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METHOD OF FUNDAMENTAL SOLUTION AND GENETIC ALGORITHMS FOR TORSION OF BARS WITH MULTIPLY CONNECTED CROSS SECTIONS

Piotr Gorzelańczyk

Higher Vocational State School Pila, Polytechnic Institute, Piła, Poland

e-mail: piotr.gorzelanczyk@pwsz.pila.pl

The torsion of bars with a multiply connected cross sections by means of the method of fundamental solutions (MFS) is considered herein. To determine the optimal parameters of MFS, genetic algorithms were used. Seven cases of cross sections are considered. The numerical results for different cross sectional shapes are presented to demonstrate the efficiency and accuracy of the method. Non-dimension torsional stiffness was calculated by means of numerical integration of the stress function for one of the cases. This stiffness is compared with the exact stiffness for the first case and with the stiffness resulting from Bredt's formulae for thin walled cross sections.

Key words: Bredt's formulae, method of fundamental solutions, multiply connected sections, genetic algorithms

1. Introduction

The solution of the torsion problem for a multiply connected cross section is more difficult than for the simply connected one. Probably, it is the reason why there are not too many papers considering this problem. For a doubly connected cross section, the exact solution exists for the annular cross section. Weinel (1932) proposed the solution for a doubly connected cross section with excentric circles. In the work (Polya and Weinstein, 1950) it was found that for the doubly connected domains with prescribed area of the hole and cross section the ring bounded by two concentric circles has the maximal torsional rigidity. In the book (Arutiunian and Abramian,1963) the description of the method of solving the torsion problem for a doubly connected cross section in which outer and inner contour are rectangles is presented. This method was proposed by Russian authors (Szerman, Abramian) in series of papers in the 50s and it is based on expansion of the stress function in Fourier series. In order to obtain an effective solution, this method requires the solution to an infinite system of linear equations. Wang (1995) presented a method for torsion analysis of two connected cross sections in shape of a flattened tube consisting of two half annular pieces and two rectangular pieces. He adopted an approximate solution in the form of truncated series of functions of its own, and to satisfy the boundary conditions he used boundary element methods. Wang (1998) generalized this method to treat arbitrary two connected cross sections consisting of circular arcs and straight lines, all with a uniform thickness. A modified Fourier series method for the torsion analysis of bars with multiply connected cross sections was presented by Kim and Yoon (1997). The effectiveness of this method was presented for polygonal cross sections with polygonal holes. Mejak (2000) presented a method for an optimal shape design of doubly connected bars in torsion. He solved the problem numerically by the finite element method. In the work (Kolodziej and Fraska, 2005), the Trefftz method using special purpose T-functions was used to solve the problem of torsion. As examples, one-connected, multiply-connected and composite cross sections of bars in the shape of regular polygons were considered. The proposed Trefftz function not only satisfies the governing equation, but also the boundary conditions on some sides. In the article, for the stress function, the boundary collocation methods and the method of smallest squares were used. Using the analytical integration, an analytical solution for the dimensionless stiffness of the bar was obtained.

A special group of papers considers thin-walled cross sections. A simple formulation for torsion analysis of thin-walled hollow bars can be found in elementary textbooks of strength of materials, as was proposed by Bredt (1896). In the work (Morassi, 1999) the author proves that Bredt's theory remains true for thin tubes with multicell cross sections more than doubly connected. A closed form expression for the torsion constant and thin-walled typical multicell profiles is presented by Lubarda (2009). Generalization of Bredt's method for moderate thick hollow tubes with polygonal shapes is given by Hematiyan and Doostfatemeh (2007).

The purpose of this paper is application of Method of Fundamental Solutions and genetic algorithms for the torsion problem with multiply connected cross sections. This method belongs to so-called meshless methods which have been more and more popular in the two last decades. The MFS was first proposed by the Georgian researchers Kupradze and Aleksidze (1964). Its numerical implementation was carried out by Mathon and Johnston (1977). The mathematical analysis (convergence and stability) of this method was considered in Bogomolny (1985), Katsura (1990), Katsura and Okamoto (1988, 1996), Kitagawa (1988, 1991). The comprehensive reviews of the MFS for various applications can be found in Fairweather and Karageorghis (1998), Fairweather *et al.* (2003), Goldberg and Chen (1998). However, as yet, the method of fundamental solutions has been applied basically for simply-connected regions. There are only few papers with application of MFS for annular region, e.g. Chen *et al.* (2006), Li (2009), Tsangaris *et al.* (2006).

In the works (Fairweather and Karageorghis, 1988a,b, 1989; Fairweather *et al.*, 2003), the location of sources is determined by minimizing the functional of mean, where the optimization parameters are also the fundamental solution weighing factors and coordinates of the location of sources. These latter issues are the evidence of nonlinearity. These authors applied the collocation method, where the number of sources and the number of collocation points is determined. Genetic algorithms for one-connected areas were applied in Kołodziej and Klekiel (2008), Nishimura *et al.* (2000, 2001, 2003) to determine the optimal positioning of sources. Last time, Karageorghis (2009) appeared in which the method of the golden mean was used to determine the optimal location of source points.

In this case, the error is multidimensional, with many local minima, numerical methods of searching for optimal solution fail. The result is that there can be a case where the solution lies outside the area and thus the process of finding the optimum can not achieve the desired result. The methods consist of careful movement from point to point in a certain area of decision-making, in accordance with the selection rule determining the next point. This is not a safe way, because it allows the location of false minima in a multi-node space exploration. For this reason, genetic algorithms were used to determine the optimum parameters. According to Goldberg (1995), genetic algorithms are search algorithms based on the mechanisms of natural selection and heredity, which were developed by Howland. Combining the evolutionary principle of survival of the fittest with a systematic, although randomized exchange of information, they create a method of finding, reluctantly giving it her proper degree of inventiveness of the human mind. Genetic algorithms can cope well where the optimized function is noisy, changes over time and has many local extremes. Using genetic algorithms in an expeditious manner, we can determine the optimal solution of the method.

This paper presents the application of this method to multiply connected cross sections namely: (I) circular with circular centered hole, (II) square with circular centered hole, (III) square with square centered hole, (IV) square with square centered hole with rounded corners with the radius $r = E/2$ (where E is the characteristic dimension of the hole), (V) square with square centered hole with rounded corners with the radius $r = 3E/4$ (VI) circular with two circular symmetrical placed holes, (VII) square with two circular symmetrical placed holes.

2. Formulation of the problem

The problem of torsion of prismatic bars with a multiply connected cross section (see Fig. 1) is formulated in terms of the stress function, which satisfies Poisson's equation (Arutiunian and Abramian,1962)

$$\nabla^2\psi = -2G\omega \qquad \text{in } \Omega \tag{2.1}$$

with the boundary condition on the outer contour

$$\psi = 0 \qquad \text{on } \varGamma_0 \tag{2.2}$$

and boundary conditions on the inner contour

$$\psi = \psi_i \qquad \text{on } \varGamma_i \qquad i = 1, 2, \dots, n \tag{2.3}$$

where $\psi(x, y)$ is the stress function, μ is the shear modulus of the bar material, ω is the angle of twist of the bar per unit length, ψ_i are unknown values of the stress function at inner contours, n – number of hollow areas.

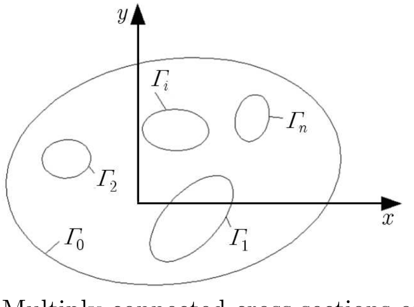

Fig. 1. Multiply connected cross sections of a bar

For determination of the unknown constants ψ_i, the following integral relations (Bredt's theorem) are given

$$\oint\limits_{\varGamma_i} \frac{\partial\psi}{\partial n}\, ds = -2\Omega_i G\omega \qquad i = 1, 2, \dots, n \tag{2.4}$$

where Ω_i is the area bounded by $\varGamma_i$.

After introducing the non-dimensional variables

$$X = \frac{x}{a} \qquad Y = \frac{y}{a} \qquad \Psi(X,Y) = \frac{\psi(x,y)}{a^2 G\omega} \tag{2.5}$$

The considered boundary value problem has the following dimensionless form: governing equation for the stress function

$$\frac{\partial^2 \Psi}{\partial X^2} + \frac{\partial^2 \Psi}{\partial Y^2} = -2 \qquad \text{in } \widetilde{\Omega} \tag{2.6}$$

with the boundary condition at the outside contour

$$\Psi = 0 \qquad \text{on } \widetilde{\Gamma}_0 \tag{2.7}$$

and the boundary condition at the inner contour

$$\Psi = \Psi_i \qquad \text{on } \widetilde{\Gamma}_i \qquad i = 1,2,\dots,n \tag{2.8}$$

and an integral relation in the form

$$\oint\limits_{\widetilde{\Gamma}_i} \frac{\partial \Psi}{\partial n}\, ds = -2\widetilde{\Omega}_i \qquad i = 1,2,\dots,n \tag{2.9}$$

where $\widetilde{\Omega}_i$ is the dimension area bounded by $\widetilde{\Gamma}_i$.

3. Method of solution

In the MFS, the approximate solution to the problem is represented in form of linear superposition of source functions (fundamental solutions) with singular points that are located outside the domain of the problem. These points, called source points, are located on a "pseudo-boundary" outside the region. The pseudo-boundary has no common points with the boundary of the region. Because the fundamental solution satisfies the differential equation at any point except for the source point, it follows that this representation exactly satisfies the governing equation whereas the boundary conditions are only satisfied approximately. Therefore, the MFS belongs to the group of Trefftz methods for which it is essential that the governing equation is exactly satisfied. The weights of coefficients which occur in the approximate solution are determined by the satisfaction of the boundary condition, usually on a set of boundary points (collocation points). Using MFS, the solution of boundary value problem formulated by (2.6)-*2.9) can now be given as the sum of the particular solution and the homogeneous solution

$$\begin{aligned}\varPsi = -\frac{1}{2}(X^2+Y^2) + \sum_{j=1}^{MZ} c_j \ln[(X-XSZ_j)^2 + (Y-YSZ_j)^2] \\ + \sum_{i=1}^{n} \sum_{k=1}^{MWi} c_k^{(i)} \ln[(X-XSW_k^{(i)})^2 + (Y-YSW_k^{(i)})^2]\end{aligned} \tag{3.1}$$

where: XSZ_j, YSZ_j are the coordinates of source points which are placed outside the region $\widetilde{\varOmega}$ (Fig. 2), $XSW_k^{(i)}$, $YSW_k^{(i)}$ are the coordinates of source points which are placed inside the inner contours $\widetilde{\varGamma}_i$, where $i = 1, 2, \ldots, n$, MZ is the number of source points outside the region $\widetilde{\varOmega}$, MWi is the number of source points inside each inner contour $\widetilde{\varGamma}_i$, c_j and $c_k^{(i)}$ are unknown coefficients.

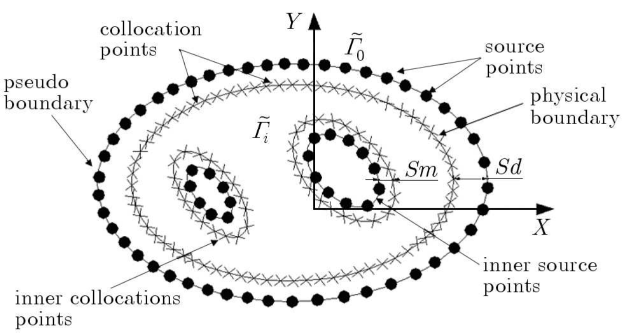

Fig. 2. Arrangement of the source points on a similar contour

The unknown coefficients c_j, $c_k^{(i)}$ and constants $\varPsi_i$ are determined by collocation of boundary conditions (2.7) and (2.8) and application of integral relations (2.9).

In order to do this, NCZ collocation points on the outer contour with coordinates XCZ_l, YCZ_l are chosen and NCW collocation points on each inner contour with the coordinates $XCW_m^{(i)}$, $YCW_m^{(i)}$ are chosen.

Substituting solution (3.1) to the boundary condition (2.7) we have $(l=1,2,\dots,NCZ)$

$$
\begin{split}
&\sum_{j=1}^{MZ} c_j \ln[(XCZ_l - XSZ_j)^2 + (YCZ_l - YSZ_j)^2] \\
&\qquad + \sum_{i=1}^{n}\sum_{k=1}^{MWi} c_k^{(i)} \ln[(XCZ_l - XSW_k^{(i)})^2 + (YCZ_l - YSW_k^{(i)})^2] \\
&\qquad = \frac{1}{2}(XCZ_l^2 + YCZ_l^2)
\end{split}
\tag{3.2}
$$

Similarly, substitution of solutions (3.1) into boundary condition (2.8) leads to a system of linear equations $(m=1,2,\dots,NCW)$

$$
\begin{split}
&\sum_{j=1}^{MZ} c_j \ln[(XCW_m^{(i)} + XSZ_j)^2 + (YCW_m^{(i)} - YSZ_j)^2] \\
&\qquad + \sum_{i=1}^{n}\sum_{k=1}^{MWi} c_k^{(i)} \ln[(XCW_m - XSW_k^{(i)})^2 + (YCW_m - YSW_k^{(i)})^2] \\
&\qquad = \frac{1}{2}(XCZ_l^2 + YCZ_l^2) + \Psi_i
\end{split}
\tag{3.3}
$$

Using Bredt's conditions (2.9) on the inner contour $\widetilde{\Gamma}_i$, we have $(i=1,2,\dots,n)$

$$
\begin{split}
&\sum_{j=1}^{MZ} c_j \oint\limits_{\widetilde{\Gamma}_i} \frac{\partial}{\partial n} \ln[(X_s + XSZ_j)^2 + (Y_s - YSZ_j)^2]\, ds \\
&\qquad + \sum_{i=1}^{n}\sum_{k=1}^{MWi} c_k^{(i)} \oint\limits_{\widetilde{\Gamma}_i} \frac{\partial}{\partial n} \ln[(X_s - XSW_k^{(i)})^2 + (Y_s - YSW_k^{(i)})^2]\, ds = 0
\end{split}
\tag{3.4}
$$

In this way, we obtain $NCZ + nNCW + n$ equations with $MZ + nMW + n$ unknowns.

For further numerical calculations, the following assumptions: $M = MZ + MWi$ and $NC = NCZ + NCW$ were made.

4. Torsional stiffness and stress

The relations between the nonzero components of the stress and stress function are given by following formulae

$$\tau_{yz} = -\frac{\partial \psi}{\partial x} \qquad \tau_{xz} = \frac{\partial \psi}{\partial y} \tag{4.1}$$

The torsional moment is given by an integral of the shear stresses over the area, which gives

$$SZ = \iint (\tau_{yz}x - \tau_{xz}y)\;dx\,dy = -\iint \frac{\partial \Psi}{\partial x}x\;dx\,dy - \iint \frac{\partial \Psi}{\partial y}y\;dx\,dy + 2\sum_{i=1}^{n} \varOmega_i \psi_i \tag{4.2}$$

After simple manipulations, we get

$$SZ = 2 - \iint \varPsi\;dx\,dy + 2\sum_{i=1}^{n} \varOmega_i \psi_i \tag{4.3}$$

Introducing the non-dimensional variables into (4.3), the torsional moment can be related to the non-dimensional stress function

$$SZ = G\omega a^4 \Big[2\iint \varPsi(X,Y)\;dX\,dY + 2\sum_{i=1}^{n} \widetilde{\varOmega}_i \varPsi_i \Big] \tag{4.4}$$

Next, the non-dimensional torsional stiffness can by expressed as

$$M_s = \frac{SZ}{G\omega a^4} = 2\iint \varPsi(X,Y)\;dX\,dY + 2\sum_{i=1}^{n} \widetilde{\varOmega}_i \varPsi_i \tag{4.5}$$

In elementary textbooks of strength of materials (Dylag *et al.*, 1999), one can find the expression for torsional stiffness for thin-walled hollow bars (Fig. 3), known as Bredt's formula

$$SZ = \frac{4GA_{mid}^2\omega}{\oint \frac{ds}{t}} \tag{4.6}$$

where A_{mid} is the area bounded by the centerline of the wall cross section, t is thickness.

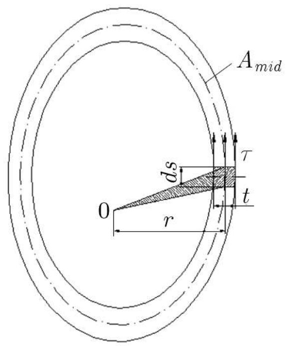

Fig. 3. Thin-walled bar with closed cross-section

5. Test examples

In order to demonstrate the exactness and the effectiveness of the proposed method, seven cases of cross sections are considered: (I) circular with circular centered hole, (II) square with circular centered hole, (III) square with square centered hole, (IV) square with square centered hole with rounded corners with the radius $r = E/2$ (where E is the characteristic dimension of the hole), (V) square with square centered hole with rounded corners with the radius $r = 3E/4$, (VI) circular with two circular symmetrical placed holes, (VII) square with two circular symmetrical placed holes.

The formulation of boundary values problems for seven considered cross sections in terms of the non-dimensional stress function $\Psi(X,Y)$ is given in Figs. 4-10. In this case, the thickness E is a geometrical parameter which changes in a permissible range, i.e. $0 < E < 0.5$ for problems I-V and $0 < E < 0.25$ for problems VI, VII.

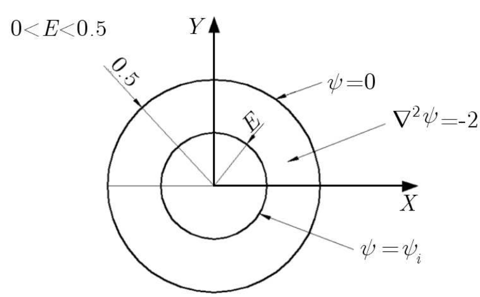

Fig. 4. Formulation of the boundary value problem for the circular with circular centered hole cross section of the bar. Problem I

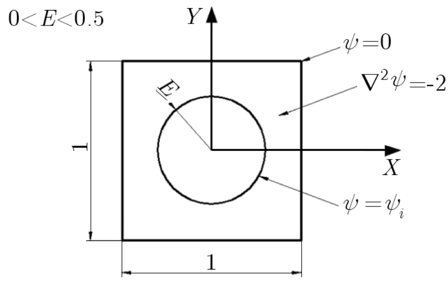

Fig. 5. Formulation of the boundary value problem for the square with circular centered hole cross section of the bar. Problem II

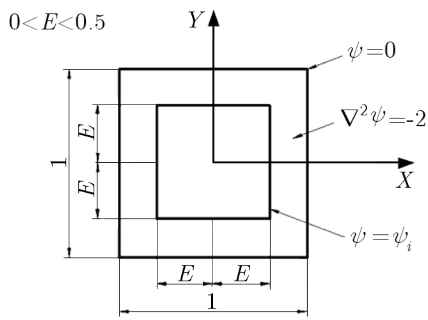

Fig. 6. Formulation of the boundary value problem for the square with square centered hole cross section of the bar. Problem III

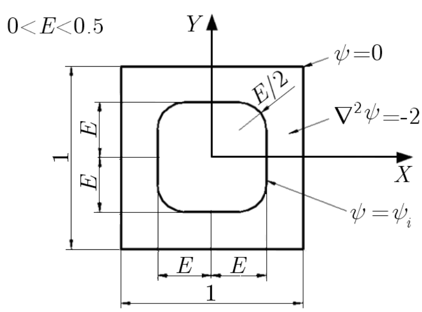

Fig. 7. Formulation of the boundary value problem for the square with square centered hole cross section of the bar with rounded corners with the radius $r = E/2$. Problem IV

In order to validate the proposed numerical method, the maximum relative error on the outer and the inner boundary can be evaluated by

$$\delta_{MAXou} = \frac{\max|\Psi_{outer}|}{\max(i)|\Psi_{mid\ outer_i}|} \qquad \delta_{MAXin_i} = \frac{\max|\Psi_{outer_i} - \Psi_{mid\ inner_i}|}{\max(i)|\Psi_{mid\ inner_i}|} \tag{5.1}$$

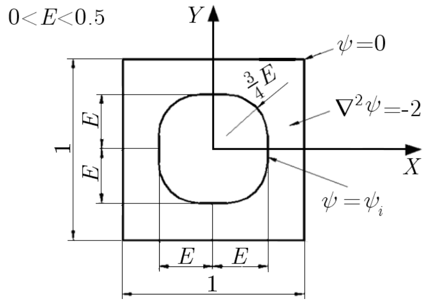

Fig. 8. Formulation of the boundary value problem for the square with square centered hole cross section of the bar with rounded corners with the radius $r = 3E/4$. Problem V

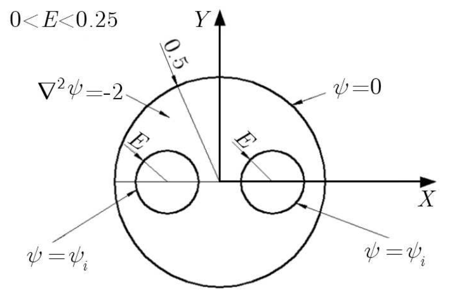

Fig. 9. Formulation of the boundary value problem for the circular with two circular symmetrically placed holes cross section of the bar. Problem VI

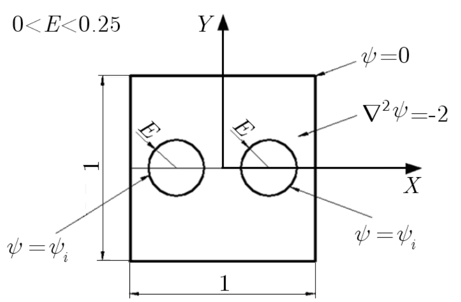

Fig. 10. Formulation of the boundary value problem for the square with two circular symmetrically placed holes section of the bar. Problem VII

where: δ_{MAXou} is the maximum error on the outer contour, δ_{MAXin_i} – maximum error on the inner contour, where $i = 1$ for the problem I-V and $i = 1, 2$ for the problem VI-VII, Ψ_{outer} – value of the stress function on the outer contour, $\Psi_{mid\ inner_i}$ – value of the stress function on the inner contour, where $i = 1$ for the problem I-V and $i = 1, 2$ for the problem VI-VII.

6. Discussion on the numerical results

The MFS applied in this paper to the problem of torsion of a prismatic bar depends on ythe number of parameters. These parameters are as follows: the distance of the outer contour containing the source points from the boundary – Sd, the distance of the inner contour containing the source points from the boundary – Sm (Fig. 3), the number of source points – NC, the number of collocation points – M and thickness of the elements – E. In the first method, for given M and NC equal to 100 for problem I-II and 120 for the case of III-VII, Sd in the range from 0.001 to 2 and Sm were searched through in the range from 0.001 to $E - 0.01$. Out of thousands of errors, the minimum value was determined which was the value of the error method. Determination of the minimum value of bug was becoming very time consuming. In this case, the error is multidimensional, with many local minima (Fig. 11) and numerical methods of searching for the optimal solution fail. For this reason, genetic algorithms to the parameters set out in Table 1 were used to determine the optimal values of M, NC, Sd and Sm for a given thickness of E. After determining the optimal values M, NC, Sd and Sm, the values of the minimum error on the internal and external contour were determined, which are presented in Tables 2-8. Based on the numerical results presented in the Tables, it can be concluded that the smallest values or error were obtained for case I, and the largest for case III (square with square centered hole). The possible explanation of the large error in this case can be the existence of the corners of the inner boundary. In the remaining cases, when the inner contour or contours were smooth (circular), the values of error were much smaller.

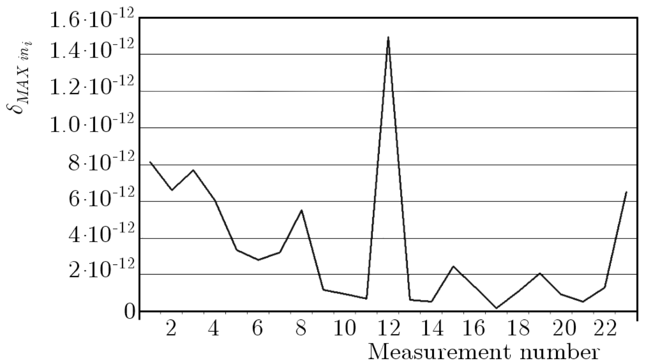

Fig. 11. Searching errors

Table 1. Parameters used for genetic algorithm calculation

Population size	40
Crossover probability	0.9
Probability of mutation	0.1
Parameter of random number generator	100
Number of generations	100
The lower limit for Sd	0.001
The upper limit for Sd	2
The lower limit for Sm	0.001
The upper limit for Sm	0.01
The lower limit for M	5
The upper limit for M	25
The lower limit for NC	5
The upper limit for NC	25

Table 2. Values of maximum local errors – case I

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
26	26	0.05	1.75	6.00E-10	0.01	4.49E-16
46	46	0.1	0.37	1.91E-06	0.01	8.33E-15
46	46	0.15	0.39	1.35E-06	0.01	9.76E-16
46	46	0.2	0.41	1.19E-06	0.01	1.45E-15
46	46	0.25	0.83	6.56E-11	0.02	1.92E-15
46	46	0.3	0.83	7.41E-11	0.07	1.14E-14
46	46	0.35	1.00	2.88E-12	0.09	8.75E-14
46	46	0.4	0.97	1.12E-11	0.01	9.50E-13
46	46	0.45	1.00	1.69E-11	0.01	1.63E-11

A* – Optimal values outer contour Sd

A** – Optimal values inner contour Sm

For the case of circular with circular centered hole, the calculation of the dimensionless torsional stiffness was conducted. The calculation of stiffness based on the approximate solution was completed by dividing the cross section into triangular elements which were integrated with the seven point Gauss quadrature. The obtained results were compared with the exact formula for dimensionless stiffness of the annular region

$$M_s = \frac{\pi}{2}\Big[\Big(\frac{1}{2}\Big)^4 - E^4\Big] \tag{6.1}$$

Table 3. Values of maximum local errors – case II

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
65	65	0.05	0.16	4.45E-02	0.04	2.08E-10
65	65	0.1	0.15	8.51E-03	0.09	1.67E-08
70	70	0.15	0.04	7.74E-02	0.12	3.61E-07
65	65	0.2	0.04	7.94E-02	0.18	4.45E-06
65	65	0.25	0.04	2.28E-01	0.24	2.83E-05
65	65	0.3	0.04	4.30E-01	0.33	1.21E-04
65	65	0.35	0.04	1.81E-01	0.39	2.74E-04
65	65	0.4	0.08	1.46E-01	0.51	1.00E-03
65	65	0.45	0.01	8.90E-01	0.41	3.18E-02

Table 4. Values of maximum local errors – case III

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
164	164	0.05	0.10	1.67E-05	0.01	5.72E-12
164	164	0.1	0.21	1.32E-05	0.01	4.44E-08
164	164	0.15	0.26	1.77E-03	0.07	7.67E-04
164	164	0.2	0.21	2.32E-05	0.01	1.24E-03
164	164	0.25	0.15	4.36E-05	0.08	5.09E-03
164	164	0.3	0.08	6.39E-05	0.08	7.18E-03
164	164	0.35	0.09	1.23E-04	0.07	9.13E-03
164	164	0.4	0.17	4.40E-05	0.01	3.20E-02
164	164	0.45	0.29	1.47E-02	0.01	4.71E-02

Table 5. Values of maximum local errors – case IV

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
96	96	0.05	0.08	7.96E-02	0.01	1.01E-05
96	96	0.1	0.52	1.42E-03	0.05	4.19E-06
96	96	0.19	0.06	3.35E-01	0.11	2.03E-05
96	96	0.2	0.07	1.05E-01	0.04	9.14E-05
96	96	0.25	0.18	2.74E-01	0.11	1.57E-05
96	96	0.4	0.04	3.98E-01	0.06	1.20E-03
224	224	0.45	0.07	5.12E-02	0.03	1.85E-03
96	96	0.49	0.01	7.48E+00	0.41	2.75E-02

A* – Optimal values outer contour Sd
A** – Optimal values inner contour Sm

Table 6. Values of maximum local errors – case V

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
96	96	0.05	0.34	1.09E-03	0.03	1.89E-10
224	224	0.06	0.07	9.72E-03	0.01	1.50E-09
96	96	0.1	0.45	8.71E-05	0.08	3.70E-09
96	96	0.11	0.52	3.63E-04	0.08	1.58E-09
96	96	0.2	0.54	7.97E-03	0.12	1.20E-07
96	96	0.22	0.32	3.72E-02	0.11	1.08E-07
224	224	0.25	0.21	4.87E-04	0.04	1.14E-06
224	224	0.45	0.18	6.32E-01	0.04	7.05E-05

Table 7. Values of maximum local errors – case VI

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
69	69	0.025	0.61	8.80E-06	0.02	1.24E-11
69	69	0.05	0.54	3.79E-05	0.03	2.23E-11
69	69	0.075	0.27	1.33E-04	0.05	6.52E-11
69	69	0.1	1.35	1.58E-04	0.13	7.29E-10
72	72	0.125	1.06	6.01E-05	0.09	8.76E-10
72	72	0.15	1.37	1.41E-04	0.10	9.78E-10
69	69	0.175	1.23	1.61E-03	0.23	2.56E-08
72	72	0.2	0.80	1.57E-03	0.12	4.05E-06
72	72	0.225	1.85	1.40E-02	0.10	4.27E-05

Table 8. Values of maximum local errors – case VII

M	NC	E	A*	δ_{MAXou}	A**	δ_{MAXin_i}
78	78	0.025	0.34	1.09E-02	0.02	3.90E-10
78	78	0.05	0.43	5.34E-03	0.04	2.81E-09
78	78	0.075	0.01	4.41E-01	0.08	2.56E-08
84	84	0.1	0.17	2.87E-02	0.07	1.43E-07
78	78	0.125	0.17	3.80E-02	0.12	6.08E-06
78	78	0.15	0.39	2.59E-01	0.14	3.97E-05
84	84	0.175	0.51	7.68E-03	0.10	3.97E-05
78	78	0.2	0.60	1.97E-02	0.12	1.15E-03
78	78	0.225	0.63	1.06E-01	0.09	4.21E-03

A* – Optimal values outer contour Sd

A** – Optimal values inner contour Sm

Next, the result was compared with the results of the dimensionless stiffness resulting from Brendt's formula for an annular region

$$M_s = \frac{\pi}{4}\Big(\frac{1}{2} + E\Big)^3\Big(\frac{1}{2} - E\Big) \tag{6.2}$$

The results of comparison are presented in Fig. 12.

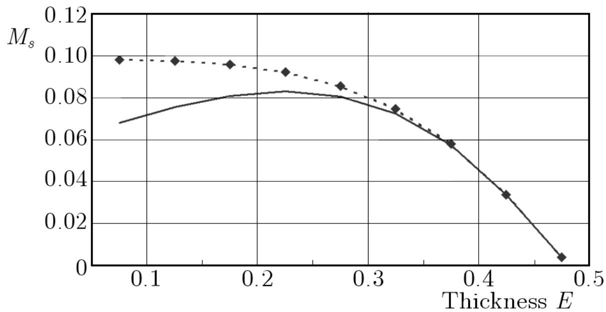

Fig. 12. Comparison of the results given by proposed here formula (4.6) – points, Bredt's formula (6.2) – solid line and exact solution (6.1) – broken line for the case of circular with circular centered hole; $M = 46$, $NC = 46$

7. Conclusion

The method of fundamental solutions was successfully applied to solve the boundary problem of torsion of bars with multiply connected cross sections. The paper considered seven cases with multiply connected cross sections, where the thickness E varied between $0 < E < 0.5$ for problems I-V and $0 < E < 0.25$ for problems VI and VII. With the application of this method, the maximum error for problems I, II, and IV-VII is smaller on the inner boundary than on the outer boundary. For problem III, the errors on the inner and outer boundary are comparable. The values of the maximum errors are really small.

Furthermore, on the basis of the numerical results for the circle with circular hole cross section of the bar, it was concluded that well known Brendt's method of calculating stiffness of thin-walled bars can be successfully applied to bars with the dimensionless thickness larger than $E = 0.4$. In all the cases studied, it was observed that the boundary condition is satisfied. Stress function values along the X axis converge to zero, reaching zero at the edge of the outer section.

In the first method for given M and NC, Sd and Sm were searched through. Out of thousands of errors, the minimum value, which was the value of the method error, was determined. Determination of the minimum value of bug was becoming very time consuming. For this reason, genetic algorithms were used to determine the optimal coefficients M, NC, Sd and Sm of the method of fundamental solutions in order to determine values of the error method. Application of genetic algorithms increased the speed of finding the minimum error in comparison with the method of searching.

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Zastosowanie metody rozwiązań podstawowych oraz algorytmów genetycznych do zagadnienia skręcania prętów o przekroju wielospójnym

Streszczenie

W artykule rozważano skręcanie pretów z wielospójnym przekrojem poprzecznym za pomocą metody rozwiązań podstawowych (MRP). Do wyznaczenia optymalnych parametrów MRP wykorzystano algorytmy genetyczne. W pracy rozważano siedem problemów testowych. Bezwymiarowe sztywności skręcania liczono za pomocą numerycznego całkowania funkcji naprężeń dla jednego z przypadków. Te sztywności porównywano ze ścisłą sztywnością dla pierwszego przypadku i ze sztywnością uzyskaną ze wzoru Bredta dla cieńkich przekrojów poprzecznych.

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