JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
**50**, 2, pp. 639-652, Warsaw 2012
**50th Anniversary of JTAM**

# ANALYTICAL SOLUTIONS FOR FREE OSCILLATIONS OF BEAMS ON NONLINEAR ELASTIC FOUNDATIONS USING THE VARIATIONAL ITERATION METHOD

Davood Younesian, Zia Saadatnia, Hassan Askari

*Center of Excellence in Railway Transportation, School of Railway Engineering, Iran University of Science and Technology, Narmak, Tehran, Iran; e-mail: younesian@iust.ac.ir*

Closed form expressions are obtained for the dynamic response of an elastic beam rested on a nonlinear foundation in this paper. The nonlinear governing equation is solved using the Variational Iteration Method (VIM). An iteration formulation is constructed based on the VIM and the dynamic responses are then obtained. Frequency responses are presented in a closed form and their sensitivity with respect to the initial amplitudes are investigated. A number of numerical simulations are then carried out and performance and validity of the solution procedure is evaluated in the time domain. It is proved that the VIM is quite a reliable and straightforward technique to solve the corresponding set of coupled nonlinear differential equations.

*Key words:* beam, nonlinear oscillation, variational iteration method, elastic foundation

## 1. Introduction

Vibration analysis of beam-type structures rested on nonlinear foundations have recently received a remarkable amount of attention due to importance and various applications of the subject. Surveying the literature reveals that the ballast dynamic behavior has been proved to be nonlinear in railway tracks. Dynamic behavior of columns, piles and pipes supported along their length, typically by soils, are also some other applications of the subject. It has been proved that the linearization in governing equations can sometimes lead to significant errors in the system modeling and identifications. Consequently, free and forced vibration analysis of beams supported by nonlinear elastic and visco-elastic foundations have been widely focused in the last decade. Applications of the Multiple Scales Method (MSM), Method of Shaw and Pierre, Method of Normal Forms and Method of King and Vakakis in free vibration analysis of a simply supported beam rested on a nonlinear elastic foundation have been summarized by Nayfeh (2000). Younesian *et al.* (2005) and Kargarnovin *et al.* (2005) employed a perturbation technique and obtained the main harmonic and also sub-harmonics of an infinite beam supported by a nonlinear viso-elastic foundation traversed by a moving load. An analytical solution was obtained by Bogacz and Czyczuła (2008) for the response of a beam on a visco-elastic foundation subjected to moving distributed loads. Vibrations of structures composed of beams and plates subjected to various types of moving loads was analytically studied by Bogacz and Frischmuth (2009). Santee and Goncalves (2006) used the Melnikov Method and obtained the frequency response and bifurcation diagrams for a harmonically excited beam rested on a nonlinear elastic foundation. Malekzadeh and Vosoughi (2009) employed the Differential Quadrature Method (DQM) and numerically studied large amplitude vibration of composite beams on nonlinear elastic foundations. A weak form Quadrature Element Method (QEM) was employed by Yihua *et al.* (2009) and the main natural frequency was numerically obtained for vibrations of beams rested on nonlinear elastic foundations. The Galerkin method was utilized by Senalp *et al.* (2010) and vibrations of finite beams on linear and nonlinear elastic foundations were numerically studied.

The orthogonality conditions for a transverse vibration mode equation of a one-dimensional plate on an elastic Winkler foundation has been studied by Zhang (2002). Baghani *et al.* (2011) and Jafari-Talookolaei *et al.* (2011) respectively employed Variational Iteration Method (VIM) and Homotopy Analysis Method (HAM) and obtained the first natural frequency for unsymmetrically laminated composite beams rested on nonlinear elastic foundations. More recently, external and internal-external resonances in finite beams rested on nonlinear visco-elastic foundations subjected to moving loads were analyzed using the MSM by Ansari *et al.* (2010, 2011). Surveying the literature demonstrates that the employed solution methods so far have some restrictions and limitations. The numerical methods are not able to provide closed form solutions, and consequently they are suitable for numerical case studies. The utilized approximate methods are also limited to obtain only the first natural frequency or they are sometimes practically inapplicable in providing higher natural frequencies. The other constraint arises from the limitations of approximate solutions in strong nonlinear systems. The present paper is aimed at covering the limitations of earlier solution procedures in dealing with the higher natural frequencies. The main objectives of the present study is to provide a straightforward solution procedure to generate a closed form solution:

A) applicable for higher mode natural frequencies,

B) applicable for a strong nonlinear system.

There is no doubt that the closed form solutions are always of importance because of their potential to provide a deeper and simpler engineering estimate and judgment. The variational Iteration Method (VIM) (He, 2000) is employed for this purpose. Application of the VIM has been proved to be quite efficient in varieties of nonlinear systems governed by ordinary and partial differential equations (Askari *et al.*, 2010; He *et al.*, 2010; Inan and Yildirim, 2010; Ozis Yildirim, 2007; Wazwaz, 2008-2010; Yildirim, 2010; Yildirim and Ozis, 2009; Younesian *et al.*, 2011). Employing the VIM, analytical expressions are obtained for the natural frequencies of free oscillation of an elastic beam rested on a nonlinear foundation in this paper. The corresponding set of coupled nonlinear differential equations is then derived from the governing partial differential equation. The associated iteration formulation is then presented and the closed form solution is consequently constructed. The frequency responses are obtained and illustratively discussed for a real mechanical system. A number of numerical simulations are then carried out and the accuracy of the solution procedure is evaluated.

## 2. Solution procedure

The case of a uniform finite beam resting on a non-linear Winkler foundation is governed by Baghani *et al.* (2011) and Jafari-Talookolaei *et al.* (2011)

$$EIw_{xxxx} + kw + \alpha w^3 + \rho A w_{tt} = 0 \tag{2.1}$$

in which the parameters $E$, $I$, $A$, and $\rho$ are the modulus of elasticity, second moment of area, cross-sectional area, and the material density of the beam, respectively. The other parameters $k$ and $\alpha$ are the respective linear and nonlinear parts of the foundation stiffness. The beam is assumed to have homogeneous mass density and cross section along its length, and it is modeled by the Euler-Bernoulli theory. The beam material is assumed to be isotropic and mechanical properties of the foundation are uniform along the beam length.

The approximate solution is constructed for a simply supported beam based on the Galerkin approximation presented by Nayfeh (2000) and Younesian *et al.* (2008)

$$W(x,t)=\sum_{n=1}^{\infty}u_n(t)\sin\frac{n\pi x}{L} \tag{2.2}$$

The solution procedure is aimed at finding the main frequencies of the symmetric and un-symmetric mode shapes while the procedure can be straightforwardly expanded for even higher frequencies

$$\begin{aligned}&\sum_{m=1}^{2}\Big\{\Big[\rho A\ddot{u}_m(t)+\Big(\frac{m^4\pi^4EI}{L^4}+k\Big)u_m(t)\Big]\sin\frac{m\pi x}{L}\Big\}\\&\qquad+\alpha\Big(\sum_{m=1}^{2}u_m(t)\sin\frac{m\pi x}{L}\Big)^3=0\end{aligned} \tag{2.3}$$

Using the orthogonality principle of the mode shapes, one can arrive at $(u_1=X, u_2=Y)$

$$\ddot{X}+\omega_1^2X=-\alpha_1X^3-\alpha_3XY^2 \qquad\qquad \ddot{Y}+\omega_2^2Y=-\alpha_2Y^3-\alpha_4X^2Y \tag{2.4}$$

in which

$$\alpha_1=\alpha_2=\frac{3\alpha}{4\rho A} \qquad\qquad \alpha_3=\alpha_4=\frac{3\alpha}{2\rho A} \tag{2.5}$$

and

$$\omega_1^2=\frac{EI\pi^4}{\rho AL^4}+\frac{k}{\rho A} \qquad\qquad \omega_2^2=\frac{16EI\pi^4}{\rho AL^4}+\frac{k}{\rho A} \tag{2.6}$$

Applying the variational iteration method (Ansari *et al.*, 2011; He, 2000; He *et al.*, 2010; Inan and Yildirim, 2010; Ozis Yildirim, 2007; Wazwaz, 2008--2010; Yildirim, 2010; Yildirim and Ozis, 2009), one can construct the following iteration formulations

$$\begin{aligned}X_{n+1}&=X_n+\frac{1}{\omega_1}\int_0^t\sin\omega_1(s-t)\Big(\frac{d^2X}{ds^2}+\omega_1^2X+\alpha_1X^3+\alpha_3XY^2\Big)\,ds\\Y_{n+1}&=Y_n+\frac{1}{\omega_2}\int_0^t\sin\omega_2(s-t)\Big(\frac{d^2Y}{ds^2}+\omega_2^2Y+\alpha_2Y^3+\alpha_4YX^2\Big)\,ds\end{aligned} \tag{2.7}$$

Based on the iteration harmonic formulations recognized by the respective amplitudes of $A$ and $B$ for $X$ and $Y$, one can consequently arrive at

$$
\begin{aligned}
X_1 &= A\cos\Omega_1 t + \frac{1}{\omega_1}\int\limits_0^t \sin\Omega_1(s-t)\\
&\quad\cdot\Big[A(\omega_1^2-\Omega_1^2)\cos\Omega_1 s + \alpha_1 A^3\cos^3\Omega_1 s + \alpha_3 AB^2\cos\Omega_1 s\cos^2\Omega_1 s\Big]\,ds\\
Y_1 &= B\cos\Omega_2 t + \frac{1}{\omega_2}\int\limits_0^t \sin\Omega_2(s-t)\\
&\quad\cdot\Big[B(\omega_2^2-\Omega_2^2)\cos\Omega_2 s + \alpha_2 B^3\cos^3\Omega_2 s + \alpha_4 BA^2\cos\Omega_2 s\cos^2\Omega_1 s\Big]\,ds
\end{aligned}
\tag{2.8}
$$

in which $\Omega_1$ and $\Omega_2$ denote the nonlinear natural frequencies. Implementing a sort of appropriate mathematical operations and simplifications, one can reach to

$$
\begin{aligned}
X_1 &= \Big[A + \frac{\alpha_3 AB^2}{2(\omega_1^2-\Omega_1^2)}\\
&\quad+\frac{\alpha_3 AB^2}{4}\Big(\frac{1}{\omega_1^2-(2\Omega_2+\Omega_1)^2}+\frac{1}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\Big)\\
&\quad+\frac{\alpha_1 A^3}{4}\Big(\frac{3}{\omega_1^2-\Omega_1^2}+\frac{1}{\omega_1^2-9\Omega_1^2}\Big)\Big]\cos\Omega_1 t-\frac{\alpha_1 A^3}{4}\Big(\frac{3\cos\Omega_1 t}{\omega_1^2-\Omega_1^2}\Big)+\frac{\cos 3\Omega_1 t}{\omega_1^2-9\Omega_1^2}\\
&\quad-\frac{\alpha_3 AB^2}{4}\Big(\frac{2\cos\Omega_1 t}{\omega_1^2-\Omega_1^2}+\frac{\cos(2\Omega_2+\Omega_1)t}{\omega_1^2-(2\Omega_2+\Omega_1)^2}+\frac{2\cos(2\Omega_2-\Omega_1)t}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\Big)\\
Y_1 &= \Big[B + \frac{\alpha_4 BA^2}{2(\omega_2^2-\Omega_2^2)}\\
&\quad+\frac{\alpha_4 BA^2}{4}\Big(\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\Big)\\
&\quad+\frac{\alpha_2 B^3}{4}\Big(\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\Big)\Big]\cos\Omega_2 t-\frac{\alpha_2 B^3}{4}\Big(\frac{3\cos\Omega_2 t}{\omega_2^2-\Omega_2^2}+\frac{\cos 3\Omega_2 t}{\omega_2^2-9\Omega_2^2}\Big)\\
&\quad-\frac{\alpha_4 BA^2}{4}\Big(\frac{2\cos\Omega_2 t}{\omega_2^2-\Omega_2^2}+\frac{\cos(2\Omega_1+\Omega_2)t}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{2\cos(2\Omega_1-\Omega_2)t}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\Big)
\end{aligned}
\tag{2.9}
$$

Eliminating the secular terms in $X$ and $Y$ directions, yields

$$
\begin{aligned}
&A + \frac{\alpha_3 AB^2}{2(\omega_1^2-\Omega_1^2)}+\frac{\alpha_3 AB^2}{4}\Big(\frac{1}{\omega_1^2-(2\Omega_2+\Omega_1)^2}+\frac{1}{\omega_1^2-(2\Omega_2-\Omega_1)^2}\Big)\\
&\quad+\frac{\alpha_1 A^3}{4}\Big(\frac{3}{\omega_1^2-\Omega_1^2}+\frac{1}{\omega_1^2-9\Omega_1^2}\Big)=0
\end{aligned}
\tag{2.10}
$$

$$B+\frac{\alpha_4 BA^2}{2(\omega_2^2-\Omega_2^2)}+\frac{\alpha_4 BA^2}{4}\Big(\frac{1}{\omega_2^2-(2\Omega_1+\Omega_2)^2}+\frac{1}{\omega_2^2-(2\Omega_1-\Omega_2)^2}\Big)$$
$$+\frac{\alpha_2 B^3}{4}\Big(\frac{3}{\omega_2^2-\Omega_2^2}+\frac{1}{\omega_2^2-9\Omega_2^2}\Big)=0$$

Equations (2.10) implicitly generate the main frequencies of the symmetric and un-symmetric modes of oscillations for the general case of $A, B \neq 0$. For a special case of

$$X(0)=A \qquad\qquad Y(0)=0 \tag{2.11}$$

one can arrive at

$$\Omega_1=\frac{1}{\sqrt{72}}\sqrt{40\omega_1^2+28\alpha_1 A^2+\sqrt{(40\omega_1^2+28\alpha_1 A^2)^2-576\omega_1^2(\alpha_1 A^2+\omega_1^2)}} \tag{2.12}$$

For the other special case of

$$X(0)=0 \qquad\qquad Y(0)=B \tag{2.13}$$

one can reach to

$$\Omega_2=\frac{1}{\sqrt{72}}\sqrt{40\omega_2^2+28\alpha_2 B^2+\sqrt{(40\omega_2^2+28\alpha_2 B^2)^2-576\omega_2^2(\alpha_2 B^2+\omega_2^2)}} \tag{2.14}$$

## 3. Numerical examples

The presented solution procedure is employed to obtain the frequency responses for a typical railway track system (Kargarnovin *et al.*, 2005; Younesian *et al.*, 2005) with strong nonlinearity. The set of nonlinear frequency Equations $(2.10)_2$, (2.11) is solved and the first and second natural frequencies of the beam are consequently obtained as two different functions of the initial conditions. Mechanical and geometrical properties of the beam-foundation system are listed in Table 1. Variations of the first and second natural frequencies are illustrated in Figs. 1 and 2. The relative error with respect to the linear solutions is also demonstrated in these figures. As it is seen, the relative error can be evaluated up to 60% and 5%, respectively for the first and second natural frequency. The corresponding latticed contour maps are illustrated in Figs. 3 and 4. The color gradient is correlated with nonlinear sensitivity of the system with respect to the initial amplitudes. The elliptic-type contours are

**Table 1.** Mechanical and geometrical properties of the beam-foundation system (Ansari *et al.*, 2010, 2011)

| Item | Notation | Value |
|---|---|---|
| Young's modulus (steel) | $E$ | 210 GPa |
| Mass density | $\rho$ | $7850\,\mathrm{kg/m^3}$ |
| Cross sectional area | $A$ | $7.69 \cdot 10^{-3}\,\mathrm{m^2}$ |
| Second moment of area | $I$ | $30.55 \cdot 10^{-6}\,\mathrm{m^4}$ |
| Beam length | $L$ | 18 m |
| First linear natural frequency | $\omega_1$ | 98.86 rad/s |
| Second llinear natural frequency | $\omega_2$ | 157.8 rad/s |
| Nonlinear stiffness coefficient | $\alpha_1 = \alpha_2$ | $8.69 \cdot 10^3$ |
| Nonlinear stiffness coefficient | $\alpha_3 = \alpha_4$ | $1.74 \cdot 10^4$ |

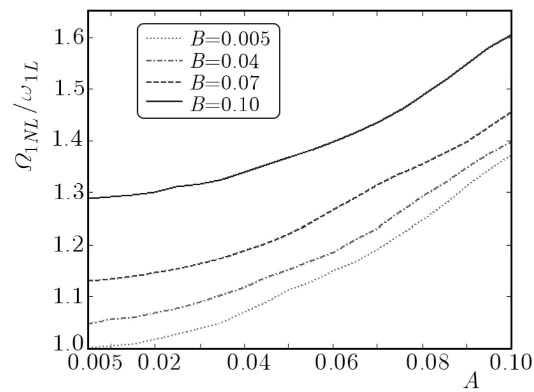


Fig. 1. The first frequency ratio (nonliner/linear) with respect to the initial amplitude

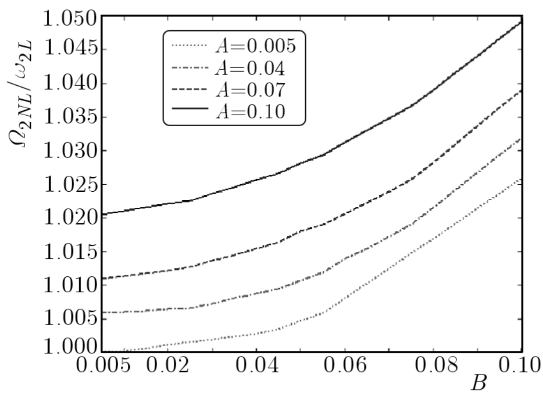


Fig. 2. The second frequency ratio (nonliner/linear) with respect to the initial amplitude

seen to have an aspect ratio very close to one and they seem to be vertical in Fig. 3 and horizontal in Fig. 4. This means that both natural frequencies have almost similar sensitivity with respect to the initial amplitudes. In order to evaluate the accuracy of the solution method in the time domain, a number of numerical simulations are carried out in the following sections.

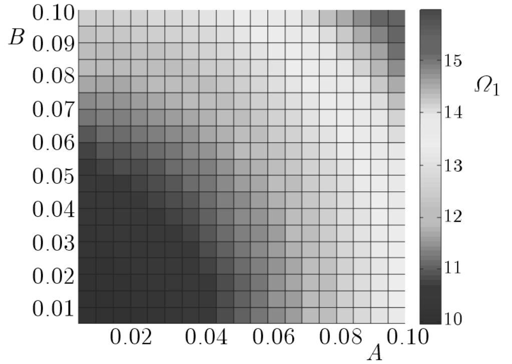


Fig. 3. First natural frequency contour map

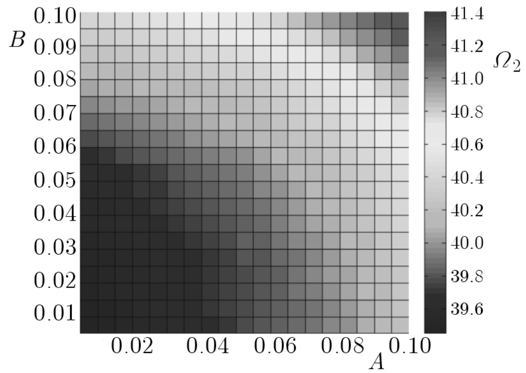


Fig. 4. Second natural frequency contour map

### 3.1. Case #1

A set of mutual infinitesimal initial conditions is assumed to be

$$X(0) = 0.01 \qquad\qquad Y(0) = 0.01 \tag{3.1}$$

in this case. Solving the set of nonlinear equations of (2.9) and employing the prescribed solution procedure for this case, the consequent closed form solutions are obtained as

$$
\begin{aligned}
x(t) &= 0.009989\cos(10.019t) + 2.702\cdot 10^{-6}\cos(30.058t)\\
&\quad + 5.563\cdot 10^{-7}\cos(88.971t) + 9.3458\cdot 10^{-6}\cos(68.932t)\\
Y(t) &= 0.0103\cos(39.4759t) + 1.4738\cdot 10^{-7}\cos(118.43t)\\
&\quad + 2.19\cdot 10^{-6}\cos(59.514t) - 3.6877\cdot 10^{-6}\cos(19.427t)
\end{aligned}
\tag{3.2}
$$

The corresponding nonlinear differential equations are numerically solved in parallel, and the numerical results are compared in the time domain in Fig. 5. The very close agreement can accordingly guarantee accuracy of the solution procedure.

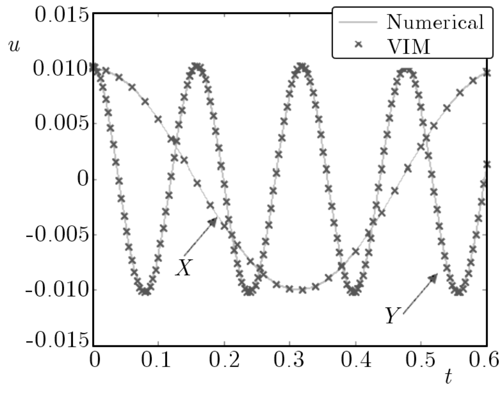


Fig. 5. Time history of the dynamic responses $(A, B = 0.01)$

### 3.2. Case #2

For a set of larger initial amplitudes, i.e.

$$
X(0) = 0.1 \qquad\qquad Y(0) = 0.1 \tag{3.3}
$$

Corresponding similar calculations have been carried out and the obtained closed form solutions are listed as

$$
\begin{aligned}
x(t) &= 0.0979\cos(15.96t) + 9.9115\cdot 10^{-4}\cos(47.88t)\\
&\quad + 4.5\cdot 10^{-4}\cos(98.76t) + 9.9536\cdot 10^{-4}\cos(66.84t)\\
y(t) &= 0.094\cos(41.5t) + 1.57\cdot 10^{-4}\cos(124.5t)\\
&\quad + 0.0011\cos(73.32t) + 0.003\cos(9.48t)
\end{aligned}
\tag{3.4}
$$

The obtained analytical solutions are compared with the numerical ones in Fig. 6. It is seen that validity of the approximate closed form solutions is still preserved in the time domain.

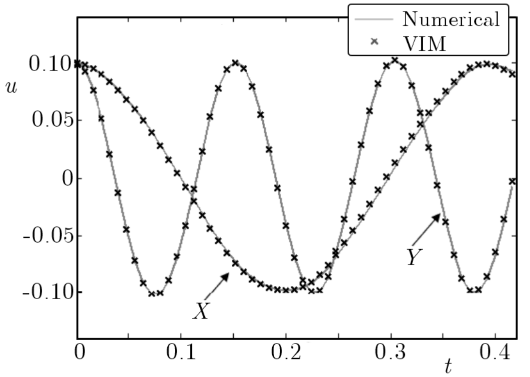


Fig. 6. Time history of the dynamic responses $(A, B = 0.1)$

### 3.3. Case #3

In order to evaluate the accuracy of the solution procedure for a non-conjugate initial condition, the following set of initial conditions is taken into account

$$X(0) = 0.1 \qquad\qquad Y(0) = 0.05 \tag{3.5}$$

The solution procedure ends up in the following closed form solutions

$$\begin{aligned}x(t) &= 0.0985\cos(13.68t) + 0.00137\cos(41.04t)\\&\quad + 1.2125\cdot 10^{-4}\cos(95.208t) + 2.4134\cdot 10^{-4}\cos(67.848t)\\y(t) &= 0.04826\cos(40.762t) + 2.0284\cdot 10^{-4}\cos(122.29t)\\&\quad + 7.0502\cdot 10^{-3}\cos(68.124t) + 0.001578\cos(13.404t)\end{aligned} \tag{3.6}$$

A very good correlation is again detected between the numerical results and analytical solutions (Fig. 7).

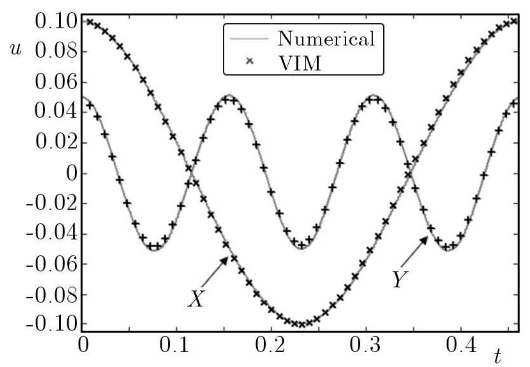


Fig. 7. Time history of the dynamic responses $(A = 0.1,\ B = 0.05)$

## 4. Conclusions

Analytical solutions were obtained for free oscillation of an elastic beam supported by a nonlinear foundation in this paper. An iteration formulation was presented based on the VIM and the dynamic responses were then obtained in a closed form. The solution procedure was directed to obtain the main symmetric and unsymmetric mode frequencies because of their importance in dynamic characteristics of structures. The solution technique was planned to be as simple as possible and also extendable to obtain higher (third and more) natural frequencies. Limitations of earlier solution procedures in dealing with the higher natural frequencies were covered by a reliable and straightforward technique. Characteristic equations were obtained as functions of initial amplitudes and the corresponding frequency responses were obtained in closed forms. An error analysis was carried out, and it was proved that any linearization could lead to a significant error especially for the main symmetric mode. It was also found that both the symmetric and un-symmetric natural frequencies had almost similar sensitivity with respect to the initial amplitudes. A number of numerical simulations were then directed to evaluate the performance and validity of the solution procedure in the time domain. It was proved that the VIM was a quite accurate and simple technique to find higher natural frequencies of similar structures with strong nonlinearities and nonlinear couplings between different modes of oscillations.

## References

1. Ansari M., Esmailzadeh E., Younesian D., 2010, Internal-external resonance of beams on non-linear viscoelastic foundation traversed by moving load, *Nonlinear Dynamics*, **61**, 163-182
2. Ansari M., Esmailzadeh E., Younesian D., 2011, Frequency analysis of finite beams on nonlinear Kelvin-Voight foundation under moving loads, *Journal of Sound and Vibration*, **330**, 1455-1471
3. Askari H., KalamiYazdi M., SaadatNia Z., 2010, Frequency analysis of nonlinear oscillators with rational restoring force via He's energy balance method and He's variational approach, *Nonlinear Science Letters A*, **1**, 425-430
4. Baghani M., Jafari-Talookolaei R.A., Salarieh H., 2011, Large amplitudes free vibrations and post-buckling analysis of unsymmetrically laminated composite beams on nonlinear elastic foundation, *Applied Mathematical Modelling*, **35**, 130-138

5. Bogacz R., Czyczula W., 2008, Response of beam on visco-elastic foundation to moving distributed load, *Journal of Theoretical and Applied Mechanics*, **46**, 763-775

6. Bogacz R., Frischmuth K., 2009, Vibration in sets of beams and plates induced by traveling loads, *Archive of Applied Mechanics*, **79**, 509-516

7. Kargarnovin M.H., Younesian D., Thompson D.J., Jones C.J.C., 2005, Response of beams on nonlinear viscoelastic foundations to harmonic moving loads, *Computers and Structures*, **83**, 1865-1877

8. He J.H., 2000, Variational iteration method for autonomous ordinary differential systems, *Applied Mathematics and Computation*, **114**, 115-123

9. He J.H., Wu G.C., Austin F., 2010, The variational iteration method which should be followed, *Nonlinear Science Letters A*, **1**, 1-30

10. Inan A., Yildirim A., 2010, Comparison between variational iteration method and homotopy perturbation method for linear and nonlinear partial differential equations with the nonhomogeneous initial conditions, *Numerical Methods for Partial Differential Equations*, **26**, 1581-1593

11. Jafari-Talookolaei R.A., Salarieh H., Kargarnovin M.H., 2011, Analysis of large amplitude free vibrations of unsymmetrically laminated composite beams on a nonlinear elastic foundation, *Acta Mech.*, **219**, 6575, DOI 10.1007/s00707-010-0439-x

12. Malekzadeh P., Vosoughi A.R., 2009, DQM large amplitude vibration of composite beams on nonlinear elastic foundations with restrained edges, *Communications in Nonlinear Science and Numerical Simulation*, **14**, 906-915

13. Nayfeh A.H., 2000, *Nonlinear Interactions*, First ed. New York: John Wiley

14. Ozis T., Yildirim A., 2007, Study of nonlinear oscillators with u1/3 force by He's variational iteration method, *Journal of Sound and Vibration*, **306**, 372-376

15. Santee D.M., Goncalves P.B., 2006, Oscillations of a beam on a non-linear elastic foundation under periodic loads, *Shock and Vibration*, **13**, 273-284

16. Senalp A.D., Arikoglu A., Ozkol I., Dogan V.Z., 2010, Dynamic response of a finite length Euler-Bernoulli beam on linear and nonlinear viscoelastic foundations to a concentrated moving force, *Journal of Mechanical Science and Technology*, **24**, 1957-1961

17. Wazwaz A.M., 2008, A study on linear and nonlinear Schrodinger equations by the variational iteration method, *Chaos, Solitons and Fractals*, **37**, 1136-1142

18. Wazwaz A.M., 2009a, *Partial Differential Equations and Solitary Waves Theory*, First ed., New York: Springer

19. Wazwaz A.M., 2009b, The variational iteration method for analytic treatment for linear and nonlinear ODEs, *Applied Mathematics and Computation*, **212**, 120-134

20. Wazwaz A.M., 2010, The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations, *International Journal of Computer Mathematics*, **87**, 1131-1141

21. Yihua M., Li Q., Hongzhi Z., Vibration analysis of Timoshenko beams on a nonlinear elastic foundation, *Tsinghua Science and Technology*, **14**, 322-326

22. Yildirim A., 2010, Application of He's variational iteration method to nonlinear integro-differential equations, *Zeitschrift für Naturforschung – Section A Journal of Physical Sciences*, **65**, 418-422

23. Yildirim A., Ozis T., 2009, Solutions of Singular IVPs of Lane-Emden type by the variational iteration method, *Nonlinear Analysis, Series A, Theory Methods and Applications*, **70**, 2480-2484

24. Younesian D., Askari H., SaadatNia Z., KalamiYazdi M., 2011, Frequency analysis of strongly nonlinear generalized Duffing oscillators using homotopy perturbation method and variational approach, *Nonlinear Science Letters A*, **2**, 9-14

25. Younesian D., Kargarnovin M.H., Esmailzadeh E., 2008, Optimal passive vibration control of Timoshenko beams with arbitrary boundary conditions traversed by moving loads, *Proceedings of the IMECHE, Part K: Journal of Multi-Body Dynamics*, **229**, 179-188

26. Younesian D., Kargarnovin M.H., Thompson D.J., Jones C.J.C., 2005, Parametrically excited vibration of a Timoshenko beam on random viscoelastic foundation subjected to a harmonic moving load, *Nonlinear Dynamics*, **45**, 75-93

27. Zhang Y.S., 2002, Orthogonality conditions of the transverse vibration mode function for a stepped one-dimensional plate on a Winkler foundation and subjected to an in-plane force, *Proceedings of the IMECHE, Part K: Journal of Multi-body Dynamics*, **216**, 271-273


## Analityczne rozwiązania problemu drgań swobodnych belek umieszczonych na sprężystym podłożu wyznaczone za pomocą iteracyjnej metody wariacyjnej

Streszczenie


W pracy przedstawiono zamknięte formy rozwiązań opisujących odpowiedź dynamiczną elastycznej belki spoczywającej na nieliniowo sprężystym podłożu. Równanie ruchu rozwiązano przy pomocy iteracyjnej metody wariacyjnej (VIM). Sformułowanie iteracyjne skonstruowane w oparciu o tę metodę pozwoliło na wyznaczenie odpowiedzi dynamicznej belki. Charakterystyki częstościowe przedstawiono w zamkniętej formie i podkreślono ich wrażliwość na wartość amplitud początkowych. Następnie zaprezentowano wyniki kilku symulacji numerycznych w celu oszacowania wydajności i dokładności zastosowanej procedury rozwiązywania równań ruchu w dziedzinie czasu. Wykazano, że metoda VIM jest wystarczająco prosta i wiarygodna w rozwiązywaniu układu sprzężonych, nieliniowych równań różniczkowych.

*Manuscript received Jane 22, 2011; accepted for print September 17, 2011*