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The influence of coordinates error of navigational marks on the accuracy of position in radar navigation

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Abstract

Terrestrial navigation is one of the method of parametric navigation, i.e. a method of fix determination. Recently the method has become quite common again thanks to automated measurements of navigational parameters (radar navigation, radio direction finders and others) and connection with the ECDIS. The accuracy of position coordinates determined by such method is affected by navigational measurement errors and the accuracy of navigational mark coordinates used for position determination. The coordinates values are obtained from nautical publications (including charted data) or from an electronic chart data base. This article presents an original method for an analysis of how the accuracy of navigational mark coordinates affects the accuracy of ship's fix. Considerations are supplemented with an example of the most common terrestrial position determination in marine navigation.

Introduction

In today's navigation the main method of position coordinates determination (more generally: state vector) is parametric navigation [1]. The so called fix is determined by various methods and technical means (terrestrial, celestial, radio, satellite navigation etc.). Considering the methods for the processing measurement (observation) results (navigational parameters), it usually deals with solving systems of nonlinear equations, relating ship position coordinates to coordinates of navigational marks and measured navigational parameters.

In classical methods for the assessment of position coordinates determination it assumes that coordinates of navigational marks (or ephemerides of celestial bodies and satellites – mobile aids to navigation – navigational marks) were determined with an accuracy higher by one order than the predicted accuracy of ship's position determination. To put it simple it may say that positions of navigational marks have no error. In reality it is not the case. It also occurs these days that positional accuracy of navigational marks is comparable with the predicted accuracy of the system. It, therefore, should be accounted in the algorithms for calculating position coordinates and position accuracy assessment. The effect of initial mark coordinates errors on the series of subsequent positions is taken into account in, e.g. [1] that, however, discusses a particular case and only in the range of changes in mean elliptical error of the point being determined.

The accuracy of navigational marks (ephemerides) coordinates affects the accuracy of the determined generalized vector of measurements (projected parameters, dead reckoned measurements). These, consequently, influence the values of covariance matrix elements of this vector. A method presented below allows to take into account the accuracy of navigational marks' (ephemerides) coordinates affecting the accuracy of the position being determined (its covariance matrix).

Influence of the navigational mark position covariance matrix on the covariance matrix of navigational parameters

Generally, a system of non-linear equations of navigational functions can be written in a form of vector function of multiple variables [1, 3, 4]:

$$\mathbf{f}(\mathbf{x}, \mathbf{zn}_1, \mathbf{zn}_2, \ldots, \mathbf{zn}_k) = \mathbf{u} \tag{1}$$

where:

- $\mathbf{x}$ – m-dimensional state vector (of ship's coordinates, searched-for position);
- $\mathbf{zn}_i$ – l-dimensional vector of the coordinates of i-th navigational mark ($i = 1, 2, \ldots, k$);
- $\mathbf{u}$ – n-dimensional vector of measured navigational parameters;
- $\mathbf{f}$ – n-dimensional vector function;

usually $m = l$ (the same navigational space);

$n \geq m$;

$k > n$ – more navigational marks than position lines (e.g. hyperbolic / elliptical systems, halop and others);

$k = n$ – number of navigational marks is equal to the number of position lines;

$k < n$ – number of navigational marks is lower than the number of position lines (two position lines from one mark).

For $n < m$ not a point solution, but an area (solution interval) will be obtained.

Assuming the presently most general case of navigational space (φ, λ, h, Δt or x, y, z, Δt), it can write equation (1) as a system of equations with multiple variables):

$$
\begin{aligned}
&f_1(x,y,z,\Delta t;x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2;\ldots;x_k,y_k,z_k,t_k)=\\
&=u_1\\
&f_2(x,y,z,\Delta t;x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2;\ldots;x_k,y_k,z_k,t_k)=\\
&=u_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&f_n(x,y,z,\Delta t;x_1,y_1,z_1,t_1;x_2,y_2,z_2,t_2;\ldots;x_k,y_k,z_k,t_k)=\\
&=u_n
\end{aligned}
\tag{2}
$$

Note that these equations will not always include all navigational marks. This depends on the kind of navigational lines (hypersurfaces).

Equation (1) solved by the Newton's method of solving nonlinear equations system will have this form:

$$\mathbf{x}_{i+1} = \mathbf{G}^{-1}\mathbf{z} = \mathbf{G}^{-1}(\mathbf{u} - \mathbf{f}(\mathbf{x}_i)) \tag{3}$$

while the least squares method yields:

$$\mathbf{x} = \left(\mathbf{G}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{G}\right)^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{z} \tag{4}$$

where: $\mathbf{G} = \mathbf{f}'(\mathbf{x})$ – Jacobian matrix of the function $\mathbf{f}$ in respect to $\mathbf{x}$,

$$\mathbf{G} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathrm{x}_1} & \frac{\partial f_1}{\partial \mathrm{x}_2} & \cdots & \frac{\partial f_1}{\partial \mathrm{x}_m} \\ \frac{\partial f_2}{\partial \mathrm{x}_1} & \frac{\partial f_2}{\partial \mathrm{x}_2} & \cdots & \frac{\partial f_2}{\partial \mathrm{x}_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial \mathrm{x}_1} & \frac{\partial f_n}{\partial \mathrm{x}_2} & \cdots & \frac{\partial f_n}{\partial \mathrm{x}_m} \end{bmatrix} \tag{5}$$

$\mathbf{u}$ – vector of direct measurements;

$\mathbf{f}(\mathbf{x}_i)$ – vector of dead reckoned measurements;

$\mathbf{z} = \mathbf{u} - \mathbf{f}(\mathbf{x})$ – generalized vector of measurements.

The position $\mathbf{x}$ coordinates vector covariance matrix is expressed by this formula [2, 5]:

$$\mathbf{P}_{\mathbf{x}} = \left(\mathbf{G}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{G}\right)^{-1} \tag{6}$$

where:

$$\mathbf{R} = \mathbf{R}_{\mathbf{u}} + \mathbf{R}_{\mathbf{f}(\mathbf{x})} - \mathbf{R}_{\mathbf{uf}(\mathbf{x})} - \mathbf{R}^{\mathrm{T}}_{\mathbf{uf}(\mathbf{x})} \tag{7}$$

If $\mathbf{u}$ and $\mathbf{f}(\mathbf{x})$ are independent, which practically is the case, then:

$$\mathbf{R} = \mathbf{R}_{\mathbf{u}} + \mathbf{R}_{\mathbf{f}(\mathbf{x})} \tag{8}$$

where:

$$\mathbf{R}_{\mathbf{f}(\mathbf{x})} = \mathbf{H}\mathbf{R}_{\mathbf{zn}}\mathbf{H}^{\mathrm{T}} \tag{9}$$

$\mathbf{H}$ – Jacobian matrix of function $\mathbf{f}$ in respect to $\mathbf{zn}_i$,

$$\mathbf{H} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{zn}_1} & \frac{\partial f_1}{\partial \mathbf{zn}_2} & \cdots & \frac{\partial f_1}{\partial \mathbf{zn}_k} \\ \frac{\partial f_2}{\partial \mathbf{zn}_1} & \frac{\partial f_2}{\partial \mathbf{zn}_2} & \cdots & \frac{\partial f_2}{\partial \mathbf{zn}_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial \mathbf{zn}_1} & \frac{\partial f_n}{\partial \mathbf{zn}_2} & \cdots & \frac{\partial f_n}{\partial \mathbf{zn}_k} \end{bmatrix} \tag{10}$$

$\frac{\partial f_i}{\partial \mathbf{zn}_j} = \operatorname{grad} f_i(\mathbf{zn}_j) = \left[\frac{\partial f_i}{\partial x_j}, \frac{\partial f_i}{\partial y_j}, \frac{\partial f_i}{\partial z_j}, \frac{\partial f_i}{\partial t_j}\right]$ – gradient of function f_i in respect to coordinates of a j-th navigational mark.

$$\mathbf{R}_{\mathbf{zn}} = \begin{bmatrix} \mathbf{R}_{\mathbf{zn}_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\mathbf{zn}_2} & \cdots & \mathbf{0} \\ \cdots & \cdots & .. & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{R}_{\mathbf{zn}_k} \end{bmatrix} \tag{11}$$

$$\mathbf{R}_{\mathbf{zn}_i} = \begin{bmatrix} \sigma^2_{x_i} & \sigma_{x_i y_i} & \sigma_{x_i z_i} & \sigma_{x_i t_i} \\ \sigma_{x_i y_i} & \sigma^2_{y_i} & \sigma_{y_i z_i} & \sigma_{y_i t_i} \\ \sigma_{x_i z_i} & \sigma_{y_i z_i} & \sigma^2_{z_i} & \sigma_{z_i t_i} \\ \sigma_{x_i t_i} & \sigma_{y_i t_i} & \sigma_{z_i t_i} & \sigma^2_{t_i} \end{bmatrix} \tag{12}$$

If each position line is determined only from one navigational mark, then:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{zn}_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \frac{\partial f_2}{\partial \mathbf{zn}_2} & \cdots & \mathbf{0} \\ \cdots & \cdots & .. & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{\partial f_n}{\partial \mathbf{zn}_k} \end{bmatrix} \tag{13}$$

Mean error of ship's position coordinates from bearing and radar range

Let us illustrate the above considerations with a simple case of position determination from bearing on and radar range (distance) to the same navigational mark [6]. This is quite frequent case of position determination in maritime navigation, where radar measurements are used. The situation is shown in figure 1.

Data:

Navigational mark: coordinates (x_0, y_0), covariance matrix elements ($\sigma_{x_0}^2, \sigma_{y_0}^2, \sigma_{x_0 y_0}$).

Range parameters: D, σ_D.

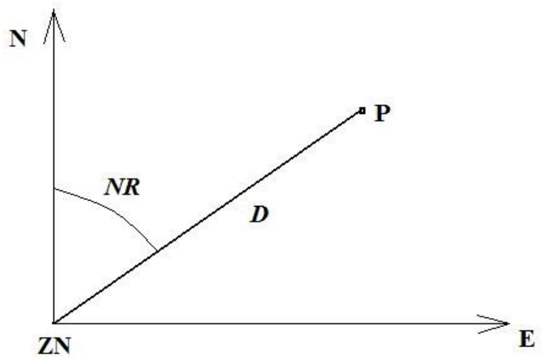

Fig. 1. A position from the bearing on and range to one navigational mark

Partial derivatives of range relative to the coordinates of the position being calculated:

$$\frac{\partial D}{\partial x} = \frac{x - x_0}{D} = \frac{\Delta x}{D} \quad \text{[nondimensional]};$$

$$\frac{\partial D}{\partial y} = \frac{y - y_0}{D} = \frac{\Delta y}{D} \quad \text{[nondimensional]}.$$

Partial derivatives of range in respect to navigational mark position coordinates:

$$\frac{\partial D}{\partial x_0} = -\frac{x - x_0}{D} = -\frac{\Delta x}{D} \quad \text{[nondimensional]};$$

$$\frac{\partial D}{\partial y_0} = -\frac{y - y_0}{D} = -\frac{\Delta y}{D} \quad \text{[nondimensional]}.$$

Bearing parameters: NR, σ_{NR}.

Partial derivatives of bearing in respect to navigational mark position coordinates:

$$\frac{\partial NR}{\partial x} = -\frac{y - y_0}{D^2} = -\frac{\Delta y}{D^2} \left[\frac{1}{\text{m}} \right];$$

$$\frac{\partial NR}{\partial y} = \frac{x - x_0}{D^2} = \frac{\Delta x}{D^2} \left[\frac{1}{\text{m}} \right].$$

Partial derivatives of bearing in respect to navigational mark position coordinates:

$$\frac{\partial NR}{\partial x_0} = \frac{y - y_0}{D^2} = \frac{\Delta y}{D^2} \left[\frac{1}{\text{m}} \right];$$

$$\frac{\partial NR}{\partial y_0} = -\frac{x - x_0}{D^2} = -\frac{\Delta x}{D^2} \left[\frac{1}{\text{m}} \right].$$

The matrix of position lines gradients (Jacobian matrix of the navigational function):

$$\mathbf{G} = \begin{bmatrix} \frac{\partial D_1}{\partial x} & \frac{\partial D_1}{\partial y} \\ \frac{\partial D_2}{\partial x} & \frac{\partial D_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\Delta x}{D} & \frac{\Delta y}{D} \\ -\frac{\Delta y}{D^2} & \frac{\Delta x}{D^2} \end{bmatrix} \tag{14}$$

The transition matrix of reckoned measurements:

$$\mathbf{H} = \begin{bmatrix} -\frac{\Delta x}{D} & -\frac{\Delta y}{D} \\ \frac{\Delta y}{D^2} & -\frac{\Delta x}{D^2} \end{bmatrix} \tag{15}$$

The covariance matrix of the measured navigational parameters:

$$\mathbf{R_u} = \begin{bmatrix} \sigma_D^2 & 0 \\ 0 & \sigma_{NR}^2 \end{bmatrix} \tag{16}$$

The distance (range) measurement is not correlated with the bearing measurement.

The covariance matrix of navigational mark coordinates:

$$\mathbf{R_{zu}} = \begin{bmatrix} \sigma_{x_0}^2 & \sigma_{x_0 y_0} \\ \sigma_{x_0 y_0} & \sigma_{y_0}^2 \end{bmatrix} \tag{17}$$

$$\mathbf{R_{f(x)}} = \mathbf{H R_{zn} H}^{\mathrm{T}} = \begin{bmatrix} \frac{1}{D^2}\left(\Delta x^2 \sigma_{x_0}^2 + 2\Delta x \Delta y \sigma_{x_0 y_0} + \Delta y^2 \sigma_{y_0}^2\right) & \frac{1}{D^3}\left[\left(\Delta x^2 - \Delta y^2\right)\sigma_{x_0 y_0} - \Delta x \Delta y\left(\sigma_{x_0}^2 - \sigma_{y_0}^2\right)\right] \\ \frac{1}{D^3}\left[\left(\Delta x^2 - \Delta y^2\right)\sigma_{x_0 y_0} - \Delta x \Delta y\left(\sigma_{x_0}^2 - \sigma_{y_0}^2\right)\right] & \frac{1}{D^4}\left(\Delta y^2 \sigma_{x_0}^2 - 2\Delta x \Delta y \sigma_{x_0 y_0} + \Delta x^2 \sigma_{y_0}^2\right) \end{bmatrix} \tag{18}$$

$$\mathbf{R} = \mathbf{R_u} + \mathbf{R_{f(x)}} = \mathbf{R_u} + \mathbf{H R_{zn} H}^{\mathrm{T}} =$$

$$= \begin{bmatrix} \sigma_D^2 + \frac{1}{D^2}\left(\Delta x^2 \sigma_{x_0}^2 + 2\Delta x \Delta y \sigma_{x_0 y_0} + \Delta y^2 \sigma_{y_0}^2\right) & \sigma_{x_0 y_0} + \frac{1}{D^3}\left[\left(\Delta x^2 - \Delta y^2\right)\sigma_{x_0 y_0} - \Delta x \Delta y\left(\sigma_{x_0}^2 - \sigma_{y_0}^2\right)\right] \\ \sigma_{x_0 y_0} + \frac{1}{D^3}\left[\left(\Delta x^2 - \Delta y^2\right)\sigma_{x_0 y_0} - \Delta x \Delta y\left(\sigma_{x_0}^2 - \sigma_{y_0}^2\right)\right] & \sigma_{NR}^2 \operatorname{arc} 1^\circ + \frac{1}{D^4}\left(\Delta y^2 \sigma_{x_0}^2 - 2\Delta x \Delta y \sigma_{x_0 y_0} + \Delta x^2 \sigma_{y_0}^2\right) \end{bmatrix} \tag{19}$$

The covariance matrix of the dead reckoned measurements: (18).

The covariance matrix of the measurement vector: (19), $\sigma_{NR}^2 \mathrm{arc}1°$ [radians].

The covariance matrix of ship's coordinates vector:

$$\mathbf{P_x} = \left(\mathbf{G}^T\mathbf{R}^{-1}\mathbf{G}\right)^{-1}.$$

The mean error of position is found as:

$$M = \sqrt{tr\mathbf{P_x}} \tag{20}$$

An example

The following are calculations of the mean error of position obtained from bearing on and radar range to one navigational mark.

The input data are as follows:

- coordinates of the navigational mark $ZN(0,0)$;
- elements of the covariance matrix ($\sigma_{x_0} = \sigma_{y_0} =$ $= 185$ m, $\sigma_{x_0 y_0} = 0$); this corresponds to the accuracy of the determined coordinates of the navigational mark of 1' order;
- parameters of range measurement: $D = 10{,}000$ m, $\sigma_D = 50$ m (typical measurement error of adar range);
- bearing measurement parameters: $NR = 180°$, $\sigma_D = 0.8°$ (typical measurement error of radar bearing);
- range measurement is independent of bearing measurement (these measurements are not correlated).

Each element is calculated as follows:

- coordinates difference: $\Delta x = 0$ m, $\Delta y = 10{,}000$ m,
- matrix $\mathbf{G} = \begin{bmatrix} 0 & 1 \\ -0.0001 & 0 \end{bmatrix}$;
- matrix $\mathbf{H} = \begin{bmatrix} 0 & -1 \\ 0.0001 & 0 \end{bmatrix}$;
- matrix $\mathbf{R_u} = \begin{bmatrix} 2500 & 0 \\ 0 & 0.0002 \end{bmatrix}$;
- matrix $\mathbf{R_{zn}} = \begin{bmatrix} 34{,}225 & 0 \\ 0 & 34{,}225 \end{bmatrix}$;
- matrix $\mathbf{R_{f(x)}} = \begin{bmatrix} 34{,}225 & 0 \\ 0 & 0.00034 \end{bmatrix}$;
- matrix $\mathbf{R} = \begin{bmatrix} 36{,}725 & 0 \\ 0 & 0.00054 \end{bmatrix}$;
- matrix $\mathbf{R}^{-1} = \begin{bmatrix} 0.000027 & 0 \\ 0 & 1861.486 \end{bmatrix}$;
- matrix $\mathbf{P} = \begin{bmatrix} 53{,}720.515 & 0 \\ 0 & 36{,}725 \end{bmatrix}$.

Therefore, the mean error of position equals:

$$M = \sqrt{tr\mathbf{P}} = 300.742 \text{ [m]}$$

The table 1 contains the results of calculated mean error of position coordinates as a function of measured range and the accuracy of navigational mark coordinates. The range measurement was assumed to have 1 nautical mile steps (from one to ten), except the values of 1 kilometre, 10 kilometres and 20 nautical miles. The accuracy of navigational mark coordinates has these values: 2 cm (practically idealized value), 18.5 cm (0.0001') as the limit value for differentiability of coordinates in GPS receivers, 1.85 m (0.001'), the value corresponding to the diameter of a navigational mark, 18.5 m

Table 1. Mean error of position coordinates as a function of measured range and the error of navigational mark coordinates

D [m]	Error of navigational mark coordinate [m]						
	0.02	0.185	1.85	18.5	50	100	185
1000	51.913	51.914	51.979	58.133	87.721	150.648	266.730
1852	56.291	56.292	56.352	62.074	90.381	152.213	267.617
3704	71.935	71.936	71.983	76.545	100.870	158.665	271.339
5556	92.294	92.294	92.331	95.930	116.267	168.873	277.431
7408	114.886	114.886	114.916	117.828	134.903	182.205	285.743
9260	138.625	138.625	138.650	141.072	155.618	198.033	296.086
10000	148.309	148.309	148.332	150.599	164.303	204.928	300.742
11112	163.010	163.011	163.031	165.097	177.686	215.806	308.257
12964	187.790	187.790	187.809	189.604	200.662	235.086	322.048
14816	212.827	212.827	212.843	214.429	224.266	255.530	337.262
16668	238.040	238.040	238.054	239.473	248.320	276.881	353.713
18520	263.378	263.378	263.931	264.674	272.705	298.944	371.238
37040	519.587	519.587	519.594	520.246	524.377	538.490	581.740

(0.01'), the value corresponding to the limit differentiability of points on navigational charts, 50 m and 100 m, which correspond to the accuracy of range measurement, respectively, in 1:1 and 2:1 ratio, while the last column includes the 185 m accuracy (0.1'), as in the calculations above.

A graphic interpretation of calculation results is given in figure 2. It is obvious that the ship's position error increases along with the distance (range) to the navigational mark, while the influence of navigational mark coordinates accuracy on the resultant ship's position is less obvious. The firm line shows changes in position accuracy as a function of measured distance for position errors up to a few meters. The dotted line corresponds to the limit (maximum) cartographic accuracy of the navigational mark. The lines above correspond to errors of navigational mark errors of 50 m, 100 m and 185 m.

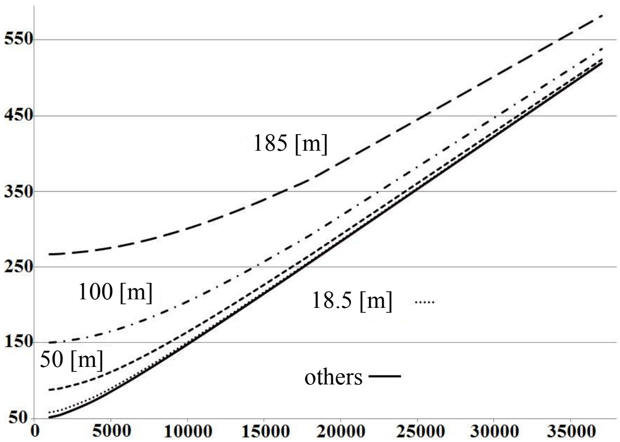

Fig. 2. The mean error of ship's position coordinates as a function of measured distance and erros of navigational mark coordinates

The diagram analysis shows that when navigational-hydrographic data from an electronic data base are used, with highly accurate coordinates of navigational marks, the influence of their errors is not significant in case of analytical calculations of coordinates. However, when errors of mark coordinates are close to or larger than measurement errors, the final result is burdened with a large error, particularly when short distances are involved. In an extreme situation the error is 215 metrów (position at a 1000 metres range).

Conclusions

The article presents a method of assessing the influence of the coordinates accuracy of navigational marks or ephemerides of celestial bodies on the accuracy of observer's position. The method consists in the transformation of covariance matrix of navigational mark coordinates (celestial body, navigational satellite) into the measurement space, and, consequently, taking it into account as a component of measurement error of navigational parameters. This method is general and can be used for an analysis of observer's position accuracy as a function of navigational mark coordinates accuracy (their covariance matrix), as well as for an analysis of predicted accuracy of designed navigational systems.

In the example illustrating the method it can see a significant influence of navigational mark coordinates accuracy on the final result, i.e. calculated coordinates of ship's position. It has to realize as well that in a general case it should take into account the time factor (simultaneous measurements) and the correct identification of navigational marks [4, 7].

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