JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
50, 3, pp. 701-715, Warsaw 2012
50th Anniversary of JTAM

IRRATIONAL ELLIPTIC FUNCTIONS AND THE ANALYTICAL SOLUTIONS OF SD OSCILLATOR

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The smooth and discontinuous (SD) oscillator is a strongly nonlinear system with an irrational restoring force proposed in P.R.E (2006), which leads to barriers for the conventional methods to investigate the dynamical behaviour directly. In this paper, two kinds of irrational elliptic functions and a kind of hyperbolic functions are defined in the real domain to formulate the analytical solutions of the system. The properties of the functions are obtained including differentiability, periodicity and parity. As the application of the defined irrational functions, the chaotic thresholds of the oscillator are also depicted by using the Melnikov method. Numerical analysis shows the efficiency of the proposed procedure.

Key words: SD oscillator, irrational nonlinearity, irrational elliptic functions, threshold of chaos

1. Introduction

The SD oscillator is a typical strong nonlinear system with an irrational restoring force (Cao *et al.*, 2006), which is widely used in engineering, such as the cable-stayed bridge, isolation system and truss-structure and so on Zhang and Sun (2005), Yang and Yang (2008), Luh and Lin (1011), Hajirasouliha *et al.* (2011), Kirsch (1989). Based upon Taylor's expansion, the conventional theoretical methods are confined by the harsh conditions of locality and smoothness. The global dynamical structure and local behaviour of the SD oscillator with irrational nonlinearity can hardly be depicted precisely. It is imperative to develop methodologies for the key problems in engineering. Methodologies can be found to get the approximate solutions of the irrational nonlinear systems in the literature. The homotopy perturbation method (He, 2006; Liao, 2004) was used to get an approximate expression for the periodic solutions of a irrational nonlinear system, and the harmonic balance method (Belendez *et al.*, 2007, 2009) via the first Fourier coefficient is used to construct two approximate frequency-amplitude relations for a conservative nonlinear oscillatory system which has an irrational restoring force. Analytical approximations were investigated for the system which has an irrational restoring force by employing a combined method of Newton's approach and the harmonic balance technique (Wu *et al.*, 2003, 2004, 2006; Sun *et al.*, 2007). A generalized Senator-Bapat (GSB) perturbation technique (Lai and Xiang, 2010) was given to solve the conservative oscillating system with an irrational nonlinearity.

Approximate qualitative analyses were given for the SD oscillator in the past studies (Cao *et al.*, 2006, 2008a,b; Tian *et al.*, 2009, 2012). A triple linear approach (Cao *et al.*, 2008a) was used to theoretically investigate the dynamics of the irrational nonlinear oscillator. The codimension-two bifurcation for the SD oscillator was proposed and studied in Tian *et al.* (2009). Periodic solution analysis was given (Cao *et al.*, 2011) by using the averaging theorem in Nayfrh (1981). However, the problem of analytical solutions for the oscillator given in Cao *et al.* (2008a) remains open.

The motivation of this paper is to provide a solution to the open problem proposed in Cao *et al.* (2008a) for irrational integrals. We propose a series of irrational elliptic functions (Greenhill, 1959; Whittaker and Watson, 1952) and hyperbolic functions defined in the real domain, which give analytical periodic solutions and homoclinic solutions of the irrational system. We discuss the chaotic threshold of the oscillator by the Melnikov methods with the hyperbolic functions defined in this paper. Meanwhile, the basic properties of the irrational elliptic functions and hyperbolic functions are gained as a part of the applied mathematic theory.

This paper is organized as follows. In Section 2, brief introduction to the open problem proposed in Cao *et al.* (2008a) is given. In the following section, Section 3, two kinds of irrational elliptic functions and a kind of hyperbolic functions are defined and the basic properties are obtained, which allows us to get the analytical solutions of the SD oscillator. Furthermore, in Section 4, the chaotic threshold and attractors of the oscillator are given by employing the Melnikov method as the application of the defined functions, and finally this paper is ended with conclusions and discussions.

2. Open problem of the SD oscillator

Consider the non-dimensional system of the unperturbed SD oscillator, written as

$$x'' + x\Big(1 - \frac{1}{\sqrt{x^2+\alpha^2}}\Big) = 0 \tag{2.1}$$

which was firstly proposed and investigated in Cao *et al.* (2008a). This system is strongly irrational nonlinear with smooth $(\alpha > 0)$ and discontinuous $(\alpha = 0)$ behaviour.

A pitchfork bifurcation occurs for the equilibria of the system at $\alpha = 1$: a pair of centres $(\pm 1, 0)$ and a saddle-like $(0,0)$ co-existed for $\alpha = 0$, a pair of centres $(\pm\sqrt{1-\alpha^2}, 0)$ and a saddle $(0,0)$ for $0 < \alpha < 1$ and a unique center $(0,0)$ for $\alpha \geqslant 1$.

Even some effective works have been achieved, a triple linear system was proposed to get the approximate solutions Cao *et al.* (2008a), an equivalent form (Tian *et al.*, 2012) was presented to get a step forward to the analytical solutions of the system and the GSB method (Lai and Xiang, 2010) was employed to solve the system, the theoretical solution of system (2.1) still remains open, the details seen in Cao *et al.* (2008a).

3. The definitions

Generally, it is difficult to get the analytical solutions of the irrational system precisely. Here definitions are given to get the analytical solutions of the oscillator in the following sections.

3.1. Irrational elliptic functions of the first kind

3.1.1. Definitions

The Hamilton function of (2.1) can be written as

$$H = U(x) + \frac{1}{2}y^2 \tag{3.1}$$

where

$$\dot{x} = y \qquad U(x) = \frac{1}{2}x^2 - \sqrt{x^2+\alpha^2} + \frac{1+\alpha^2}{2}$$

The phase portraits are plotted for different values of $H(x,y)$ as shown in Fig. 1, the details seen in the corresponding captions.

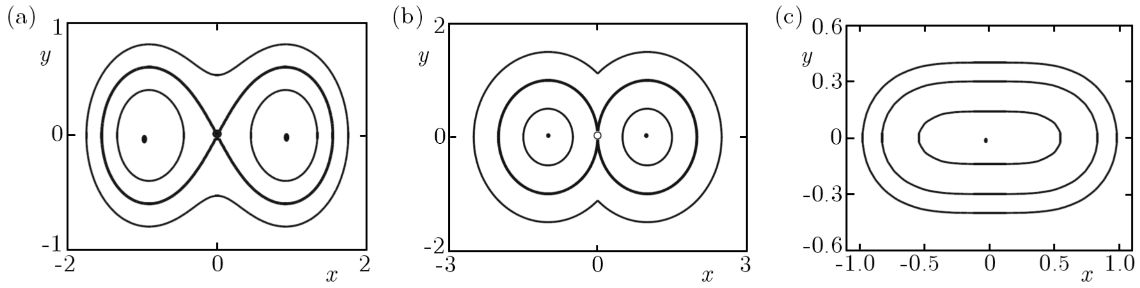

Fig. 1. Phase portraits: (a) for $\alpha = 0.4$ with the pair of centres and saddle, (b) for $\alpha = 0$ with the pair of centres and saddle-like equilibrium (Cao *et al.*, 2006) and (c) for $\alpha = 1$ with the unique centre, respectively

Introducing a notation $k^2/2 = H$, it follows

$$H = \frac{1}{2}k^2 = \frac{1}{2}y^2 + U(x) \tag{3.2}$$

Denoting x_0 as the maximum intersect point of orbits with the x-axis, it follows that $-2\sqrt{1-\alpha} \leqslant x_0 \leqslant 2\sqrt{1-\alpha}$ for $k \leqslant 1-\alpha$ and the orbits can be classified by k which is determined by x_0, as shown in Table 1.

Table 1. Unperturbed Orbits classified by parameter k

Orbits	Centres	Small periodic orbits	Homoclinic orbits	Large periodic orbits
k	$k=0$	$0<k<1-\alpha$	$k=1-\alpha$	$k>1-\alpha$
$2H$	$2H=0$	$0<2H<(1-\alpha)^2$	$2H=(1-\alpha)^2$	$2H>(1-\alpha)^2$

Time τ from $(x_0,0)$ to (x,y) along the lower branch of the periodic orbit can be obtained by the following integral

$$\tau = -\int\limits_{x_0}^{x} \frac{1}{\sqrt{k^2-(\sqrt{x^2+\alpha^2}-1)^2}}\, dx \tag{3.3}$$

Letting

$$x = \pm\sqrt{(k\cos\varphi+1)^2-\alpha^2} \tag{3.4}$$

where $\varphi \in [0,\pi]$, $k \in [0, 1-\alpha)$, it follows that the time τ expressed in (3.3) is rewritten as

$$\tau = \int\limits_{0}^{\varphi} \frac{1}{\sqrt{1-\frac{\alpha^2}{(1+k\cos\varphi)^2}}}\, d\varphi \tag{3.5}$$

and denote

$$I(k,\alpha) = \int\limits_{0}^{\pi} \frac{1}{\sqrt{1-\frac{\alpha^2}{(1+k\cos\varphi)^2}}}\, d\varphi \tag{3.6}$$

The definition of $\varphi = am\tau$ is said to the angular form of τ, and the irrational elliptic functions of the first kind are defined as follows

$$\begin{aligned} &sd(\tau,k,\alpha) \triangleq \sin\varphi = \sin(am\tau) \qquad ad(\tau,k,\alpha) \triangleq \cos\varphi = \cos(am\tau) \\ &hd(\tau,k,\alpha) \triangleq \sqrt{[1+k\cos(am\tau)]^2-\alpha^2} \end{aligned} \tag{3.7}$$

3.1.2. Properties

The fundamental properties of irrational elliptic functions of the first kind defined above are listed bellow by denoting $hd\tau$, $sd\tau$, $ad\tau$ instead of $hd(\tau,k,\alpha)$, $sd(\tau,k,\alpha)$, $ad(\tau,k,\alpha)$ without confusion.

(1) **Identity**

$$sd^2\tau + ad^2\tau = 1 \qquad hd^2\tau + \alpha^2 = (1+kad\tau)^2$$

(2) **Parity**

From integral (3.5) and the above definitions, one can obtain that angle φ is an odd function of the time variable τ, $sd\tau$ is an odd function of τ and $hd\tau$, $ad\tau$ are even functions of τ, then

$$sd(-\tau) = -sd\tau \qquad hd(-\tau) = hd\tau \qquad ad(-\tau) = ad\tau$$

(3) **Differentiability**

From integral (3.5) and above definitions, the differentiation of the angle φ and the irrational elliptic functions of the first kind $sd\tau$, $hd\tau$, $ad\tau$ are given as follows

$$\begin{aligned} &\frac{\partial\varphi}{\partial\tau} = \frac{hd\tau}{1+kad\tau} \qquad &&\frac{\partial sd\tau}{\partial\tau} = \frac{ad\tau hd\tau}{1+kad\tau} \\ &\frac{\partial ad\tau}{\partial\tau} = -\frac{sd\tau hd\tau}{1+k\sqrt{1-sd^2\tau}} \qquad &&\frac{\partial hd\tau}{\partial\tau} = -ksd\tau \end{aligned}$$

(4) **Periodicity**

The periodicity of the irrational functions can be obtained and written as the following by considering the periodicity of the integrand of (3.5) of period of 2π. When

$$\begin{aligned} &\varphi = am\tau \qquad &&\tau \to \tau + 2I(k,\alpha) \\ &am(\tau+2I(k,\alpha)) = am\tau + 2\pi \qquad &&am(-\tau+2I(k,\alpha)) = -am\tau + 2\pi \end{aligned}$$

then

$$\begin{aligned} &sd(\tau+2I(k,\alpha)) = sd\tau \qquad &&sd(-\tau+2I(k,\alpha)) = -sd\tau \\ &ad(\tau+2I(k,\alpha)) = ad\tau \qquad &&ad(-\tau+2I(k,\alpha)) = ad\tau \end{aligned}$$

which leads to that $2I(k,\alpha)$ is the period of $sd\tau$ and $ad\tau$ and follows that $2I(k,\alpha)$ is the period of $hd\tau$, that is

$$hd(\tau+2I(k,\alpha)) = hd\tau \qquad hd(-\tau+2I(k,\alpha)) = hd\tau$$

The graphs of the elliptic functions are plotted in Fig. 2, (a) for $sd(t)$, (b) for $ad(t)$ and (c) for $hd(t)$, respectively, for parameters $\alpha = 0.6$ and $k = 0.1$.

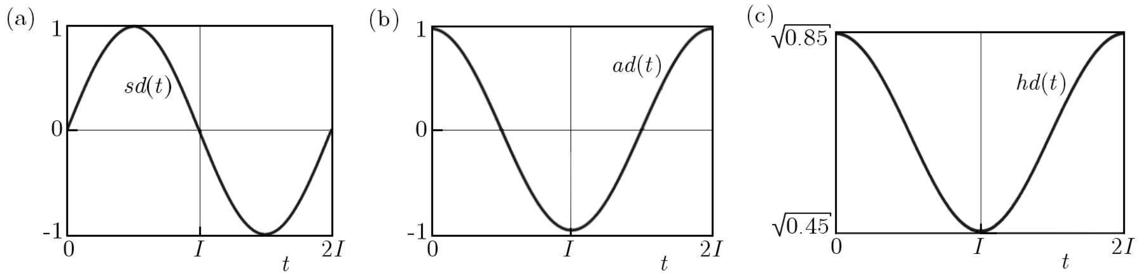

Fig. 2. Graphs for the elliptic functions of the first kind: (a) for $sd(t)$, (b) for $ad(t)$ and (c) for $hd(t)$ when parameters $\alpha = 0.6$ and $k = 0.1$

Special values of the irrational elliptic functions of the first kind are listed in Table 2.

Table 2. Special values of the irrational elliptic functions of the first kind

$\varphi = 0,\ \tau = 0$	$\varphi = \pi,\ \tau = I(k,\alpha)$
$sd(0) = 0,\ ad(0) = 1$	$sd(I) = 0,\ ad(I) = -1$
$hd(0) = \sqrt{(1+k)^2 - \alpha^2}$	$hd(I) = \sqrt{(1-k)^2 - \alpha^2}$

The Taylor expansion of the integrand of Eq. (3.6) is written as

$$
\begin{aligned}
\frac{1}{\sqrt{1 - \frac{\alpha^2}{(1+k\cos\varphi)^2}}} &= \frac{1}{\sqrt{1-\alpha^2}} - \frac{k\alpha^2}{\sqrt{(1-\alpha^2)^3}}\cos\varphi + \frac{3k^2\alpha^2}{2\sqrt{(1-\alpha^2)^5}}\cos^2\varphi \\
&\quad - \frac{5k^3}{2\sqrt{(1-\alpha^2)^7}}\cos^3\varphi + \frac{5k^4\alpha^2(4+3\alpha^2)}{8\sqrt{(1-\alpha^2)^9}}\cos^4\varphi - \frac{3k^5\alpha^2(8+12\alpha^2+\alpha^4)}{8\sqrt{(1-\alpha^2)^{11}}}\cos^5\varphi + \cdots
\end{aligned}
\tag{3.8}
$$

which leads to

$$
I(k,\alpha) = \frac{3k^2\alpha^2\pi}{4\sqrt{(1-\alpha^2)^5}} + \frac{\pi}{\sqrt{1-\alpha^2}} - \frac{15k^4\alpha^2(3\alpha^4+\alpha^2-4)\pi}{64\sqrt{1-\alpha^2}} + \cdots \tag{3.9}
$$

Equation (3.5) is rewritten as

$$
\tau = \int\limits_0^z \frac{1}{\sqrt{1 - \frac{\alpha^2}{(1+k\cos\varphi)^2}}}\, d\varphi \tag{3.10}
$$

which leads to that τ can be written as follows

$$
\tau = az + bz^3 + cz^5 + \cdots \tag{3.11}
$$

where

$$
a = \frac{1+k}{\sqrt{1+2k+k^2-\alpha^2}} \qquad b = \frac{k\alpha^2}{6\sqrt{(1+2k+k^2-\alpha^2)^3}}
$$
$$
c = \frac{k\alpha^2(\alpha^2+8k^2+7k-1)}{120\sqrt{(1+2k+k^2-\alpha^2)^5}}
$$

Then, the anti-series form of τ in terms of the variable z in Eq. (3.11) can be written as

$$
z = A\tau + B\tau^3 + E\tau^5 + \cdots \tag{3.12}
$$

where

$$A=\frac{1}{a} \qquad B=-\frac{b}{a^4} \qquad E=\frac{3b^2-ac}{a^7}$$

The series forms of the irrational elliptic functions are given as follows

$$sd\tau = s_1\tau - s_3\tau^3 + s_5\tau^5 + \cdots \tag{3.13}$$

where

$$\begin{aligned}
s_1 &= \frac{\sqrt{1+2k+k^2-\alpha^2}}{1+k}\\
s_3 &= \frac{\sqrt{1+2k+k^2-\alpha^2}}{3!(1+k)^4}(1+3k+3k^2+k^3-\alpha^2)\\
s_5 &= \frac{\sqrt{1+2k+k^2-\alpha^2}}{5!(1+k)^7}[15k^4+6k^5+k^6+(\alpha^2-1)^2\\
&\quad -3k^2(5+4\alpha^2)+k^3(20+7\alpha^2)+k(6+3\alpha^2-9\alpha^4)]
\end{aligned} \tag{3.14}$$

$$ad\tau = 1 - a_2\tau^2 + a_4\tau^4 - a_6\tau^6 + \cdots$$

where

$$\begin{aligned}
a_2 &= \frac{(1+k)^2-\alpha^2}{2!(1+k)^2}\\
a_4 &= \frac{(1+k)^2-\alpha^2}{4!(1+k)^5}[1+3k^2+k^3-\alpha^2+3k(1+\alpha^2)]\\
a_6 &= \frac{1}{6!(1+k)^8}[8k^7+k^8-(\alpha^2-1)^3+8k(\alpha^2-1)^2(1+3\alpha^2)-8k^5(13\alpha^2-7)\\
&\quad -k^6(31\alpha^2-28)+k^4(70-109\alpha^2+75\alpha^4)-8k^3(7-2\alpha^2+13\alpha^4)\\
&\quad +k^2(28+31\alpha^2-14\alpha^4-45\alpha^6)]
\end{aligned}$$

Furthermore, the series form of the analytical solutions of the periodic orbits is

$$hd\tau = p_0 - p_2\tau^2 + p_4\tau^4 - p_6\tau^6 - \cdots \tag{3.15}$$

where

$$\begin{aligned}
p_0 &= \sqrt{(1+k)^2-\alpha^2} \qquad p_2 = \frac{k\sqrt{(1+k)^2-\alpha^2}}{2!(1+k)}\\
p_4 &= \frac{k\sqrt{(1+k)^2-\alpha^2}}{4!(1+k)^4}(1+3k+3k^2+k^2-\alpha^2)\\
p_6 &= \frac{k\sqrt{(1+k)^2-\alpha^2}}{6!(1+k)^7}[15k^4+6k^5+k^6+(1-\alpha^2)^2+3k^2(5+4\alpha^2)\\
&\quad +k^3(20+7\alpha^2)+k(6+3\alpha^2-9\alpha^4)]
\end{aligned}$$

3.1.3. The analytical expression of the periodic orbits

When $0 \leqslant k < 1-\alpha$, the analytical expression of the periodic orbits can be written as

$$x = \pm hd(\tau, k, \alpha) \tag{3.16}$$

The first and the second derivatives of the solution x are obtained by the differential properties of irrational elliptic functions of the first kind, respectively

$$\frac{dx}{d\tau} = \mp ksd\tau \qquad \frac{d^2x}{d\tau^2} = \mp hd\tau\Big(1 - \frac{1}{\sqrt{hd^2\tau + \alpha^2}}\Big) = -x\Big(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}\Big) \tag{3.17}$$

which leads to $x = \pm hd(\tau, k, \alpha)$ is the periodic solution inside the homoclinic orbits of the oscillator. Moreover, the period of these orbits is given by $T_k = 2I(k, \alpha)$, and T_k increases monotonically in k with

$$\lim_{k\to 0} T_k = \frac{2\pi}{\sqrt{1-\alpha^2}} \qquad \lim_{k\to 1-\alpha} T_k \to \infty$$

For $I(k, \alpha)$, when $k = 0$, then

$$\lim_{k\to 0} I(k, \alpha) = \int\limits_0^\pi \frac{1}{\sqrt{1-\alpha^2}}\, dx = \frac{\pi}{\sqrt{1-\alpha^2}} \tag{3.18}$$

and when $\alpha = 0$, then $I(k, \alpha) = \pi$.

3.2. The hyperbolic functions

3.2.1. Definition

The Hamiltonian $k^2/2 = y^2/2 + U(x)$ represents the energies of the homoclinic orbits when $k = 1 - \alpha$. Let

$$\tau = \int\limits_0^\varphi \frac{1}{\sqrt{1 - \frac{\alpha^2}{[1+(1-\alpha)\cos\varphi]^2}}}\, d\varphi \tag{3.19}$$

and the hyperbolic functions of the oscillator are defined as follows

$$\begin{aligned} &\varphi = tam\tau \qquad \lim_{k\to 1-\alpha} sd(\tau, k, \alpha) \triangleq tsd(\tau, \alpha) \\ &\lim_{k\to 1-\alpha} ad(\tau, k, \alpha) = \sqrt{1 - tsd^2(\tau, \alpha)} \triangleq tad(\tau, \alpha) \\ &\lim_{k\to 1-\alpha} hd(\tau, k, \alpha) = \sqrt{[(1-\alpha)tad(\tau, \alpha) + 1]^2 - \alpha^2} \triangleq thd(\tau, \alpha) \end{aligned} \tag{3.20}$$

where $0 \leqslant \alpha < 1$ and $0 \leqslant k < 1 - \alpha$, and the graphs of the hyperbolic functions $thd(t)$ and $tsd(t)$ are shown in Fig. 3.

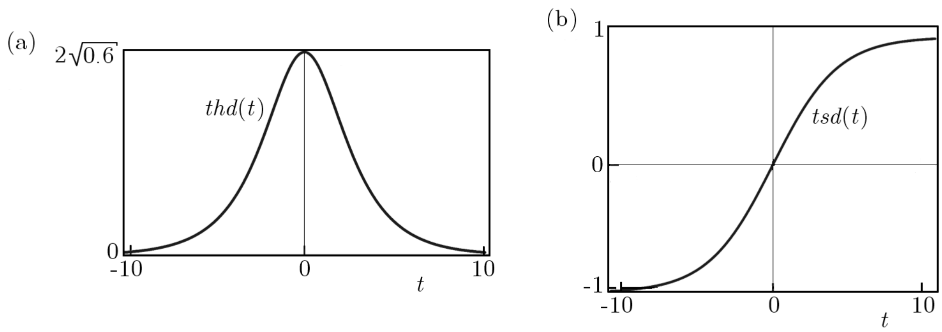

Fig. 3. Graphs of the hyperbolic function for $\alpha = 0.4$: (a) for $thd(t)$ and (b) for $tsd(t)$

3.2.2. Properties

The fundamental properties of the hyperbolic functions are obtained as following.

(1) **Identity**

$$tsd^2\tau + tad^2\tau = 1 \qquad thd^2\tau + \alpha^2 = [1 + (1-\alpha)tad\tau]^2$$

(2) **Parity**

It can be seen that $tsd\tau$ is an odd function of τ and $thd\tau$, $tad\tau$ are even functions of τ due to the odd property of φ, that is

$$tsd(-\tau) = -tsd\tau \qquad thd(-\tau) = thd\tau \qquad tad(-\tau) = tad\tau$$

(3) **Differentiability**

Differentiation of the hyperbolic functions are gained by the definitions and identity properties as follows

$$\frac{\partial\varphi}{\partial\tau} = \frac{thd\tau}{1+(1-\alpha)tad\tau} \qquad \frac{\partial tsd\tau}{\partial\tau} = \frac{tad\tau thd\tau}{1+(1-\alpha)tad\tau}$$
$$\frac{\partial tad\tau}{\partial\tau} = -\frac{tsd\tau thd\tau}{1+(1-\alpha)\sqrt{1-tsd^2\tau}} \qquad \frac{\partial thd\tau}{\partial\tau} = -(1-\alpha)tsd\tau$$

3.2.3. Analytical expressions of the homoclinic orbits

When $k = 1-\alpha$, the analytical expressions of the homoclinic orbits can be written as

$$x = \pm thd(\tau, k, \alpha) \tag{3.21}$$

The first and second derivatives of the solution are obtained by the differential properties of the hyperbolic functions, respectively

$$\begin{aligned} \frac{dx}{d\tau} &= \mp(1-\alpha)tsd\tau \\ \frac{d^2x}{d\tau^2} &= \mp thd\tau\Big(1 - \frac{1}{\sqrt{thd^2\tau+\alpha^2}}\Big) = -x\Big(1-\frac{1}{\sqrt{x^2+\alpha^2}}\Big) \end{aligned} \tag{3.22}$$

which leads to that $x = \pm thd(\tau, k, \alpha)$ is the analytical solution of the homoclinic orbits of the SD oscillator.

3.3. Irrational elliptic functions of the second kind

3.3.1. Definition

When $\alpha \geqslant 1$, $(0,0)$ is the unique centre and the phase portrait is shown in Fig. 1c. The Hamiltonian is given as follows

$$H = \frac{1}{2}k^2 = \frac{1}{2}y^2 + U(x) \geqslant \frac{(1-\alpha)^2}{2} \tag{3.23}$$

where $k > \alpha - 1$ and $\alpha \geqslant 1$, then the time τ from $(x_0, 0)$ to (x, y) along the lower branch of the periodic orbits can be obtained by the following

$$\tau = -\int_{x_0}^{x} \frac{1}{\sqrt{k^2 - (\sqrt{x^2+\alpha^2}-1)^2}}\, dx \tag{3.24}$$

where x_0 is the maximum intersect point of the orbits with the x-axis. Let

$$x = \pm\sqrt{(k\cos\varphi + 1)^2 - \alpha^2} \tag{3.25}$$

where $\varphi \in [0, \varphi_0]$ for $x \in [0, x_0]$ and $\varphi_0 = \arccos[(\alpha - 1)/k]$. Integration (3.24) can be simplified as

$$\tau = \int\limits_0^{\varphi} \frac{1}{\sqrt{1 - \frac{\alpha^2}{(1+k\cos\varphi)^2}}}\, d\varphi \tag{3.26}$$

where the time variable τ is a function of k, φ, α.

Denote

$$W(k,\alpha) = \int\limits_0^{\varphi_0} \frac{1}{\sqrt{1 - \frac{\alpha^2}{(1+k\cos\varphi)^2}}}\, d\varphi \tag{3.27}$$

Define

$$\varphi = Am\tau \tag{3.28}$$

which means φ as the angle of τ. Furthermore, irrational elliptic functions of the second kind are defined as follows

$$\begin{aligned} Sd(\tau,k,\alpha) &\triangleq \sin\varphi = \sin(Am\tau) \\ Ad(\tau,k,\alpha) &\triangleq \cos\varphi = \cos(Am\tau) \\ Hd(\tau,k,\alpha) &\triangleq \sqrt{[1 + k\cos(Am\tau)]^2 - \alpha^2} \end{aligned} \tag{3.29}$$

3.3.2. *Properties*

The fundamental properties of irrational elliptic functions of the second kind defined above are listed bellow by denoting $Hd\tau$, $Sd\tau$, $Ad\tau$ instead of $Hd(\tau,k,\alpha)$, $Sd(\tau,k,\alpha)$, $Ad(\tau,k,\alpha)$ without confusion.

(1) **Identity**

$$Sd^2\tau + Ad^2\tau = 1 \qquad Hd^2\tau + \alpha^2 = (1 + kAd\tau)^2$$

(2) **Parity**

It can be seen that $Sd\tau$ is an odd function of τ and $Hd\tau$, $Ad\tau$ are even functions of τ due to odd property of φ, that is

$$Sd(-\tau) = -Sd\tau \qquad Hd(-\tau) = Hd\tau \qquad Ad(-\tau) = Ad\tau$$

(c) **Differentiability**

Derivatives of φ, $Sd\tau$, $Ad\tau$, $Hd\tau$ with respect to the time variable τ are given as follows

$$\frac{\partial\varphi}{\partial\tau} = \frac{Hd\tau}{1 + kAd\tau} \qquad \frac{\partial Sd\tau}{\partial\tau} = \frac{Ad\tau Hd\tau}{1 + kAd\tau}$$

$$\frac{\partial Ad\tau}{\partial\tau} = -\frac{Sd\tau Hd\tau}{1 + k\sqrt{1 - Sd^2\tau}} \qquad \frac{\partial Hd\tau}{\partial\tau} = -kSd\tau$$

(4) **Periodicity**

For the symmetry of the phase portraits, the period T of the solution is four times of the time taken from $x = x_0$ to $x = 0$, which leads to the period of $Hd\tau$ is $4W(k,\alpha)$, that is

$$Hd(\tau + 4W(k,\alpha)) = Hd\tau \qquad Hd(-\tau + 4W(k,\alpha)) = Hd\tau$$

The differential properties of the irrational elliptic functions of the second kind leads to the period of $Sd\tau$ is $4W(k,\alpha)$ as follows

$$\frac{\partial Hd(\tau + 4W(k\alpha))}{\partial\tau} = -kSd(\tau + 4W(k,\alpha)) = \frac{\partial Hd\tau}{\partial\tau} = -kSd\tau \tag{3.30}$$

that is

$$Sd(\tau + 4W(k,\alpha)) = Sd\tau \qquad Sd(-\tau + 4W(k,\alpha)) = -Sd\tau$$

Furthermore, the period of $Ad\tau$ can be obtained as $4W(k,\alpha)$, that is

$$Ad\tau = \sqrt{1 - Sd^2\tau} = \sqrt{1 - Sd^2(\tau + 4W(k,\alpha))} = Ad(\tau + 4W(k,\alpha)) \tag{3.31}$$

and

$$Ad(\tau + 4W(k,\alpha)) = Ad\tau \qquad Ad(-\tau + 4W(k,\alpha)) = Ad\tau$$

Special values of the irrational elliptic functions of the second kind are listed in Table 3.

Table 3. Special values of the irrational elliptic functions of the second kind

$\varphi = 0,\ \tau = 0$	$\varphi = \pi,\ \tau = W(k,\alpha)$
$Sd(0) = 0,\ Ad(0) = 1$	$Hd(W) = 0$
$Hd(0) = \sqrt{(1+k)^2 - \alpha^2}$	$Ad(W) = \frac{\alpha-1}{k},\ Sd(W) = \sqrt{1 - \frac{(\alpha-1)^2}{k^2}}$

3.3.3. Analytical expressions of the periodic orbits

When the parameter $k > \alpha - 1$, the expressions for the periodic orbits of the irrational system can be obtained as

$$x = \pm Hd(\tau, k, \alpha) \tag{3.32}$$

The first and the second derivatives of the solution are obtained by the differential properties of irrational elliptic functions of the second kind, respectively

$$\frac{dx}{d\tau} = \mp kSd\tau \qquad \frac{d^2x}{d\tau^2} = \mp Hd\tau\Big(1 - \frac{1}{\sqrt{Hd^2\tau + \alpha^2}}\Big) = -x\Big(1 - \frac{1}{\sqrt{x^2 + \alpha^2}}\Big) \tag{3.33}$$

which leads to that $x = \pm Hd(\tau, k, \alpha)$ is the analytical solution of periodic orbits outside the centre $(0,0)$ of the oscillator. Moreover, the period of these orbits is given by

$$T_k^o = 4W(k,\alpha)$$

4. Chaotic threshold

The perturbed SD oscillator with viscous damping and an external harmonic forcing can be written as follows

$$x'' + 2\xi x' + x\Big(1 - \frac{1}{\sqrt{x^2+\alpha^2}}\Big) = f_0\cos(\omega\tau) \tag{4.1}$$

where $x' = y = dx/d\tau$, and which can be written as

$$x' = y \qquad y' = -2\xi y - x\Big(1 - \frac{1}{\sqrt{x^2+\alpha^2}}\Big) + f_0\cos(\omega\tau) \tag{4.2}$$

4.1. Melnikov method of the homoclinic orbits

The Melnikov function (Guckenheimer and Holmes, 1983; Garcia-Margallo and Bejarano, 1998) of system (4.2) is given as

$$\begin{aligned} M_\pm(\tau_1) &= \int\limits_{-\infty}^{\infty} f(\mathbf{x}_\pm(\tau)) \wedge g(\mathbf{x}_\pm(\tau), \tau+\tau_1)\, d\tau = \int\limits_{-\infty}^{\infty} \{-2\xi y_\pm(\tau)^2 + y_\pm(\tau) f_0 \cos[\omega(\tau+\tau_1)]\}\, d\tau \\ &= -2\xi \int\limits_{-\infty}^{\infty} y_\pm(\tau)^2\, d\tau + f_0 \int\limits_{-\infty}^{\infty} y_\pm(\tau)\cos[\omega(\tau+\tau_1)]\, d\tau \end{aligned} \tag{4.3}$$

The following discussion concentrates on $M_+(\tau_1)$ for symmetry of the system. When $0 \leqslant \alpha < 1$, one can get the displacement and velocity of the homoclinic orbit as

$$x = thd(\tau) \qquad y = x' = -(1-\alpha)tsd(\tau) \tag{4.4}$$

Then, the Melnikov function of the system can be obtained as follows

$$M_\pm(\tau_1) = -4\xi A \pm 2f_0\sin(\omega\tau_1)B \tag{4.5}$$

where

$$\begin{aligned} A &= (1-\alpha)^2 \int\limits_0^{\infty} tsd(\tau)^2\, d\tau \\ B &= (1-\alpha) \int\limits_0^{\infty} tsd(\tau)\sin(\omega\tau)\, d\tau = (1-\alpha)\pi \mathrm{i}\mathrm{Res}[tsd(\tau)\sin\omega\tau, \mathrm{i}H(\alpha)] \end{aligned}$$

where $\mathrm{Res}[f,z]$ means the residue of f at the point z.

It can be seen that $M_\pm(\tau_1) = 0$ has simple zero for τ_1 if and only if the following inequality holds

$$\Big|\frac{f_0}{\xi}\Big| \geqslant 2\frac{|A|}{|B|} \tag{4.6}$$

when the inequality is satisfied, the Poincaré map of Eq. (4.2) might be chaotic in the sense of Smale horseshoe, the intersection between the stable and unstable manifolds. Furthermore, the thresholds of chaos occurrence are plotted in Fig. 4a: with solid line for $\alpha = 0$, $k = 1$, thin for $\alpha = 0.4$, $k = 0.6$ and dashed for $\alpha = 0.8$, $k = 0.2$, respectively. Chaos might occur if the parameters over the corresponding curves.

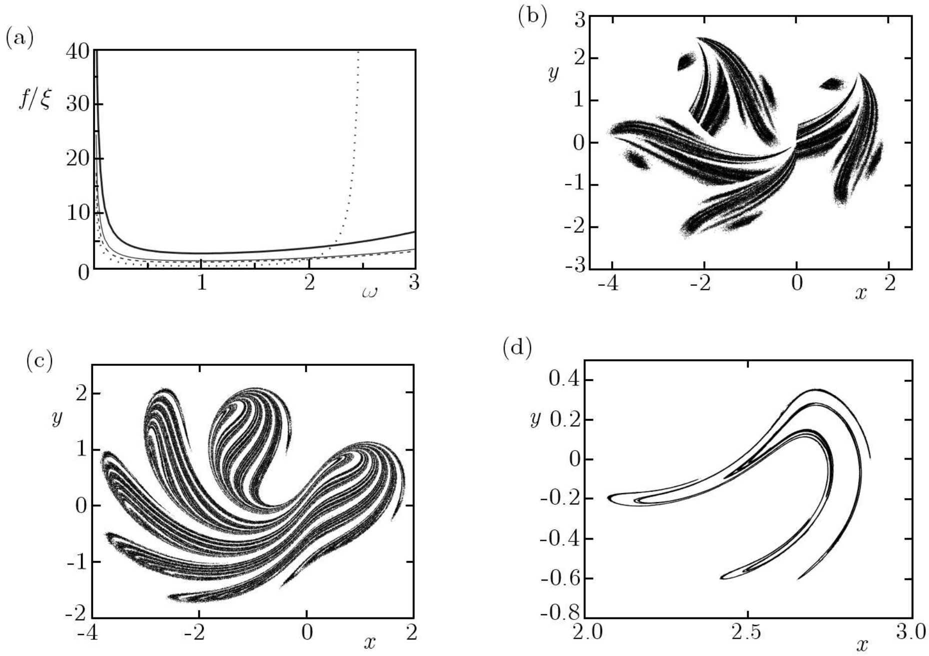

Fig. 4. Chaotic thresholds and attractors: (a) chaotic thresholds of the SD oscillator plotted with the solid curve for $\alpha = 0$, $k = 1$, thin for $\alpha = 0.4$, $k = 0.6$, dashed for $\alpha = 0.8$, $k = 0.2$ and dotted for $\alpha = 1$, $k = 0.4$, respectively; (b) attractors for $\alpha = 0$, $\xi = 0.015$, $f_0 = 1.1$, $\omega = 1.3$; (c) for $\alpha = 0.4$, $\xi = 0.02$, $f_0 = 0.70$, $\omega = 1.06$ and (d) for $\alpha = 1$, $\xi = 0.015$, $f_0 = 0.80$, $\omega = 1.3$, respectively

4.2. Melnikov method of the subharmonic orbits

One can get the subharmonic Melnikov function of the subharmonic orbits when $\alpha = 1$, that is

$$M_1^+(\tau_1) = M_1^-(\tau_1) = \int_0^T \{-2\xi(y_k(\tau))^2 + y_k(\tau) f_0 \cos[\omega(\tau + \tau_1)]\}\, d\tau \tag{4.7}$$

which can be simplified as follows

$$M_1^+(\tau_1) = -8\xi P + 4 f_0 Q \tag{4.8}$$

where

$$P = k^2 \int_0^{W(k,\alpha)} Sd(\tau)^2\, d\tau \qquad Q = k \int_0^{W(k,\alpha)} Sd(\tau) \sin(\omega\tau)\, d\tau$$

It can be seen that $M_1^+(\tau_1) = 0$ has simple zero for τ_1 if and only if the following inequality holds

$$\left|\frac{f_0}{\xi}\right| \geqslant 2\frac{|P|}{|Q|} \tag{4.9}$$

when the inequality is satisfied, the subharmonic orbits would occur under the sense of Smale horseshoe transform. Furthermore, the curve of threshold is plotted for $\alpha = 1$, $k = 0.4$ with the dotted curve in Fig. 4a.

4.3. Attractors

Numerical analysis is carried out by using the forth order Runge-Kutta method (Cartwright and Piro, 1992), the attractors of the SD oscillator in different conditions are given in Figs. 4b and 4c, respectively: (b) for $\alpha = 0$, $\xi = 0.015$, $f_0 = 1.1$, $\omega = 1.3$; (c) for $\alpha = 0.4$, $\xi = 0.02$, $f_0 = 0.70$, $\omega = 1.06$ and (d) for $\alpha = 1$, $\xi = 0.015$, $f_0 = 0.80$, $\omega = 1.3$, which verified the conditions given by the thresholds found in this paper.

5. Conclusions

Two kinds of irrational elliptic functions and a kind of irrational hyperbolic functions have been defined for the SD oscillator which is controlled by the parameter α in this paper. The fundamental properties of identity, parity, differentiability and the periodicity of the functions have been obtained and presented. The periodic solutions of the oscillator have been formulated by the irrational elliptic functions and the homoclinic solutions of the oscillator have been given by the hyperbolic functions. The chaotic threshold of the irrational system has been obtained by employing the Melnikov functions with respect to the hyperbolic functions defined in this paper. The irrational elliptic functions defined herein may be applied to formulate a general irrational nonlinear system to get the precise solution; as a part of applied mathematics, further studies for more properties of the irrational functions defined in this paper are being investigated by the current authors. The properties of the defined irrational functions in complex domain remain open.

Acknowledgement

The first two authors acknowledge the financial supports, under the Grant 10872136, 11072065 and 10932006, National Natural Science Foundation of China.

Appendix

The method used to get the pole of the hyperbolic function $tsd(\tau)$ of Eq. (3.19) is shown as follows.

With the half-angle formulae, let

$$t = \tan\frac{\varphi}{2} \qquad \frac{d\varphi}{dt} = \frac{2}{1+t^2}$$
$$\cos\varphi = \frac{1-t^2}{1+t^2} \qquad \sin\varphi = \frac{2t}{1+t^2}$$

Then integral (3.19) is rewritten as

$$\tau = 2\int\limits_0^t \frac{1+t^2+(1-\alpha)(1-t^2)}{(1+t^2)\sqrt{[1+t^2+(1-\alpha)(1-t^2)]^2-(1+t^2)^2\alpha^2}}\,dt \tag{A.1}$$

Whittaker (1937) proved that the integral of motion is still real when t is replaced by $\sqrt{-1}t$ and the initial conditions $\beta_1,\dots,\beta_n$ by $\sqrt{-1}\beta_1,\dots,\sqrt{-1}\beta_n$, respectively, if the force is independent of time. The expression thus obtained represents the same motion with the same initial condition of the system. In this way, the variable is extended to the imaginary axis in the following.

Letting

$$\tau = \mathrm{i}u \qquad t = \mathrm{i}s$$

then, integral (A.1) is written as

$$u = 2\int_0^s \frac{1 - s^2 + (1-\alpha)(1+s^2)}{(1-s^2)\sqrt{[1-s^2+(1-\alpha)(1+s^2)]^2 - (1-s^2)^2\alpha^2}}\, ds \tag{A.2}$$

which leads to the following definitions

$$tsd(\tau,\alpha) = tsd(\mathrm{i}u,\alpha) = \frac{2t}{1+t^2} = \mathrm{i}\frac{2s}{1-s^2} = \mathrm{i}\frac{tsd(u,\alpha)}{tad(u,\alpha)}. \tag{A.3}$$

Denoting

$$H(\alpha) = 2\int_0^1 \frac{1 - s^2 + (1-\alpha)(1+s^2)}{(1-s^2)\sqrt{[1-s^2+(1-\alpha)(1+s^2)]^2 - (1-s^2)^2\alpha^2}}\, ds \tag{A.4}$$

leads to that $u = H(\alpha)$ when $s = 1$ and $tsd(\tau,\alpha)$ has a pole $\tau = \mathrm{i}H(\alpha)$ when $s = 1$.

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Niewymierne funkcje eliptyczne i rozwiązania analityczne dla oscylatora typu SD

Streszczenie

Oscylator typu gładkiego i nieciągłego (smooth and discontinuous — SD) jest silnie nieliniowym układem mechanicznym z niewymierną siłą restytucyjną opisaną przez P.R.E. w 2006 r. Jej charakter stanowi barierę dla konwencjonalnych metod badania dynamiki oscylatorów SD w sposób bezpośredni. W pracy zdefiniowano dwa rodzaje niewymiernych funkcji eliptycznych i jeden typ hiperbolicznych w dziedzinie liczb rzeczywistych do wyznaczenia rozwiązań analitycznych rozważanego układu. Właściwości tych funkcji obejmują różniczkowalność, okresowość i parzystość. Tak sformułowanych funkcji, jako przykład ich zastosowania, uzyto do określenia zakresów występowania drgań chaotycznych oscylatora przy wykorzystaniu metody Mielnikowa. Symulacje numeryczne potwierdziły efektywność zaproponowanej metody.

Manuscript received March 12, 2012; accepted for print March 22, 2012