Scientific Journals
Maritime University of Szczecin

Zeszyty Naukowe
Akademia Morska w Szczecinie

2012, 29(101) pp. 109–116

2012, 29(101) s. 109–116

Selected issues of fractional calculus in mathematical modelling of measuring transducers used in transportation facilities

Wybrane zagadnienia rachunku różniczkowo-całkowego rzędów niecałkowitych w modelowaniu matematycznym przetworników pomiarowych stosowanych w obiektach transportowych

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Key words: modelling, differential calculus

Abstract

The paper presents the possibility of modelling transducers used in transport facilities using fractional calculus and analyses of the dynamic properties in terms of time and frequency for the measuring transducer with a seismic mass. Pointed out the benefits of fractional calculus in the description of the dynamics of transducers used in transportation facilities. Simulation studies were performed in the development environment of MATLAB&Simulink.

Słowa kluczowe: modelowanie, rachunek różniczkowy

Abstrakt

W artykule przedstawiono możliwość modelowania przetworników pomiarowych stosowanych w obiektach transportowych przy zastosowaniu rachunku różniczkowo-całkowego rzędów niecałkowitych (ang. *fractional calculus*). W pracy dokonano analizy właściwości dynamicznych w ujęciu czasowym i częstotliwościowym dla przetwornika pomiarowego z masą sejsmiczną. Wskazano na zalety zastosowania rachunku różniczkowo-całkowego rzędów niecałkowitych w opisie dynamiki przetworników stosowanych w obiektach transportowych. Badania symulacyjne wykonano w środowisku programistycznym MATLAB&Simulink.

Introduction

The differential and integral calculus of fractional order better known in English as *fractional calculus* or in French as *analyse fractionnaire* is a particular case of the scientific knowledge on derivatives and integrals contained in the classical mathematical analysis [1, 2, 3]. Thus, fractional calculus covers derivatives and integrals of integer and fractional (non-integer) orders, in other words of optional orders.

A recent dynamic development of investigations on the use of fractional calculus for the analysis of dynamic systems [3] encouraged the authors of this paper to attempt its use for the analysis and modelling of measuring transducers with a seismic mass.

Mathematical model of measuring transducer with seismic mass

Architecture of the measuring transducer with a seismic mass has been discussed in detail in the papers [4, 5, 6, 7].

The equation for the transducer's seismic mass motion is derived from the equation of equilibrium of forces:

$$F_b(t) + F_r(t) + F_s(t) = 0 \qquad (1)$$

where:

$F_b(t)$ – inertial force,
$F_r(t)$ – damping force,
$F_s(t)$ – the spring reaction force.

Table 1 presents the components of the transducer's seismic mass motion equation and characteristic parameters.

Table 1. Components of the equilibrium of forces equation
Tabela 1. Składowe równania równowagi sił

Equation components (1)	
inertial force	$F_b(t) = m\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)$
damping force	$F_r(t) = B_t\left[\dfrac{\mathrm{d}}{\mathrm{d}t}y(t) - \dfrac{\mathrm{d}}{\mathrm{d}t}x(t)\right]$
spring reaction force	$F_s(t) = k_s\left[y(t) - x(t)\right]$
Transducer parameters	
amplification factor	$k = \dfrac{1}{k_s}$
natural frequency	$\omega_0 = \sqrt{\dfrac{k_s}{m}}$
damping degree	$\zeta = \dfrac{B_t}{\sqrt{2k_s m}}$

The differential equation describing the absolute motion of the transducer's seismic mass takes the following form:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 2\zeta\omega_0\frac{\mathrm{d}}{\mathrm{d}t}y(t) + \omega_0^2 y(t) = = 2\zeta\omega_0\frac{\mathrm{d}}{\mathrm{d}t}x(t) + \omega_0^2 x(t) \qquad (2)$$

After the relative seismic mass displacement has been introduced to equation (2):

$$w(t) = y(t) - x(t) \qquad (3)$$

The latter can be rewritten as:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}w(t) + 2\zeta\omega_0\frac{\mathrm{d}}{\mathrm{d}t}w(t) + \omega_0^2 w(t) = = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t) \qquad (4)$$

Figure 1 depicts over time responses of a typical 2nd order transducer to step inputs. The responses are shown for different values of the damping degree. For the given ω_0, an increase in ζ reduces oscillation.

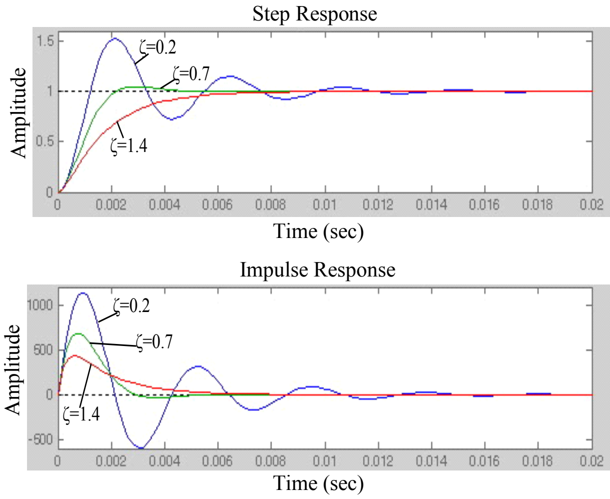

Fig. 1. Time characteristics of the 2nd order measuring transducer
Rys. 1. Charakterystyki czasowe przetwornika pomiarowego drugiego rzędu

Figure 2 presents the transducer's logarithmic frequency characteristics. These characteristics indicate that the 2nd order transducer introduces distortions due to the fact that it does not transmit signals of different frequencies in the same way.

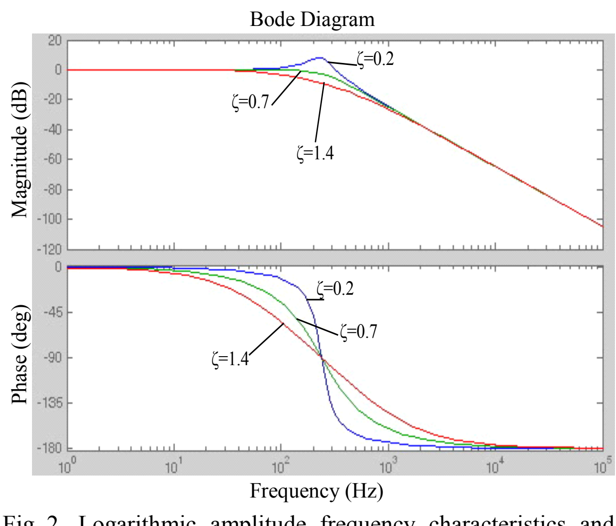

Fig. 2. Logarithmic amplitude frequency characteristics and logarithmic phase frequency characteristics of the 2nd order measuring transducer
Rys. 2. Logarytmiczne charakterystyki częstotliwościowe (amplitudowa i fazowa) przetwornika pomiarowego drugiego rzędu

Measuring transducers with a seismic mass, depending on the selection of parameters characterising their dynamic properties, can serve for measurement of such quantities as displacement, speed, or acceleration. Displacement $x(t)$ is the input quantity in these transducers. Depending on the selected parameters k_S, m and B_t a transducer can be used for the measurement of different quantities. Hence:

– assuming low values of k_S and B_t, and high value of m, equation (4) can be written down in the following way:

$$\frac{d^2}{dt}w(t) \cong -\frac{d^2}{dt^2}x(t) \tag{5}$$

Then the transducer measures displacement and fulfills the role of a vibrometer.

– assuming low values of k_S and m and high value of B_t a transducer for velocity (speed) measurements is obtained:

$$2\zeta\omega_0 \frac{d}{dt}w(t) \cong -\frac{d^2}{dt^2}x(t) \tag{6}$$

– assuming high value of k_S, low values of m and B_t, can be written as follows:

$$\omega_0^2 w(t) \cong -\frac{d^2}{dt^2}\ddot{x}(t) \tag{7}$$

Then the transducer measures acceleration (accelerometer). In practical measurements of vibrations, a transducer with a seismic mass to measure the rail vehicle acceleration is used. Parameters of speed and displacement are determined by means of elements integrating a signal from the accelorometer. Simulation of operations of the transducer described by equation (4) was carried out while assuming the following values of parameters: $\omega_0 = 15$ [rad/s], $\xi = 1.7$ and $k = 1$ [m/N] in the MATLAB &Simulink environment.

Dynamic behaviour of the transducer described by the differential equation of integer order (4) and taking into account parameters can be presented in the form of the transfer function:

$$G(s) = \frac{-s^2}{s^2 + 51s + 225} \tag{8}$$

Figure 3 depicts the step and impulse responses of the transducer characterized by the transfer function (8).

Figure 4 shows logarithmic amplitude- and phase frequency characteristics of the transducer. Amplitude amplification equal 0 db is obtained from ca. 100 Hz, at a simultaneous phase displacement from 180° to 200°.

Figure 5 presents a transducer's response to the sinusoidal input of 400 Hz in frequency. The transducer's delayed response to the input signal is clearly seen here.

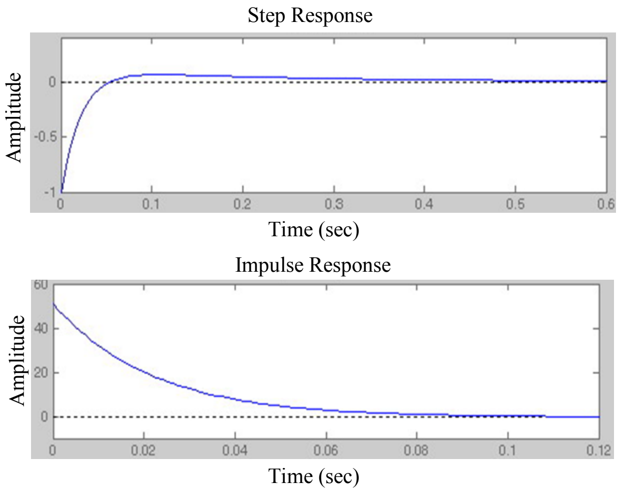

Fig. 3. Step response (top) and impulse response (bottom) of the transducer
Rys. 3. Odpowiedź skokowa i impulsowa przetwornika

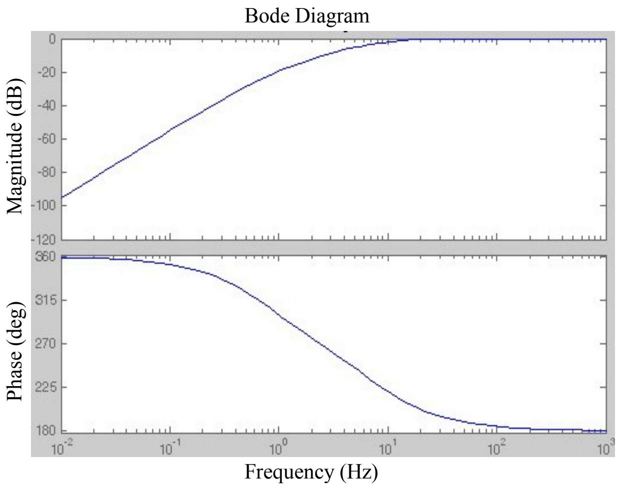

Fig. 4. Transducer's amplitude and phase frequency characteristics
Rys. 4. Charakterystyka amplitudowa i fazowa przetwornika

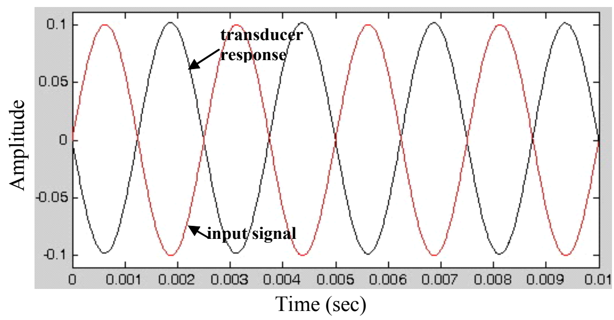

Fig. 5. Transducer's response to sinusoidal function
Rys. 5. Odpowiedź przetwornika na wymuszenie sinusoidalne

Model of measuring transducer with seismic mass described by fractional calculus of integer orders

Modelling of an actual transducer or a measuring system consisting of multiple devices requires consideration of response dynamics of each of them [8]. Knowing the input signal and the response signal one can describe the system's dynamic behaviour in the form of a differential equation. Accuracy of the model achieved in such a way depends mainly on the identification method used. The use of a differential and integral equation in the process of identification creates new opportunities of obtaining a model which reflects the dynamic behaviour of the investigated object more accurately.

This section presents a model of the transducer described by fractional calculus but of integer orders [12].

Equation (4) describing the transducer can be written down in the form of the differential equation:

$$a_2 w_k + a_1 w_{k-1} + a_0 w_{k-2} = b_2 x_k + b_1 x_{k-1} + a_0 x_{k-2} \quad (9)$$

or as the matrix equation:

$$\begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ w_{k-2} \end{bmatrix} = \begin{bmatrix} b_2 & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \end{bmatrix} \quad (10)$$

Differential equation (9) in the integral derivative notation assumes the form:

$$\begin{aligned} &A_2 \Delta_k^{(2)} w_k + A_1 \Delta_{k-1}^{(1)} + A_0 w_{k-2} = \\ &= B_2 \Delta_k^{(2)} w_k + B_1 \Delta_k^{(1)} x_{k-1} + B_0 w_{k-2} \end{aligned} \quad (11)$$

where $\Delta_k^{(n)}$ is the reverse difference of the discrete function, defined as follows:

$$\Delta_k^{(n)} f(k) = \sum_{j=0}^{k} a_j^{(n)} f(k-j) \quad (12)$$

After equation (12) has been taken into account, equation (11) in matrix notation takes the following form:

$$\begin{aligned} &\begin{bmatrix} a_2 & -a_1 - 2a_0 & a_2 + a_1 + a_0 \end{bmatrix} \begin{bmatrix} \Delta_k^{(2)} w_k \\ \Delta_k^{(1)} w_k \\ \Delta_k^{(0)} w_k \end{bmatrix} = \\ &= \begin{bmatrix} b_0 & -b_1 - 2b_0 & b_2 + b_1 + b_0 \end{bmatrix} \begin{bmatrix} \Delta_k^{(2)} x_k \\ \Delta_k^{(1)} x_k \\ \Delta_k^{(0)} x_k \end{bmatrix} \end{aligned} \quad (13)$$

After the measuring transducer's responses to the sinusoidal input signal had been compared, the transducer was described by three models:

– classical continuous model described by the transfer function:

$$G(s) = \frac{-s^2}{s^2 + 51s + 255} \quad (14)$$

– classical discrete model obtained from the continuous model, described by the discrete transmittance:

$$G(z) = \frac{-z^2 + 2z - 1}{z^2 - 1.975z + 0.9748} \quad (15)$$

– quasi-fractional discrete model determined by the integral-derivative notation from equation (13) and discrete transmittance (15):

$$G(z) = \frac{-z^2 + 0.02524z - 6.294 \cdot 10^{-5}}{z^2 - 3.161 \cdot 10^{-5} z + 1.11 \cdot 10^{-16}} \quad (16)$$

Investigation of the model responses were carried out in the MATLAB&Simulink environment. Figure 6 depicts a block diagram of the measuring system.

Responses of all models to the sinusoidal input signal with frequency of 100 rad/s are shown in figure 7.

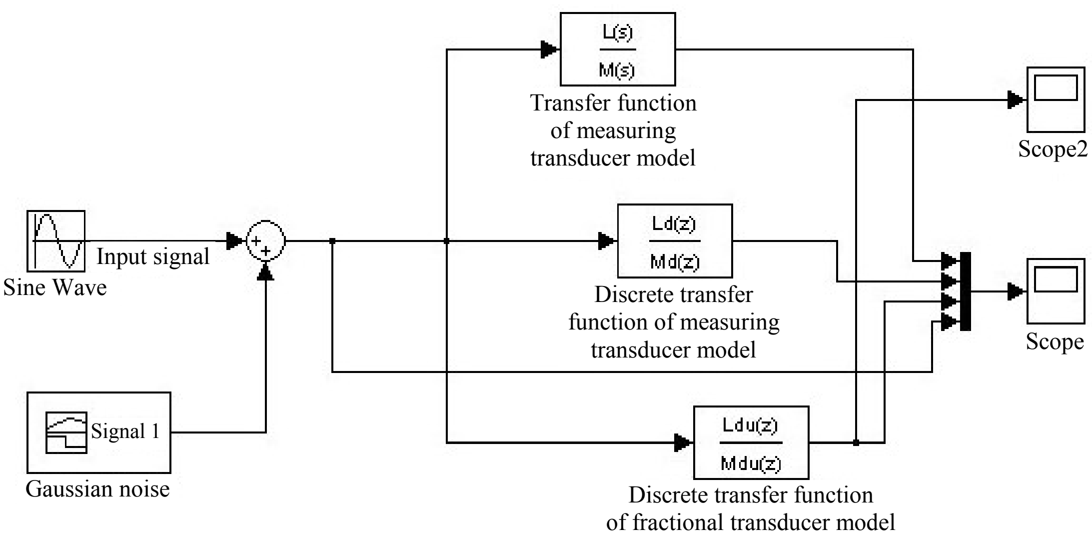

Fig. 6. Block diagram of the measuring system [4]
Rys. 6. Schemat blokowy układu pomiarowego [4]

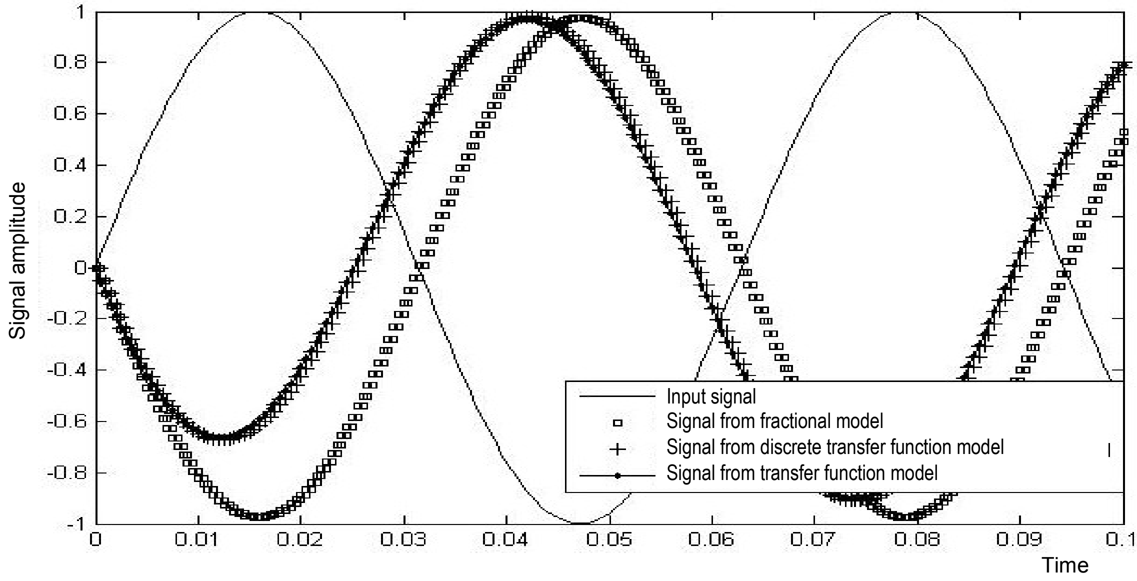

Fig. 7. Comparisons of responses of the measuring transducer models [9]
Rys. 7. Porównanie odpowiedzi modeli przetwornika pomiarowego [9]

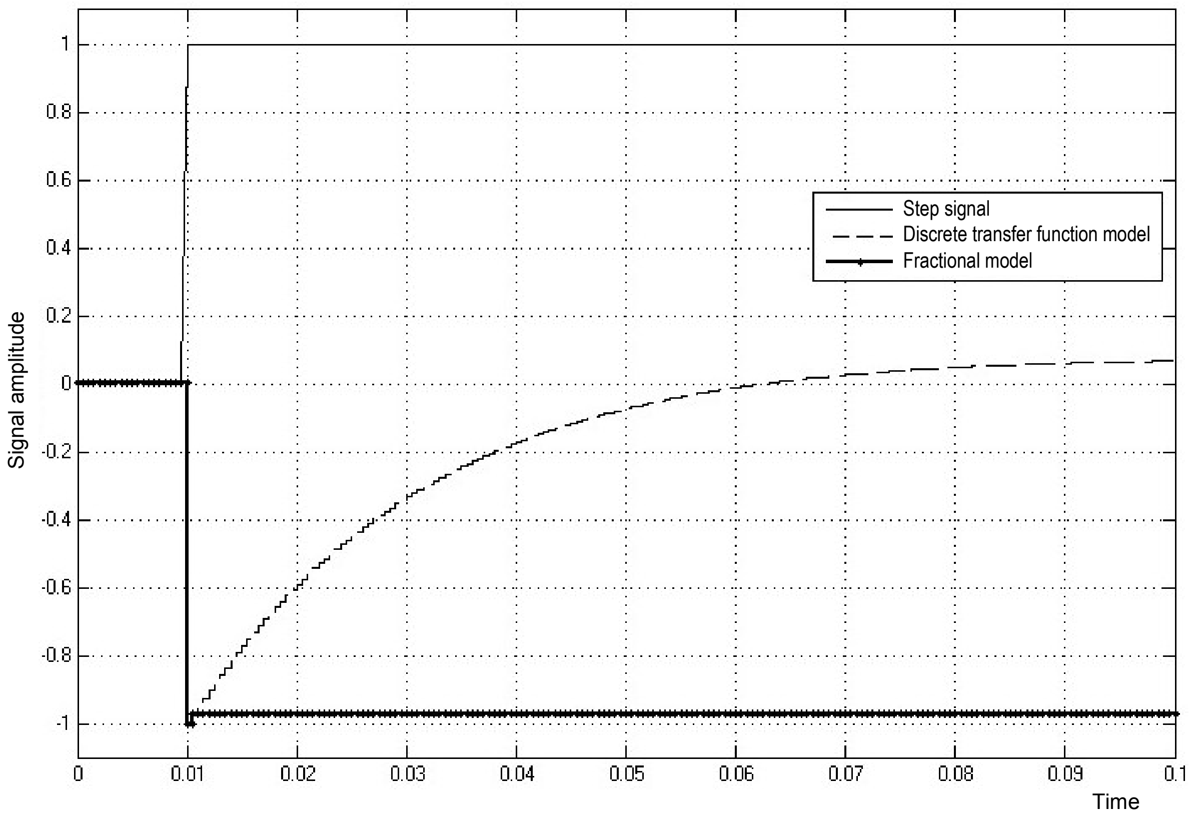

Fig. 8. Comparisons of the measuring transducer models in response to step input [9]
Rys. 8. Porównanie odpowiedzi modeli przetwornika na wymuszenie skokowe [9]

Signals from the models are displaced in phase in relation to the input signal. It is worth noticing that the model described by transmittance (16) correctly reproduces the value of the input signal amplitude.

Figure 8 shows comparisons of the presented models' responses ((15) and (16)) to the step input. The classical model's response passes into a steady state after the time of 0.6 s from the moment the signal occurs. In the case of the integral-derivative model, the steady state occurs after the time of 0.005 s and its value is close to the value of the input amplitude with the apposite sign.

Figure 9 compares responses of the presented models (15) and (16) to the impulse input. The classical model response passes into the steady state

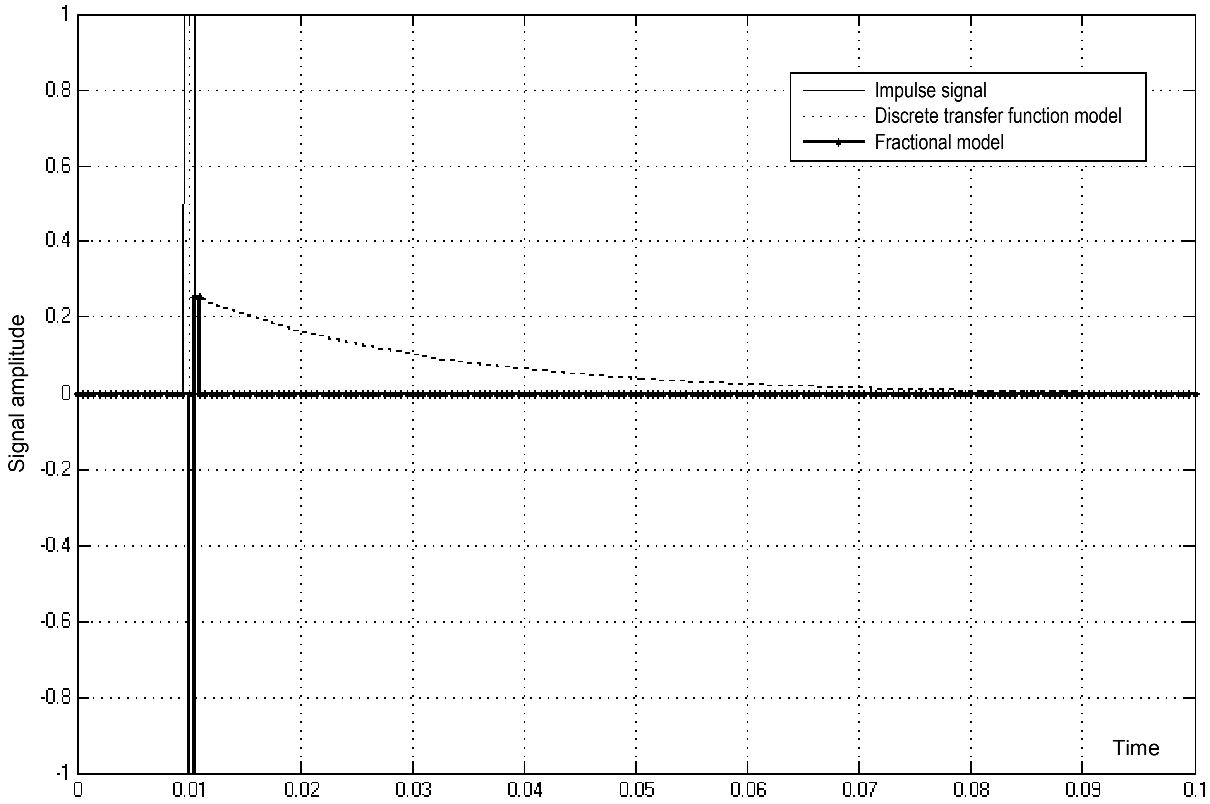

Fig. 9. Comparison of the measuring transducer models responses to the impulse input [9]
Rys. 9. Porównanie odpowiedzi modeli przetwornika na wymuszenie impulsowe [9]

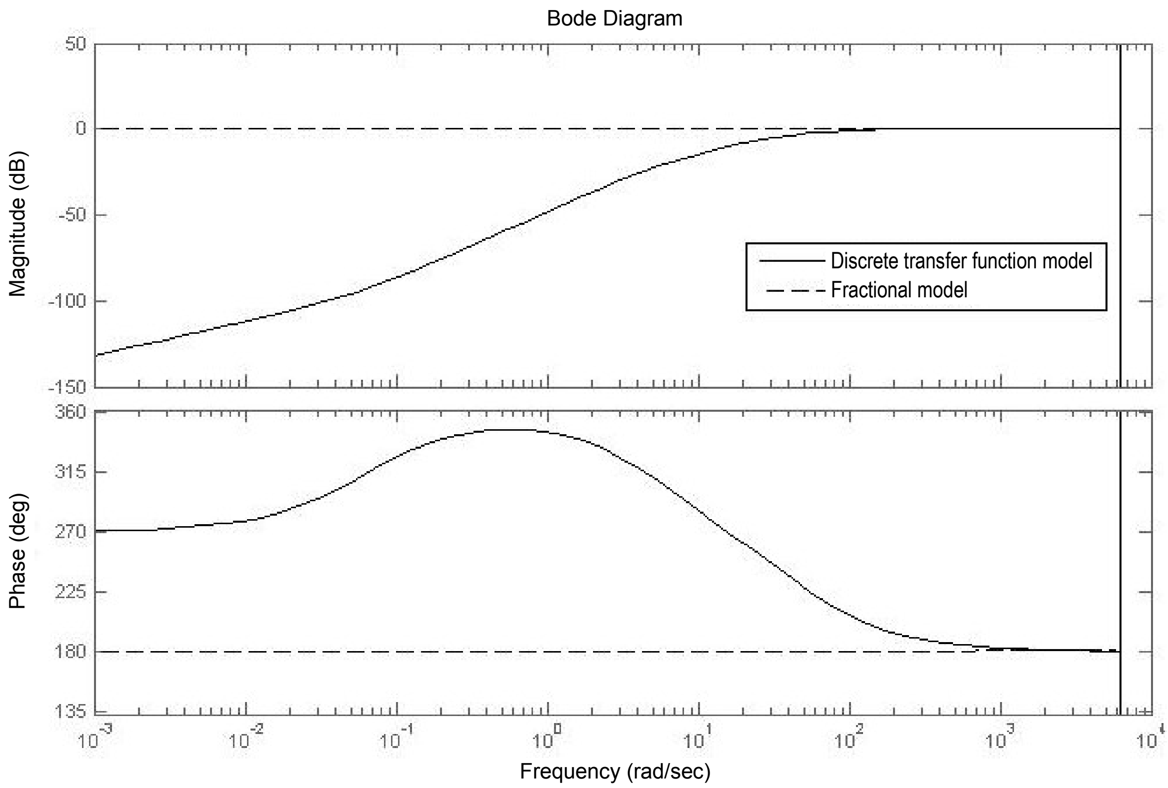

Fig. 10. Comparison of Bode diagrams of the measuring transducer models [4]
Rys. 10. Porównanie charakterystyk Bodego modeli przetwornika pomiarowego [4]

after the time of 0.07 s from the moment the signal occurs. In the case of the integral-derivative model this time is reduced to the value of 0.001 s.

Figure 10 compares frequency characteristics (Bode diagrams) for the discrete model of the measuring transducer and the discrete model of the measuring transducer determined by the integral-derivative notation.

For the characteristics presented, amplitude amplification of the fractional model equal 0 db is achieved for the frequency from 0.001 rad/s, and for the "classical" model – from 100 Hz, at a permanent phase displacement equal 180°.

It can be concluded from the above said that the description of the transducer with a seismic mass by means of the fractional calculus is an advantage in the case these transducers are used for vibration tests of rail vehicles. Vibration tests of rail vehicles are one of many types of tests aiming to confirm safety of such vehicles and parameters defining their usefulness for specific transport applications [7]. Requirements concerning dynamic behaviour of rail vehicles from the point of view of safety parameters, railway track fatigue and ride quality are described in the UIC 518 document of the *Union Internationale des Chemins de Fer* (International Union of Railways) organisation. The standards included in it are part of the documentation approving a rail vehicle for traffic admittance in Europe. The UIC 518 standard specifies the range of measured frequencies of the measured quantity to be from 0.4 Hz to 10 Hz.

On the basis of the Bode diagram characteristics (Fig. 10) it can be inferred that for the measuring transducer model determined by the fractional calculus method, in comparison to the model determined in the classical manner, the range of the input signal amplitude processing is extended by low frequencies.

Models of measuring transducer described by the fractional calculus of non-integer order

The section below presents the transducer described by the fractional calculus of non-integer (fractional) orders.

Provided that a non-integer derivative will better describe the dynamic behaviour of the element responsible for damping, equation (4) is written down as follows:

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}t^2}w(t)+2\zeta\omega_0\frac{\mathrm{d}^{(v_1)}}{\mathrm{d}t^{(v_1)}}w(t)+\omega_0^2 w(t)=\\&=-\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)\end{aligned}\tag{17}$$

In the MATLAB&Simulink environment, the measuring transducer of transfer function was modelled:

$$G(s)=\frac{-s^2}{s^2+10s+50}\tag{18}$$

Figure 11 compares how amplitude and phase characteristics of the integral derivative model of the transducer change depending on the order value of the derivative v_1 at A_1 factor (19) responsible for the damping value. These values were changed within the range from 0.2 to 1.8 by a 0.2 step. The remaining values of the orders of derivatives are integer values identical with those in the "classical" notation.

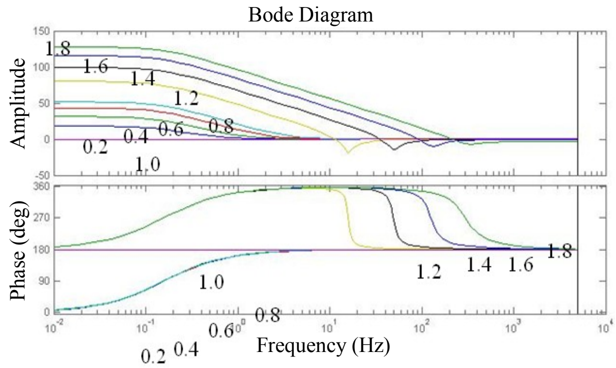

Fig. 11. Amplitude and phase characteristics of the transducer's integral derivative model depending on the values of the non-integer order [9]

Rys. 11. Charakterystyki amplitudowa i fazowa modelu pochodno-całkowego w zależności od wartości niecałkowitego rzędu [9]

While comparing amplitude and phase characteristics of the transducer model written down by means of the classical equation and the integral-derivative model, one can state that the range of amplitude and phase frequency changes within comparable frequencies in the case of the integral-derivative model changes within a narrow range. One can also notice an increase in amplitude amplification depending on the value of the order of the derivative regardless of the fact whether the order value is rising or falling. In the case of phase characteristics an increase in the order value causes increased phase displacement in relation to the order value equal 1. A decrease in the order entails an appropriate decrease in phase displacement.

Generalising equation (17) to equation (19):

$$\begin{aligned}&A_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)+A_1\frac{\mathrm{d}}{\mathrm{d}t}y(t)+A_0y=\\&=B_2\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(t)+B_1\frac{\mathrm{d}}{\mathrm{d}t}x(t)+B_0x(t)\end{aligned}\tag{19}$$

and considering the fact that the integer order derivatives in fractional calculus are a peculiar case of non-integer derivatives, equation (12) can be written down as follows:

$$A_2 \frac{\mathrm{d}^{(\nu_2)}}{\mathrm{d}t^{(\nu_2)}} y(t) + A_1 \frac{\mathrm{d}^{(\nu_1)}}{\mathrm{d}t^{(\nu_1)}} y(t) + A_0 \frac{\mathrm{d}^{(\nu_0)}}{\mathrm{d}t^{(\nu_0)}} y = \\ = B_2 \frac{\mathrm{d}^{(\mu_2)}}{\mathrm{d}t^{(\mu_2)}} x(t) + B_1 \frac{\mathrm{d}^{(\mu_1)}}{\mathrm{d}t^{(\mu_1)}} x(t) + B_0 \frac{\mathrm{d}^{(\mu_0)}}{\mathrm{d}t^{(\mu_0)}} x(t) \tag{12}$$

Figure 12 shows frequency characteristic families for $v_2 = 2$ and subsequent v_1.

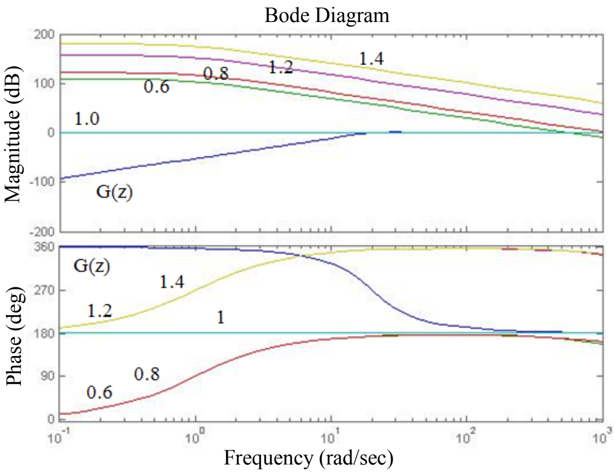

Fig. 12. Logarithmic frequency characteristics for $v_2 = 2$ and subsequent v_1 [10]

Rys. 12. Logarytmiczne charakterystyki częstotliwościowe dla $v_2 = 2$ i kolejnych v_1 [10]

Conclusions

Checking up how the measuring transducer models described by fractional calculus reflect actual measuring transducers requires further investigations. So does checking up they reflect the dynamic behaviour of the input signal processing more accurately than the model described by the integer order differential equation.

Although the authors of this paper pointed only at advantages of using the fractional calculus for measuring transducers for vibration tests of rail vehicles, the fractional calculus can also be used successfully for describing the dynamic behaviour of measuring transducers used in diagnostics of various transport facilities.

Global research into such physical phenomena as liquid permeation through porous substances, electric load transfer through an actual insulator, or heat transfer through a heat barrier, descriptions of friction or properties of viscoelastic materials showed that fractional calculus describes this type of phenomena more accurately than classical mathematical analysis. Thus, the derivative and integral of optional orders open a number of possibilities in the field of the dynamic system identification and creation of new, hitherto unattainable, algorithms of the measuring system control.

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