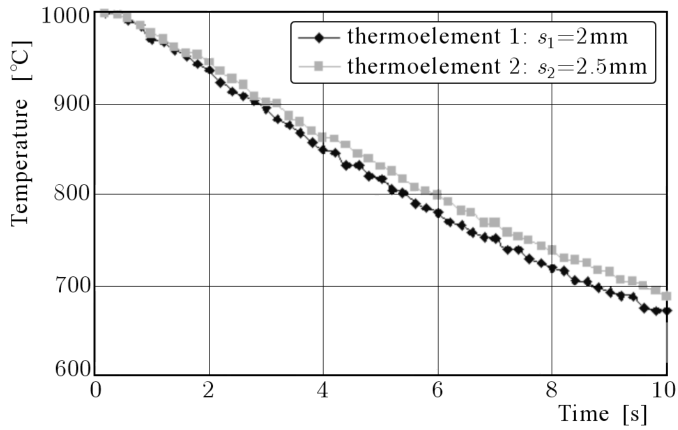


Fig. 6. Change of temperature in time at two points of measurement for $\Delta t = 0.2\,\mathrm{s}$

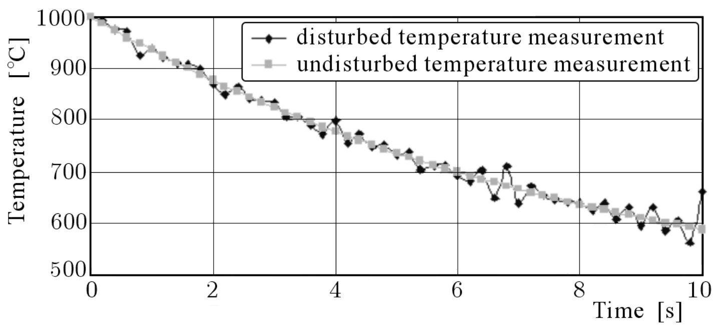


Fig. 7. Temperature on the inner boundary for disturbed and undisturbed temperature measurement at the inner points distant by $s_1 = 2\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

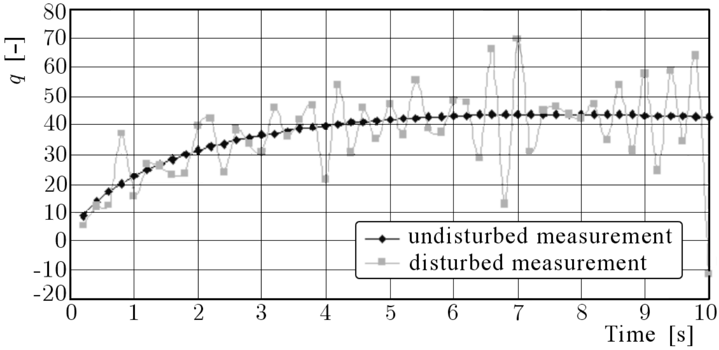


Fig. 8. Dimensionless heat flux on the inner boundary for disturbed and undisturbed temperature measurement at the inner points distant by $s_1 = 2\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

lution, in which a function of hyperbolic type, $\exp(At)f(x)$, appears. In the hyperbolic splines, functions $\exp(As)p(s)$ are used, with $s$ being time or spatial variable and $p(s)$ standing for a polynomial of the third order. For $c = 0$, a hyperbolic spline becomes a classic spline.

Figs. 9 to 12 present the temperature and heat flux on the inner boundary obtained as a result of the smoothing.

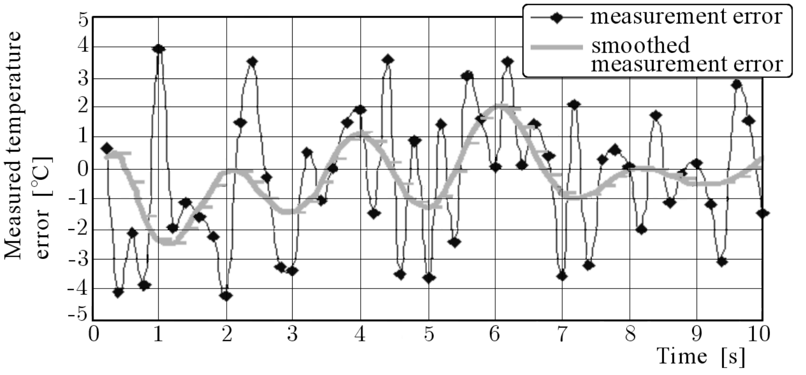


Fig. 9. Distribution of the random error of temperature measurements in the point distant by $s_1 = 2.0\,\mathrm{mm}$ from the inner boundary

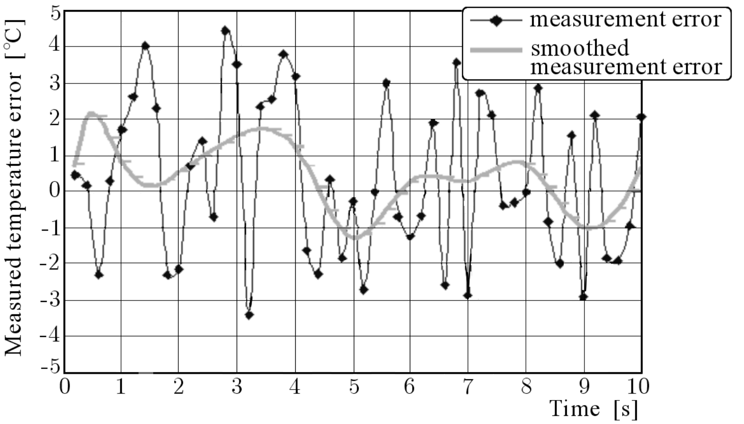


Fig. 10. Distribution of the random error of temperature measurements in the point distant by $s_2 = 2.5\,\mathrm{mm}$ from the inner boundary

For the thermoelements distant by $s_1 = 2.0\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the inner boundary and for the time step $\Delta t = 0.2\,\mathrm{s}$, the sequential approach leads to an unstable solution. Displacing the first thermoelement to the point $s_1 = 1.5\,\mathrm{mm}$ ensured stability of the inverse problem solution.

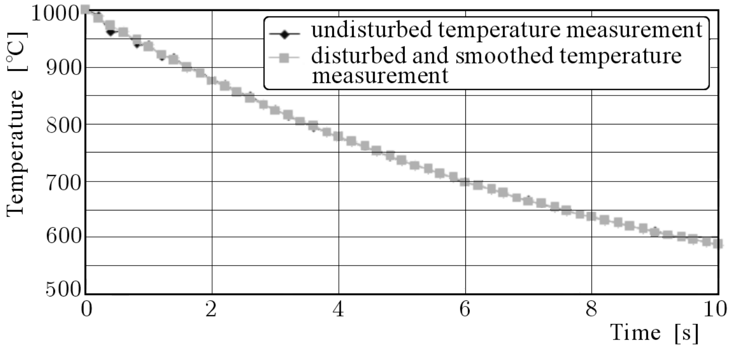


Fig. 11. Temperature on the inner boundary for undisturbed and disturbed smoothed temperature measurement at the inner points distant by $s_1 = 2.0\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

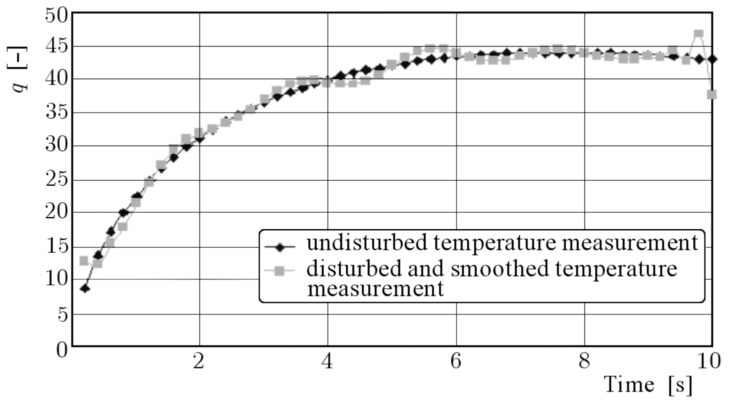


Fig. 12. Dimensionless heat flux on the inner boundary for undisturbed and disturbed smoothed temperature measurement at the inner points distant by $s_1 = 2\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

The results for the sequential approach for $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ are presented in Figs. 13 to 15 and for the global approach – in Figs. 16 to 19. Both solutions are compared in Figs. 20 and 21.

The calculations show that in the case of sequential approach, placing the thermoelements far from the inner boundary may lead to an unstable solution for the initial moments of time. The disadvantageous property is not observed in the global approach.

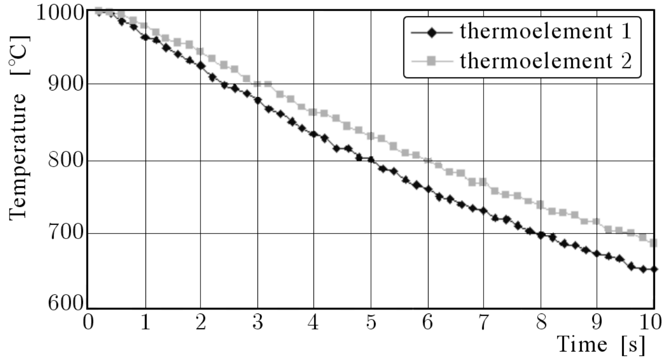


Fig. 13. Change of temperature in time at two points of measurement for $\Delta t = 0.2\,\mathrm{s}$

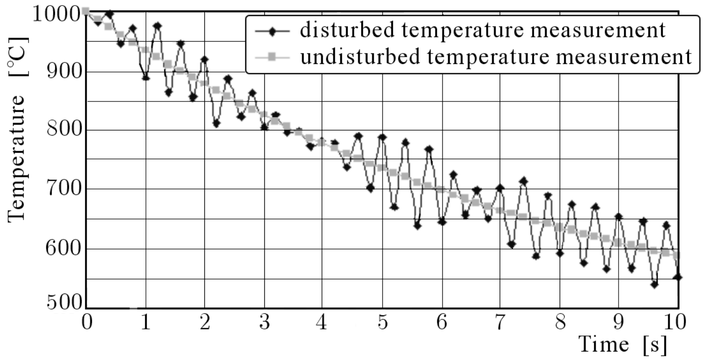


Fig. 14. Temperature on the inner boundary for disturbed and undisturbed temperature measurement at the inner points distant by $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and sequential approach

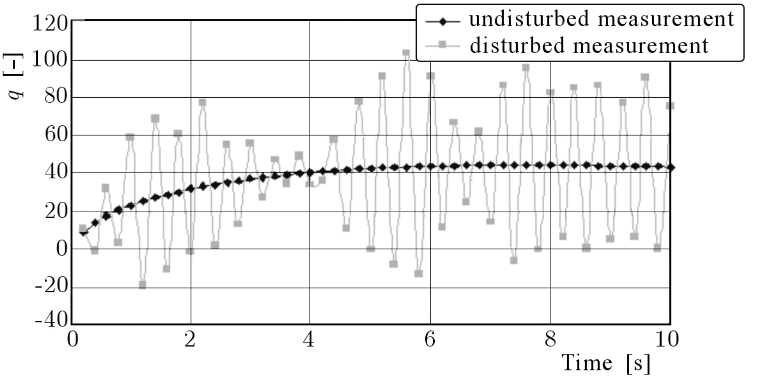


Fig. 15. Dimensionless heat flux on the inner boundary for undisturbed and disturbed temperature measurement at the inner points distant by $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and sequential approach

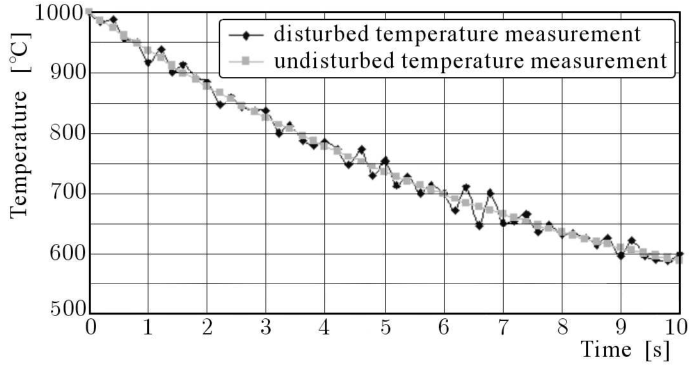


Fig. 16. Temperature on the inner boundary for disturbed and undisturbed temperature measurement at the inner points distant by $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

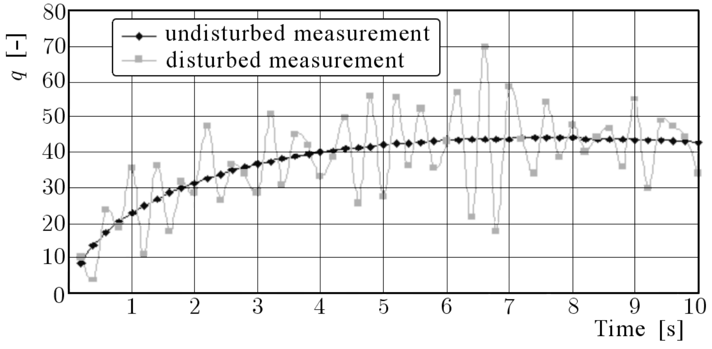


Fig. 17. Dimensionless heat flux on the inner boundary for undisturbed and disturbed temperature measurement at the inner points distant by $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

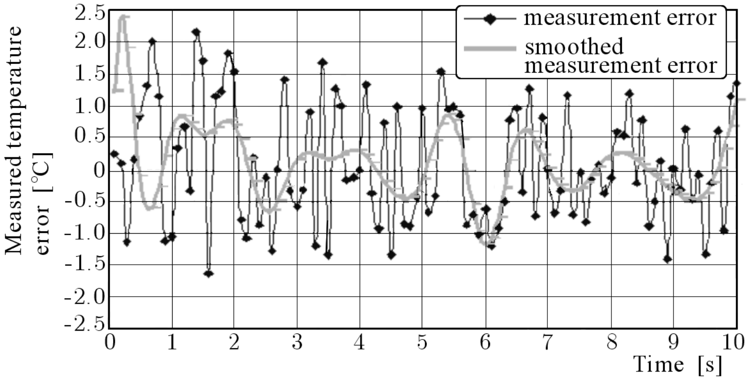


Fig. 18. Distribution of the random error of temperature measurements in the point distant by $s_2 = 2.5\,\mathrm{mm}$ from the inner boundary

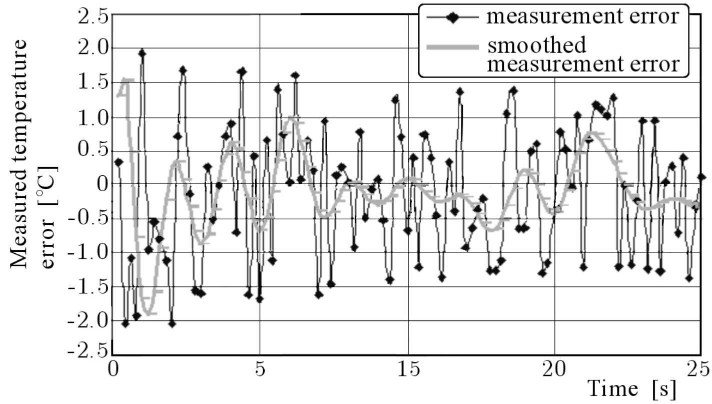


Fig. 19. Distribution of the random error of temperature measurements in the point distant by $s_2 = 1.5\,\mathrm{mm}$ from the inner boundary

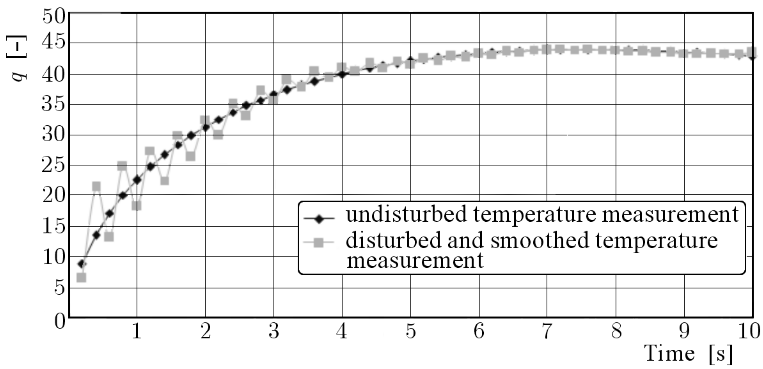


Fig. 20. Dimensionless heat flux on the inner boundary for undisturbed and disturbed smoothed temperature measurement at the inner points distant by $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and sequential approach

## 5. Location of the measurement points versus accuracy of temperature determination

Consider now how the measurement points location affects accuracy of temperature identification. In order to simplify the consideration, the case of a flat slab will be analysed. Conclusions hold also for a thin cylindrical layer which

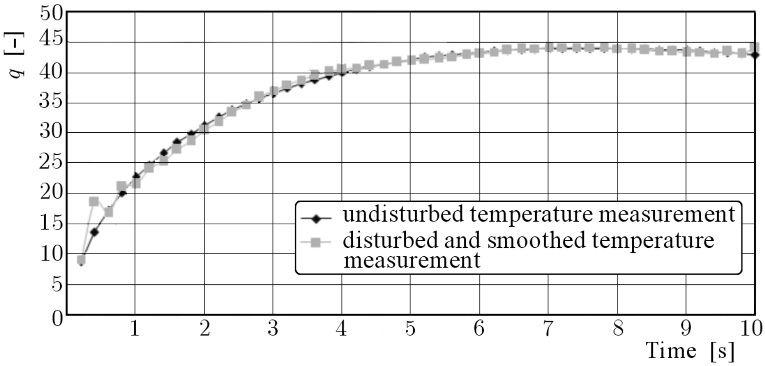


Fig. 21. Dimensionless heat flux on the inner boundary for undisturbed and disturbed smoothed temperature measurement at the inner points distant by $s_1 = 1.5\,\mathrm{mm}$ and $s_2 = 2.5\,\mathrm{mm}$ from the boundary for $\Delta t = 0.2\,\mathrm{s}$ and global approach

can be approximately described as a flat slab. Instead of equation (2.5) with conditions (2.6) to (2.8) we consider dimensionless equation

$$\frac{\partial \vartheta}{\partial \tau} = \frac{\partial^2 \vartheta}{\partial \xi^2} \qquad \tau > 0, \quad \xi \in (0,1) \tag{5.1}$$

with conditions

$$\begin{aligned} &\vartheta(\xi,0) = \vartheta_0(\xi) = 0 && \left.\frac{\partial \vartheta}{\partial \xi}\right|_{\xi=0} = 0 \\ &\vartheta(\xi^*,\tau) = \vartheta^*(\tau) > 0 && 0 \leqslant \xi^* \leqslant 1 \end{aligned}$$

Here, for simplicity we put $\vartheta(\xi,0) = \vartheta_0 = 0$ and $g(\tau) = 0$ (comp. (2.6) and (2.8)), and comparing with (2.7), $\xi^*$ stands for $\xi_1$ and $\vartheta^*$ stands for $\vartheta_f$.

In order to find temperature distribution at the beginning of the heating process, we apply the backward finite difference to approximate the time derivative in Eq. (2.1)

$$\begin{aligned} &\frac{\partial \vartheta(\xi,\tau)}{\partial \tau} \approx \frac{\vartheta(\xi,\tau) - \vartheta(\xi,\tau-\Delta\tau)}{\Delta\tau} = \alpha^2[\vartheta(\xi,\tau) - \vartheta(\xi,\tau-\Delta\tau)] \\ &\alpha^2 = \frac{1}{\Delta\tau} \end{aligned} \tag{5.2}$$

Then, the approximate solution to the problem reads

$$\vartheta(\xi,\Delta\tau) = \vartheta^* \frac{\cosh(\alpha\xi)}{\cosh(\alpha\xi^*)} \tag{5.3}$$

Consider the influence of the measured temperature noise $\delta\vartheta^*$ and inaccuracy of the thermoelement location $\delta\xi^*$ on the value of temperature. From (5.3), we find

$$\vartheta(\xi,\Delta\tau)+\delta\vartheta(\xi,\Delta\tau)=(\vartheta^*+\delta\vartheta^*)\frac{\cosh(\alpha\xi)}{\cosh[\alpha(\xi^*+\delta\xi^*)]} \tag{5.4}$$

Then, subtracting (5.3) from (5.4), we arrive at the temperature error

$$\delta\vartheta=\vartheta^*\frac{\cosh(\alpha\xi)}{\cosh(\alpha\xi^*)}\Big(\frac{\cosh(\alpha\xi^*)}{\cosh[\alpha(\xi^*+\delta\xi^*)]}-1\Big)+\delta\vartheta^*\frac{\cosh(\alpha\xi)}{\cosh[\alpha(\xi^*+\delta\xi^*)]} \tag{5.5}$$

and then we easily find the relative temperature error

$$\frac{\delta\vartheta}{\vartheta}=\frac{\cosh(\alpha\xi^*)}{\cosh[\alpha(\xi^*+\delta\xi^*)]}-1+\frac{\delta\vartheta^*}{\vartheta^*}\frac{\cosh(\alpha\xi^*)}{\cosh[\alpha(\xi^*+\delta\xi^*)]} \tag{5.6}$$

The first derivative of (5.4) with respect to $\xi$ leads to the following description of the heat flux error

$$\delta\frac{\delta\vartheta(\xi,\Delta\tau)}{d\xi}=(\vartheta^*+\delta\vartheta^*)\frac{\alpha\sinh(\alpha\xi)}{\cosh[\alpha(\xi^*+\delta\xi^*)]}-\frac{d\vartheta(\xi,\Delta\tau)}{d\xi}=\alpha\tanh(\alpha\xi)\delta\vartheta \tag{5.7}$$

Hence, we easily obtain the relative heat flux error

$$\frac{\delta\frac{d\vartheta}{d\xi}}{\frac{d\vartheta}{d\xi}}=\frac{\delta\vartheta}{\vartheta} \tag{5.8}$$

The maximum value of the error $\delta\vartheta$ appears for $\xi=1$, i.e. on the boundary with an unknown thermal condition. The relative heat flux error is proportional to the relative temperature error. Notice that the error $\delta\vartheta/\vartheta$ does not depend on the variable $\xi$.

Consider now a flat slab with thickness $h=100\,\mathrm{mm}$ made of steel with thermal coefficients $c=500\,\mathrm{J/(kgK)}$, $\lambda=30\,\mathrm{W/(mK)}$, $\rho=7800\,\mathrm{kg/m^3}$. The time step is $\Delta t=0.1\,\mathrm{s}$. Then the Fourier number $\Delta\tau=0.00007623$ and $\alpha=\sqrt{1/\Delta\tau}=114.0175$. Moreover, let the inaccuracy of the thermoelement location be $\delta x^*=0.2\,\mathrm{mm}$ and $x^*=99\,\mathrm{mm}$. Hence, we obtain $\xi^*=0.99$ and $\delta\xi^*=0.002$. For the thermal shock ($\xi^*=\xi=1$ and $\vartheta(\xi=1,\tau)=1$) we have $\vartheta(\xi,\Delta\tau)=\cosh(\alpha\xi)/\cosh\alpha$ (comp. (5.3)) and for $\xi^*=0.99$, we obtain

$$\vartheta^*=\vartheta(\xi^*,\Delta\tau)=\frac{\cosh(\alpha\xi^*)}{\cosh\alpha} \tag{5.9}$$

Hence, for the accurate $\vartheta^*$, we arrive at the formula

$$\vartheta(\xi,\Delta\tau) = \vartheta^* \frac{\cosh(\alpha\xi)}{\cosh(\alpha\xi^*)} = \frac{\cosh(\alpha\xi)}{\cosh\alpha} = \frac{\mathrm{e}^{-\alpha(1-\xi)} + \mathrm{e}^{-\alpha(1+\xi)}}{1+\mathrm{e}^{-2\alpha}} \tag{5.10}$$

From the input data we find that $\alpha\xi^* \sim 100$ and, therefore, we can write with a negligible error

$$\frac{\cosh(\alpha\xi)}{\cosh[\alpha(\xi^*+\delta\xi^*)]} = \mathrm{e}^{\alpha(\xi-\xi^*-\delta\xi^*)} \frac{1+\mathrm{e}^{-2\alpha\xi}}{1+\mathrm{e}^{-2\alpha(\xi^*+\delta\xi^*)}} = \mathrm{e}^{\alpha(\xi-\xi^*-\delta\xi^*)} \tag{5.11}$$

Therefore, the errors of the temperature determination read

$$\begin{aligned} \delta\vartheta &= \mathrm{e}^{\alpha(\xi-\xi^*)}[\vartheta^*(\mathrm{e}^{-\alpha\delta\xi^*}-1) + \delta\vartheta^*\mathrm{e}^{-\alpha\delta\xi^*}] \\ \frac{\delta\vartheta}{\vartheta} &= \mathrm{e}^{-\alpha\delta\xi^*} - 1 + \frac{\delta\vartheta^*}{\vartheta^*}\mathrm{e}^{-\alpha\delta\xi^*} \end{aligned} \tag{5.12}$$

For heat flux errors, we obtain

$$\delta\Big(\frac{d\vartheta}{d\xi}\Big) = \alpha\tanh(\alpha\xi)\mathrm{e}^{\alpha(\xi-\xi^*)}[\vartheta^*(\mathrm{e}^{-\alpha\delta\xi^*}-1) + \delta\vartheta^*\mathrm{e}^{-\alpha\delta\xi^*}] \tag{5.13}$$

Notice that the temperature distribution as well as heat flux error distribution is reinforced exponentially for $\xi > \xi^*$ and is damped exponentially for $\xi < \xi^*$. The conclusion is that the closer $\xi^*$ from the boundary $\xi = 1$, the smaller value achieves the multiplier $\exp[\alpha(\xi-\xi^*)]$, $\xi > \xi^*$. For $\xi = 1$, $\alpha = 114.0175$, $\xi^* = 0.99$, $\exp[\alpha(1-\xi^*)] = 3.1273$ and for $\xi^* = 0.98$, $\exp[\alpha(1-\xi^*)] = 9.78$. For the heat flux, the coefficient of reinforcement is equal to $\alpha\exp[\alpha(1-\xi^*)]$.

One more conclusion results from the above analysis, namely that the number of measurement points in the slab (or a cylindrical layer) along the line perpendicular to the boundary should be as small as possible, because – according to the last conclusion – all inaccuracies of the measurements will be reinforced in points placed closer to the boundary with the unknown thermal condition than the measurement points.

Finally, we can conclude that the most important seems to be proper smoothing of the input data. If the smoothing is appropriate to the considered problem, the results will be encumbered with an error not exceeding the inaccuracy of the input data.

## 6. Conclusions

In the paper two methods of solving the inverse transient heat conduction problems are presented and compared. Both methods, the sequential and global one, rely on analytic form of the direct problem solution. The problem is formulated for a flat slab, a cylinder layer and a spherical one, but the detailed consideration is presented for the cylindrical layer only. In the direct problem solution, the unknown temperature of the inner boundary of the cylindrical layer is an element of convolution. In the sequential method, the successive values of the inner boundary temperature are calculated one by one after each measurement of temperature at chosen inner points of the considered body. In the global one, all values of the inner boundary temperature are calculated simultaneously based on all measured temperatures at the inner points.

In the case of sequential method, the inner boundary temperature is sensitive to temperature measurement errors. Also for the initial moments of time and when the thermoelement is placed too far from the inner boundary, the results are not satisfactory and instability of the solution appears.

The global method is free of these disadvantages.

For larger time steps, both methods lead to practically the same results.

From the numerical calculation a rather obvious conclusion follows, namely, that the smoothing of the results of temperature measurement is important for the solution quality. The mean-square smoothing of the temperature measurement with the use of hyperbolic splines (Kosma, 1999), leads to a natural approximation of real temperatures because the heat conduction equation solution may be expressed with the use of hyperbolic functions (Ciałkowski, 2006).

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## Sekwencyjna i globalna metoda rozwiązania zagadnienia odwrotnego dla równania przewodnictwa ciepła

Streszczenie


W pracy przedstawiono rozwiązanie zagadnienia odwrotnego w oparciu o analityczną postać rozwiązania zagadnienia prostego w postaci splotowej. Ta postać analityczna $T(r,t)$ jest powierzchnią, która jest dopasowywana do przebiegów temperatury pomierzonej (obciążonej błędem) w punktach wewnętrznych. Dla przebiegów szybkozmiennych rozwiązanie zagadnienia odwrotnego jest bardzo wrażliwe na błędy pomiarów (małe czasy próbkowania). Dla otrzymania wiarygodnych wyników zastosowano metodę sekwencyjnego (z kroku na krok) i globalnego rozwiązywania zagadnienia odwrotnego w połączeniu z wygładzaniem wyników pomiarowych za pomocą hiperbolicznych funkcji sklejanych. Wyniki obliczeń numerycznych potwierdzają efektywność przedstawionych metod.


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