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Application of the gradient method for polarimetric data inversion

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Abstract
Gradient procedure is suggested, which allows to fit multi-parameter theoretical model to multi-channel experimental data obtained by means of polarimetric measurements. Procedure deals with nonlinear system of differential equations that describe the evolution of an electromagnetic wave in a hot plasma and provides sufficiently fast convergence to stationary values. The procedure is illustrated with numerical calculations in the case of one and two measurement channels, that allow to retrieve two and four of plasma parameters respectively.

Introduction

There is a growing interest in developing a reliable method for the measurement of the internal magnetic field in high temperature, magnetically confined plasmas. A special need for such diagnostic arises in the investigation of the tokamak devices in which the measurement of the poloidal field distribution would yield the plasma current density profile. This information is essential for understanding confinement, stability and energy balance of the tokamak plasma.

A precise control of the plasma position is a key issue in order to avoid damages on the first wall of the device. Such a control is essential when high-power long-duration plasmas have to be performed as on the Tore Supra tokamak. The current carried by the plasma can be localized using magnetic measurements (pick-up coils) outside the plasma. The plasma boundary can thus be identified and controlled on real time in less than a few milliseconds. In a tokamak plasma the distribution of the plasma current plays an important role because the resulting poloidal magnetic field determines the confinement properties and is crucial for the stability of the tokamak plasma. The appearance and growth of numerous instabilities are closely connected to the existence of certain rational surfaces in the plasma, as well as subtle local modifications of the poloidal magnetic field.

In order to get information on the current distribution inside the plasma, more sophisticated calculation must be performed. Because magnetic measurements are no longer sufficient to constrain the solution when detailed information on current distribution inside the plasma are mandatory, other measurements must be introduced as external constraints in the solver. At the beginning of 2008 a five-channel vertical DCN laser system has been constructed on the EAST device using both the interference and polarization effects. The system is used mainly for getting electron density profile, poloidal current profile and for density feedback control simultaneously.

The use of polarization phenomena in the plasma diagnostic encountered in practice the major difficulty. It results from the fact that in the general case when the effects of Faraday and Cotton-Mouton determining the state of polarization of the sample beam are comparable, the equations showing the changes of the measured polarization angles, depending on plasma parameters are partial differential equations [1, 2]. As a result, the lack of simple analytical relation between changes in the polarization state of the beam sample, and the plasma parameters, forces us to use unconventional methods to reproduce plasma parameters on the basis of experimental results.

Majority of plasma models used for polarimetric data inversion deal with very simple, if not primitive configuration of electron density and static magnetic field [3]. The papers [4, 5, 6] have studied "non-conventional" procedure for polarimetric data inversion. This procedure has implied fitting the two- parameters plasma model with toroidal magnetic field to experimental values of angular parameters of polarization ellipse. In the present work as in previous articles [4, 5, 6] we use new approach in plasma polarimetry: angular variables technique (AVT), which deals with angular parameters of polarization ellipse. This allows simplifying the problem and avoiding Stokes vector formalism.

Gradient algorithm for experimental data inversion in frame multi-parameters plasma model

Polarization state of electromagnetic wave traditionally is characterized by azimuthal – ψ and ellipticity – χ angles. Evolution of these parameters along the ray is described by the equations of angular variables technique (AVT) [5]:

$$\begin{aligned} \frac{\partial \psi}{\partial \sigma} &= 0.5\,(\Omega_3 - (\cos[2\psi[\sigma]]\Omega_1 + \\ &\quad + \sin[2\psi[\sigma]]\Omega_2)\tan[2\chi[\sigma]]) \\ \frac{\partial \chi}{\partial \sigma} &= 0.5\,(\sin[2\psi[\sigma]]\Omega_1 - \cos[2\psi[\sigma]]\Omega_2) \end{aligned} \tag{1}$$

Here, σ is length along the ray and $\Omega_{1,2,3}$ are plasma parameters, widely used in plasma polarimetry [7], coefficients $\Omega_{1,2}$ correspond to Cotton-Mouton effects, and coefficient Ω_3 – to Faraday phenomenon [2, 4, 6]:

$$\begin{aligned} \Omega_1 &= C_1\lambda^3\left(B_x^2 - B_y^2\right)N_e \\ \Omega_2 &= C_2\lambda^3 2\left(B_x B_y\right)N_e \\ \Omega_3 &= C_3\lambda^2 B_z\, N_e \end{aligned} \tag{2}$$

within the SI system the constants C_i are: $C_1 = 2.45681\cdot10^{-11}$; $C_2 = 2.45681\cdot10^{-11}$; $C_3 = 5.26241\cdot10^{-13}$ respectively.

It is noteworthy that AVT equations (1) are equivalent to the equations of Stokes vector formalism (SVF) [7], but are much more convenient as compared to SVF equations: two equations in AVT instead of three equations in SVF.

Let $\psi(\mathbf{p})$ and $\chi(\mathbf{p})$ are solutions of AVT equations (1) for i-th polarimetric channel and $\mathbf{p} = [p_1, p_1, ..., p_N]$ is a set of N plasma parameters to be determined by fitting angular variables ψ and χ to experimental polarimetric data.

Equating $\psi_i(\mathbf{p})$ and $\chi_i(\mathbf{p})$ to experimental observations ψ_{iex} and χ_{iex}, we obtain $2i$ equations for N parameters $(p_1, p_2, ..., p_N)$:

$$\psi_i(\mathbf{p}) = \psi_{iex};\quad \chi_i(\mathbf{p}) = \psi_{iex} \tag{3}$$

In general we do not have analytical solutions of AVT equations, so we are enforced to apply one of the numerical methods, namely – gradient approach [8, 9]. Let us introduce the error function $\Phi(p)$, which is quadratic measure of inconsistency between theoretical and experimental values:

$$\Phi^{(k)} = \sum_{i=1}^{n} (\psi_i(k) - \psi_{iex})^2 + (\chi_i(k) - \chi_{iex})^2 \tag{4}$$

Here:

- $\Phi(k)$ – the error function for the k-th step;
- $\chi_i(k)$ – the polarization angles for the i-th channel at the k-th step;
- ψ_{iex} – the experimental values of the polarization angles obtained for the i-th channel.

In consequence the equation:

$$\Phi(\mathbf{p}) = 0 \tag{5}$$

which minimizes inconsistency between theoretical and observation data, is equivalent to eq. (3).

Let $p_i(0)$ be starting value of parameter p_i in procedure of consequent approximations. According to gradient method, increment $\delta_i^{(1)}$ at the first step should be proportional to gradient $\nabla\Phi$ of the error function taken with minus sign:

$$\delta_i(1) = -H_i(1)\,G_i(1) \tag{6}$$

Here G means the gradient of the error function:

$$G = \nabla\Phi(p_1, p_2, ..., p_n) \tag{7}$$

whose components are respectively:

$$\begin{aligned} \frac{\partial\Phi}{\partial p_i} &\approx \frac{\Delta\Phi}{\Delta p_i} = \\ &= \frac{\Phi(p_1,...,p_i+\Delta p_i,...,p_n) - \Phi(p_1,...,p_i,...,p_n)}{\Delta p_i} \end{aligned} \tag{8}$$

It is reasonable to choose the coefficient H_i according to inequality:

$$H_i \le \frac{1}{2M}\frac{\Phi(p_0)}{G_i^2} \tag{9}$$

where M is total number of parameters, envisaged for fitting.

As a result:

$$\delta_i(1) \le -\frac{1}{2M}\frac{\Phi(p_0)}{G_i} \tag{10}$$

so that increment:

$$\Delta\Phi(1) \le \sum_{i=1}^{M}\frac{\partial\Phi}{\partial p_i}\delta_i(1) \tag{11}$$

of the error function $\Phi(p)$ will not exceed:

$$\Delta\Phi(1) \le \sum_{i=1}^{M}\frac{\partial\Phi}{\partial p_i}\delta_i(1) = \frac{1}{2}\Phi(p_0) \tag{12}$$

Such increment provides monotonous decreasing of the error function its exponential convergence to zero.

Thus, the first iteration for parameter p_i will be:

$$p_i(1) = p_i(0) + \delta_i(1) \tag{13}$$

Repeating these operations with starting value $p_i(1)$ we arrive to the second iteration $p_i(2)$, then to $p_i(3)$ and so on. According to general theory, sequence $p_i(1)$, $p_i(2)$, ..., $p_i(N)$ tends to solution of eqs. (3) and (5).

In the next section gradient procedure will be applied for polarimetric data inversion in conditions only two parameters: $M = 2$ and subsequently we shall perform numerical simulations for more complicate model with $M = 4$ parameters.

Numerical illustration in case of two parameters

In this section we intend to demonstrate an efficiency of the suggested method in the simplest magnetic configuration, when sounding ray lays in the horizontal (equatorial) plane of the toroidal system. In this cane influence of the poloidal field is small enough and we can restrict our selves by considering only toroidal field (see Fig. 2).

Performing numerical simulations, we solve the eq. (1) with the following values of parameters, modeling the ITER project [7].

i) The maximum value of magnetic field is accepted to be $B_0 = 5.3$ T, and magnetic profile $y(\xi)$ is supposed to correspond to the field of toroidal solenoid:

$$y(\xi) = \frac{r_{\min}}{r_{\min} + \xi} \tag{14}$$

where $r_{\min}$ is a minor radius of the toroidal camera.

ii) Sounding frequency will be $\omega = 1.5 \cdot 10^{13}$ Hz. Corresponding ware length $\lambda = 125$ μm is comparable to the wave length $\lambda = 195$ μm, applied in JET.

iii) The angle between the ray and magnetic vector **B** will be 88.2º, so that $\cos\alpha = 0.031$.

iv) The maximum electron density max N_e will be 10^{14} cm^{-3}. Density profile $x(\xi)$ we approximate by Gaussian curve:

$$x(\sigma) = \exp\left[-\frac{(\sigma - \sigma_0)^2}{g^2}\right] \tag{15}$$

with $g = 3$. We have chosen the Gaussian profile (22 with a view to illustrate our procedure only). Dealing with the real plasma, profile $x(\sigma)$ should be extracted from Thomson or Lidar scattering data. In principle, profile $x(\sigma)$ night be bimodal, that is with two local maxima.

v) The values of azimuthal and ellipticity angles, simulating the results of experimental measurements will by calculated from eqs. (1) with the initials values $\psi(\sigma = 0) = \pi/4$ and $\chi(\sigma = 0) = 0$, as it frequently used in polarimetry.

vi) The inner radius of toroidal camera is taken $r_{\min} = 2$ m and external radius $r_{\max} = 8$ m, so that the length of the ray path in camera is 6 m. The ray crosses the axis of toroidal camera at the angle $\alpha = 88.2$º, so that $\cos\alpha = 0.031$. At this condition the Faraday and Cotton-Mouton effects are comparable with each other. Thus, the method of inversion is valid, even if neither Faraday nor Cotton-Mouton effect are small.

vii) Parameters $\bar{X}_{st}$ and $\bar{Y}_{st}$, corresponding to the maximum values of magnetic field $B = 5.3$ T and $N_e = 10^{14}$ T, are respectfully $\bar{X}_{st} = 0.0014$, $\bar{Y}_{st} = 0.063$. The probing values $\bar{X}_0$ and $\bar{Y}_0$ were then to be at 30–40% less than stationary values X_{st} and Y_{st}.

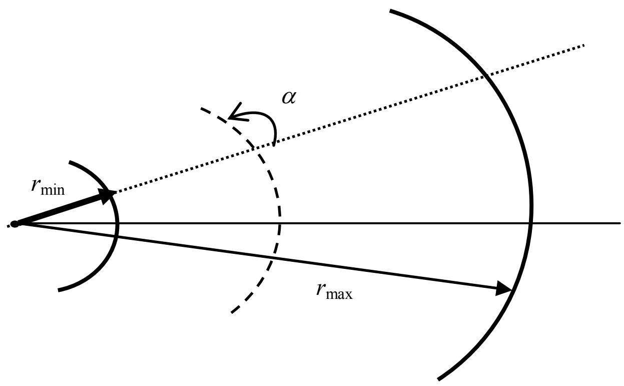

Fig. 1. Position of the sounding ray (bold dashed line) in the equatorial plane of tokamak. Here $r_{\min}$ and $r_{\max}$ are inner and external radius of toroidal camera

Data obtained by numerical simulation for parameters $\bar{X}$, $\bar{Y}$ and for producing function $\Phi(\bar{X},\bar{Y})$ are presented in the table 1, whereas three first steps on the *XY* plane are shown on the figure 2.

Table 1. Evolution of parameters $\bar{X}$, $\bar{Y}$ values of azimuthal and ellipticity angles and function $\Phi(\bar{X},\bar{Y})$ in frame of the gradient method

Step	X/Y	φ	ψ	χ
0	0.03/0.03	2.03	–0.772	0,0062
1	0.01/0.04	0.137	0.056	0.285
2	0.005/0.05	0.104	0.365	0,276
3	0.002/0.06	0.0051	0.591	0,175
4	0.0015/0.065	0.0008	0.630	0,155
5	0.0014/0.062	$1.31\cdot10^{-6}$	0.647	0,132
experiment		0	0.648	0.132

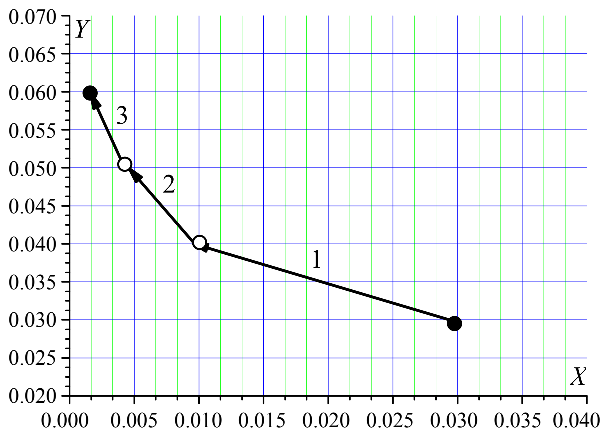

Fig. 2. Three first steps in plane (*X*, *Y*)

After the third step we are so close the real values that points referring to consecutive steps practically would be no distinguishable for the used scale.

According to table 1, iterative procedure sufficiently fast approaches to the stationary point $\bar{X}_{\text{st}} = 0.0014$, $\bar{Y}_{\text{st}} = 0.062$, relative inaccuracy $\left(\bar{X}-\bar{X}_{\text{st}}\right)/\bar{X}_{\text{st}}$ and $\left(\bar{Y}-\bar{Y}_{\text{st}}\right)/\bar{Y}_{\text{st}}$ become less 2–3% already after 3–5 iterations.

Thus the gradient method used jointly with knowledge-based model of toroidal plasma provides effective inversion of polarimetric data.

Numerical simulations in case of four parameters

In order to demonstrate the correctness of the gradient method the simple model of plasma configuration has been chosen: plasma with circular cross-section of the magnetic flux surfaces and parabolic density profile $N_e = N_0(1-\rho^2)$, where $\rho = r/a$ is the normalized radius of the flux surface in the plasma with minor radius *a* (Fig. 1). In the case of a large aspect-ratio circular plasma $a \langle\langle R$, with a toroidal current density distribution $j = j_0(1-\rho^2)^v$ providing the total current j_0, the magnetic field components, at the point with the radius *R* and normalized radius ρ, are [6]:

$$\begin{aligned} B_R &= B_0 \frac{R_0}{R} \\ B_\theta &= \frac{\mu_0 I_0}{2\pi a} \frac{1-\left(1-\rho^2\right)^{v+1}}{\rho} \end{aligned} \qquad (16)$$

where B_R and B_θ are the toroidal and the poloidal magnetic field and B_0 is the toroidal magnetic field on the magnetic axis of the plasma with the major radius R_0.

The plasma parameters have been chosen similar to that of the large thermonuclear plasma devices, like JET or ITER: $B_0 = 6$ T, $R_0 = 3$ m, $a = 1.80$ m, $N_e = 0.75\cdot10^{20}$ m^{-3}, $I = 3.2$ MA, $v = 2.6$. For such a cords the magnetic field components in the beam reference frame are: $B_x = B_R$, $B_y = B_\theta \sin\varphi$ and $B_z = -B_\theta \cos\varphi$.

Accepting configuration of two vertical polarimetric channels (see Fig. 1), we place the sources (DCN lasers with wavelengths $\lambda = 195$ μm and initial polarization $\psi_0 = \pi/4$, $\chi_0 = 0$ and receivers at the points:

$$R_1 = 0.3 \text{ m}; \quad R_2 = 0.8 \text{ m} \qquad (17)$$

in a poloidal plane of a tokamak.

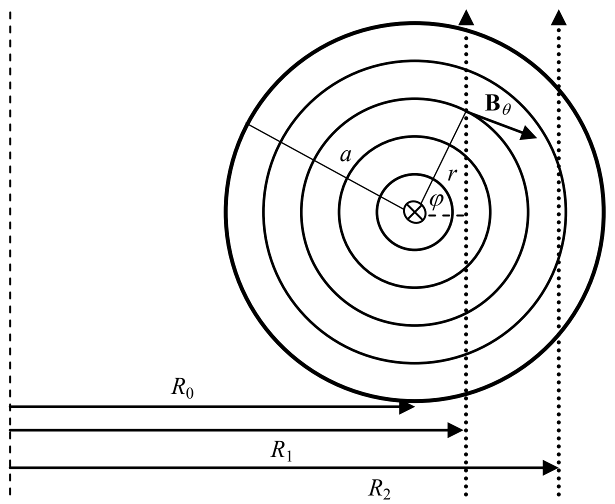

Fig. 3. Schematic of a vertical lines of sight in the plasma with circular flux surfaces

Table 2. Evolution of plasma parameters p_m in frame of gradient procedure, applied for polarimetry data inversion

Parameters p_i	Unit	p_i^*	Consequent approximations				
			$p_i(0)$	$p_i(1)$	$p_i(2)$	$p_i(6)$	$p_i(7)$
$p_1 = N_e$	10^{20} m^{-3}	0.75	0.5	0.742	0.763	0.753	0.75
$p_2 = I$	MA	3.2	3.0	3.028	3.34	3.22	3.21
$p_3 = a$	m	1.8	1.5	1.54	2.02	1.78	1.8
$p_4 = \nu$	1	2.6	2.0	2.1	2.33	2.57	2.59
error function Φ	1	0.00	0.2428	0.0272	0.009	$5{\cdot}10^{-5}$	$5.86{\cdot}10^{-7}$

According to equations (1) "experimental" values obtained for the polarization angle for corresponding canals are:

$$\psi_{1\mathrm{ex}} = 1.16585; \qquad \psi_{2\mathrm{ex}} = 1.451516$$
$$\chi_{1\mathrm{ex}} = 440777; \qquad \chi_{2\mathrm{ex}} = 0.272166 \qquad (18)$$

Now, our task is to reproduce the real (of interest to us) plasma parameters on the basis of these data obtained from polarimetric measurements, using the gradients method.

Based on the genuine parameters p_i^*, we calculate polarimetric variables $\psi_i(\mathbf{p})$ and $\chi_i(\mathbf{p})$ from AVT equation (1) and consider them as "experimental" data $\psi_{i\mathrm{ex}}(\mathbf{p})$ and $\chi_{i\mathrm{ex}}(\mathbf{p})$. Total current I in eq. (14) we also identify with experimental value, obtained from magnetic measurements. Toroidal magnetic field, having only z-component, we also assume to be known from magnetic measurements.

Then we take starting parameters $p_i(0)$ and perform gradient procedure of consequent approximations, described in previous section "genuine" p_i^* and starting,values are presented in table 2 along with consequent approximations $p_i(1)$, $p_i(2)$, ..., $p_i(N)$. Table 2 presents also evolution of the error function Φ.

Starting parameters $p_i = (0)$ were once again chosen to be about 30% lesser than "genuine" ones to test the convergence of gradient method to proper values.

The first and second columns in the table 2 show parameters p_i and their units. "Genuine" parameters p_i^* are present in the third column.

It follows from table 2, that values $p_i(7)$ achieved at ninth step of iteration procedure differ from "true" values p_i^* at most at 2–3%.

Changes of the azimuthal ψ and ellipticity angle χ along the radius during several selected steps obtained in the first and second channel are presented on the figures 4–7.

Analysis of these graphs shows the full correlation between the error function Φ, and the variability of the two polarization angles ψ, χ, which are becoming more and more similar to conditions set out in the experiment.

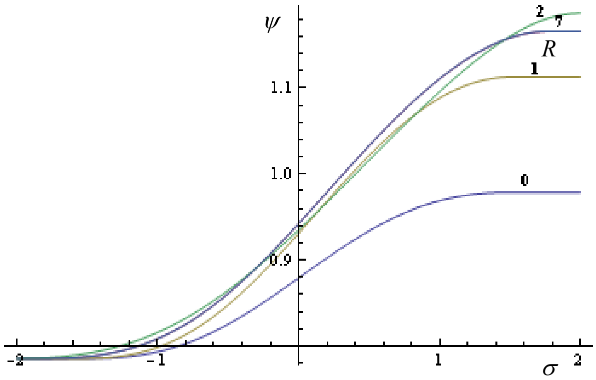

Fig. 4. Changes of the azimuthal angle ψ along the radius during several successive approximations (0 – initial conditions, R – the real plasma parameters corresponding to the measurement results) obtained in the first channel

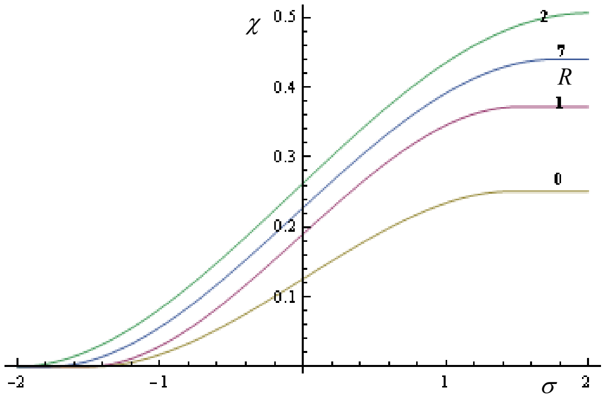

Fig. 5. Changes of the ellipticity angle χ along the radius during several successive approximations (0 – initial conditions, R – the real plasma parameters corresponding to the measurement results) obtained in the first channel

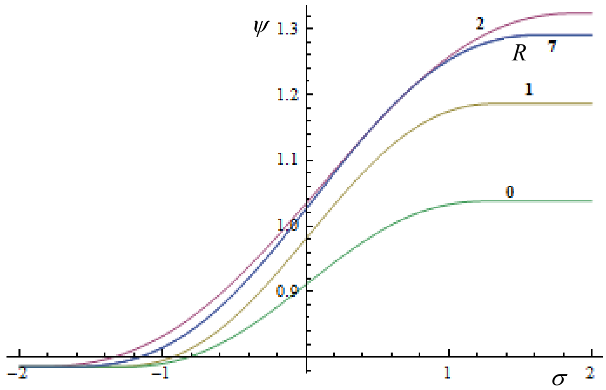

Fig. 6. Changes of the azimuthal angle ψ along the radius during several successive approximations (0 – initial conditions, R – the real plasma parameters corresponding to the measurement results) obtained in the second channel

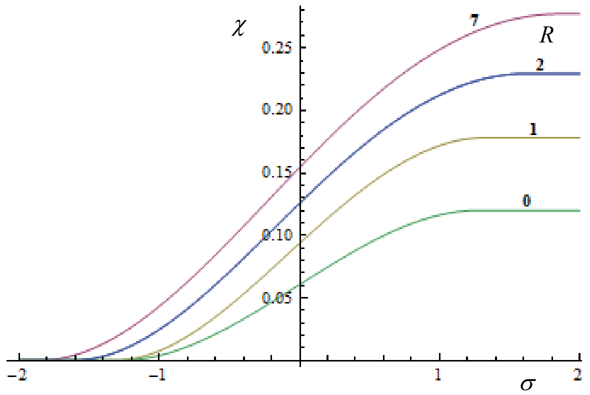

Fig. 7. Changes of the ellipticity angle χ along the radius during several successive approximations (0 – initial conditions, *R* – the real plasma parameters corresponding to the measurement results) obtained in the second channel

Conclusion

The need for maximum reliable data on the parameters of nuclear plasma forces to search for new methods applied in diagnostics. At first the equilibrium reconstruction code used in tokamaks experiments (EFIT at JET) to map diagnostic information and to derive basic plasma properties like current density and safety factor was based on magnetic probe measurements only. On the basis of the resulting data, the system was able to solve the Grad-Shafranov equation by adjusting the flux fit function. Naturally, the accuracy of the results depended on the accessibility and quality of the diagnostic information. However, as shown by the experiment data obtained exclusively by means of magnetic measurements do not allow for accurate reproduction profiles of the safety factor and current density in particular plasma scenarios. Therefore, it became necessary to supplement the data thus obtained by the additional internal diagnostic information. One of the elements of such additional system is setup for the interferometer – polarimeter which allows a measurement of the line integrated density and Faraday rotation along the same a straight line. Incorporating the Faraday rotation mainly changes the plasma elongation. In shear reversed plasma incorporating the Faraday rotation data as an additional constraint results in an increase the safety factor by 10%. However, the use of pure Faraday rotation causes distinct restriction of the measurement, and therefore in practice (at example JET) may be used a few channels only.

In the proposed method has been taken into account both, the influence of Faraday rotation and the Cotton – Mouton effect. This allows to use practically all channels for polarimetric measurements. It is important, therefore, that each additional channel allows to determine two additional parameters of the plasma. The problem with the lack of analytical relations between measured polarization angles and changes of plasma parameters that occurs with the general case is solved by applying the gradient method, which allows for rapid reconstruction of plasma parameters with any accuracy practically.

This paper considers both, 2 and 4 parameter plasma model only as illustration of multi-parameter approach. In fact, amount of parameters can be increased or reduced in dependence on the aim of plasma modeling. If we would like to describe fine details of plasma configuration we may involve slight asymmetry in density distributions both, in horizontal and in vertical directions, to account potential influence of toroidal form of camera or influence of diverter area.

Additional parameters in electron density distributions can be used to fit the model to Thomson scattering data, which bear information on local density variations. Of special interest are additional parameters, which night describe plasma configuration in diverter area.

Summarizing the results of numerical simulations, we may say that gradient procedure has approved its efficiency in condition of 4 parameters model it has demonstrated sufficiently fast convergence to "genuine" data and acceptable accuracy. Of course, accuracy of the gradient procedure in conditions of real plasma, which is not necessarily close to chosen model requires further studies. As was mentioned above starting parameters $p_i(0)$ were chosen to be 30–50% larges or lesser than the "genuine" ones only to verify their convergence to the "genuine" parameters and to test the gradient method. In practice, having available the results obtained by other methods, we can always choose the starting point much closer to actual conditions, which of course greatly reduce the number of necessary steps, so in consequence, we will focus exclusively on the measurement accuracy.

Therefore, we have reason to believe that the presented method should lead to greater use of polarimetry in the diagnostic of thermonuclear plasma making it an effective instrument for the comprehensive diagnostic applied in tokamaks.

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