JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
49, 2, pp. 369-384, Warsaw 2011

"PIES" IN PROBLEMS OF 2D ELASTICITY WITH BODY FORCES ON POLYGONAL DOMAINS

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The paper presents a thorough review of the effective approach to solving problems of plane elasticity with body forces of different types. The proposed method bases on generalization of the parametric integral equation system (PIES), which was successfully applied to solving boundary problems without body forces. The main aim of the mentioned generalization was to create such an approach which does not require physical discretization of the domain, or division it into cells, like it is done in the classic boundary element method (BEM). First, only problems defined on polygons were considered. The paper also contains the analysis of the accuracy of obtained solutions in comparison with analytical or other numerical results.

Key words: parametric integral equation system (PIES), elasticity, body forces, Bézier surfaces

1. Introduction

The main problem connected with solving boundary problems which model practical problems of e.g. mechanics, acoustics or building engineering is their computational complexity. Despite fast development of the hardware, numerical simulation of such problems requires significant resources, and also, despite some automation, the workload for input data preparation. Especially, it is related with a mesh which defines the considered domain and is necessary for the modelling of a problem solved by the finite element method (FEM) (Zienkiewicz, 1977). For that reason, a very important and substantial task is to look for new methods that would improve the effectiveness of modelling of practical boundary problems and would optimize the process of their numerical solution.

Such an improvement was achieved taking into account BEM (Banerjee and Butterfield, 1981; Brebbia *et al.*, 1984; Burczyński, 1995). That method is characterised by the lack of necessity of domain discretization, only the boundary is divided into elements. It reduces the size of solved tasks in relation to the most common and popular method that is FEM. However, taking into consideration problems modelled by the Poisson equation or problems of elasticity with body forces, that advantage is no longer valid. Such problems require calculation of the domain integral, which is connected with the division of the domain into so-called cells (Brebbia *et al.*, 1984).

For that reason, it is important to look for a method which would completely eliminate the traditional boundary and domain discretization. In research carried out by authors for solving boundary problems, PIES was obtained. It is a modified version of the classic boundary integral equation (BIE) (Zieniuk, 2001; Zieniuk and Bołtuć, 2006b). That modification consists in the fact that boundary geometry defined by proper curves was included directly into the mathematical formalism of PIES. The applied strategy results in an effective way of boundary geometry modelling by any curves known from computer graphics.

Until now, the method was successfully applied to solving 2D problems modelled by the Laplace (Zieniuk, 2001; Zieniuk *et al.*, 2004), Helmholtz (Zieniuk and Bołtuć, 2006a) or Navier-Lame (without body forces) (Zieniuk and Bołtuć, 2006b, 2008) equations. Very encouraging results were obtained, and the method proved to be effective in respect of the practical modelling of the domain and including boundary conditions, but also in respect of the accuracy of obtained numerical solutions.

The main aim of this paper is to obtain and analyse the effectiveness of the general strategy used in PIES for problems modelled by the Navier-Lame equation with body forces. In the paper, only polygonal domains were considered. They are modelled and modified using corner points. The major novelty in the paper is the fact that regardless of the need of integration over a domain, it is not divided into cells, like it is done in the case of the classic BEM. Effective and accurate calculation of the integral over the global domain is one of the main aims of this paper.

The paper also presents results of tests over the stability and accuracy of the obtained solutions taking into account different types of body forces, various shapes of domains, and also other parameters which could have influence on the results accuracy. The obtained solutions were compared with analytical and numerical results generated by other computer methods.

2. PIES for elasticity problems with body forces

As was mentioned, PIES was obtained in (Zieniuk, 2001) as a result of analytical modification of the traditional BIE for the two-dimensional Laplace equation. PIES is not defined on the boundary as the traditional BIE, but on the straight line in the global parametric reference system. The length of that line for a polygonal domain is equal to its perimeter.

PIES for the Navier-Lame equation without body forces was obtained in (Zieniuk and Bołtuć, 2006b) and it takes the following form

$$\frac{1}{2}\boldsymbol{u}_p(s_1) = \sum_{r=1}^{n} \int\limits_{s_{r-1}}^{s_r} [\overline{\boldsymbol{U}}^*_{pr}(s_1,s)\boldsymbol{p}_r(s) - \overline{\boldsymbol{P}}^*_{pr}(s_1,s)\boldsymbol{u}_r(s)]J_r(s)\,ds \tag{2.1}$$

where

$$J_r(s) = \sqrt{\Big(\frac{\partial \varGamma_r^{(1)}(s)}{\partial s}\Big)^2 + \Big(\frac{\partial \varGamma_r^{(2)}(s)}{\partial s}\Big)^2} \qquad \begin{array}{l} p = 1,2,\dots,n \\ s_{p-1} \leqslant s_1 \leqslant s_p \\ s_{r-1} \leqslant s \leqslant s_r \end{array}$$

and n – is the number of segments (the number of the polygon sides).

The first integrand function $\overline{\boldsymbol{U}}^*_{pr}(s_1,s)$ is called the fundamental boundary solution (it is the modified fundamental solution), whilst the second is the singular boundary solution. The solutions in an explicit form were presented in Zieniuk and Bołtuć (2006b) and they include the boundary geometry defined by means of parametric curves $\boldsymbol{\varGamma}(s) = \{\varGamma_r^{(1)}(s), \varGamma_r^{(2)}(s)\}^\top$ in their mathematical formalism.

The study was undertaken to apply PIES to solving problems modelled by the Navier-Lame equation, but those for which body forces were not omitted. Using the strategy of BIE modification outlined in Zieniuk and Bołtuć (2006b), for the Navier-Lame equation with body forces, the following generalized form of PIES was obtained

$$\begin{aligned} \frac{1}{2}\boldsymbol{u}_p(s_1) &= \sum_{r=1}^{n} \int\limits_{s_{r-1}}^{s_r} [\overline{\boldsymbol{U}}^*_{pr}(s_1,s)\boldsymbol{p}_r(s) - \overline{\boldsymbol{P}}^*_{pr}(s_1,s)\boldsymbol{U}_r(s)]J_r(s)\,ds + \\ &+ \int\limits_{\varOmega} \overline{\overline{\boldsymbol{U}}}^*_p(s_1,\boldsymbol{x})\boldsymbol{b}(\boldsymbol{x})J(\boldsymbol{x})\,d\varOmega(\boldsymbol{x}) \end{aligned} \tag{2.2}$$

where $\boldsymbol{x} = \{v,w\}$, $\boldsymbol{b}(\boldsymbol{x})$ is the vector of body forces and

$$J(\boldsymbol{x}) = \frac{\partial B^{(1)}(\boldsymbol{x})}{\partial w}\frac{\partial B^{(2)}(\boldsymbol{x})}{\partial \nu} - \frac{\partial B^{(1)}(\boldsymbol{x})}{\partial \nu}\frac{\partial B^{(2)}(\boldsymbol{x})}{\partial w}$$

$\boldsymbol{B}(\boldsymbol{x}) = \{B^{(1)}(\boldsymbol{x}), B^{(2)}(\boldsymbol{x})\}^\top$ are parametric surfaces used for the domain modelling. Due to the fact that only 2D problems are considered, in the surfaces definition one can omit the third dimension.

The integrands $\overline{\boldsymbol{U}}^*_{pr}(s_1, s)$ and $\overline{\boldsymbol{P}}^*_{pr}(s_1, s)$ from (2.2) take the same form as in (2.1), whilst the function $\overline{\overline{\boldsymbol{U}}}^*_p(s_1, \boldsymbol{x})$ $(p = 1, 2, \ldots, n)$ from the second integral (over the domain) is given by

$$\overline{\overline{\boldsymbol{U}}}^*_p(s_1, \boldsymbol{x}) = -\frac{1}{8\pi(1-\nu)\mu} \begin{bmatrix} (3-4\nu)\ln(\overline{\boldsymbol{\eta}}) - \dfrac{\overline{\eta}_1^2}{\overline{\boldsymbol{\eta}}^2} & -\dfrac{\overline{\eta}_1\overline{\eta}_2}{\overline{\boldsymbol{\eta}}^2} \\ -\dfrac{\overline{\eta}_1\overline{\eta}_2}{\overline{\boldsymbol{\eta}}^2} & (3-4\nu)\ln(\overline{\boldsymbol{\eta}}) - \dfrac{\overline{\eta}_2^2}{\overline{\boldsymbol{\eta}}^2} \end{bmatrix} \tag{2.3}$$

where

$$\overline{\boldsymbol{\eta}} = \sqrt{\overline{\eta}_1^2 + \overline{\eta}_2^2} \qquad \overline{\eta}_1 = B^{(1)}(\boldsymbol{x}) - \Gamma_p^{(1)}(s_1) \qquad \overline{\eta}_2 = B^{(2)}(\boldsymbol{x}) - \Gamma_p^{(2)}(s_1)$$

and $\boldsymbol{\Gamma}(s_1) = \{\Gamma_p^{(1)}(s_1), \Gamma_p^{(2)}(s_1)\}^\top$ are parametric curves known from computer graphics, whilst $\boldsymbol{B}(\boldsymbol{x})$ are surfaces which are used for the domain modelling.

As can be seen in (2.2) next to the interval integrals defined on the straight line in the parametric reference system additionally, there is the integral over domain Ω.

PIES presented by (2.1) was widely studied in (Zieniuk and Bołtuć, 2006b) taking into account the possibility of effective modelling and modification of the boundary geometry using corner points. Segments of the boundary between corner points were defined by Bézier curves of the first degree. With equation (2.2) an additional problem is connected, concerned with the modelling and integration over the domain.

3. Modelling of a boundary and domain in PIES

The modelling of the shape of the domain in boundary problems which are numerically solved usually bases on discretization. In the case of the most popular method FEM, it is the discretization domain, whilst in BEM, only the boundary. In own researches, the authors looked for a strategy which would totally eliminate the necessity of the division of the boundary or domain into elements. PIES developed by Zieniuk (2001), Zieniuk and Bołtuć (2006a,b), includes the boundary geometry in its mathematical formalism. For the modelling of the shape, any parametric curve used successfully in computer graphics

(Farin, 1990; Foley *et al.*, 2001) can be applied. The modelling of an exemplary polygon in PIES in comparison with the classic element methods BEM and FEM is presented in Fig. 1.

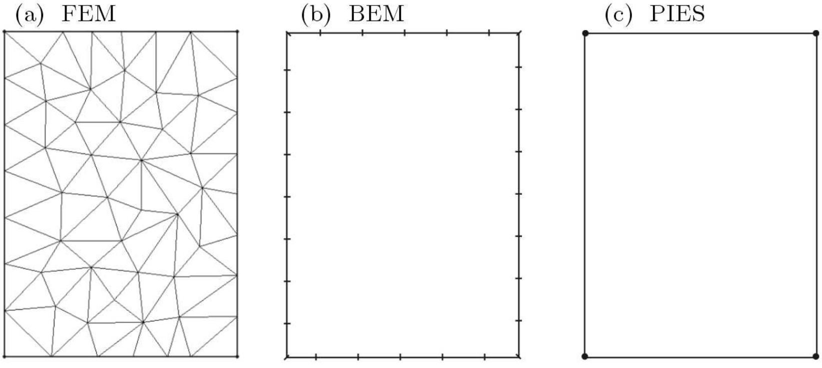

Fig. 1. Modelling of polygon domains in: (a) FEM – 84 triangular finite elements, 55 nodes, (b) BEM – 28 linear boundary elements, 28 nodes, (c) PIES – 4 Bézier curves of the first degree, 4 corner points

As shown in Fig. 1c, for modelling of the boundary geometry in PIES, only corner points of the polygon are required. The concept has proved to be very effective, because the modelling process becomes easier, and the number of data comparing with FEM and BEM becomes smaller. The studies by Zieniuk (2001), Zieniuk and Bołtuć (2006a,b, 2008) showed that such a modelling does not affect negatively the accuracy of solutions obtained by PIES.

However, it was decided to examine problems which in the case of BEM eliminated its major advantage – the lack of discretization. These are problems described by the Poisson equation or the Navier-Lame equation with body forces. The problem that arises in such issues is the need for domain integrals calculation. BEM requires the division of the domain into smaller sub-areas called cells, calculation of local integrals over these areas and, finally, summing all the values in order to obtain the global value of the integral. It is a difficult and time-consuming process, and from the practical point of view, analogous to the definition of finite elements in FEM. Thus, there is another challenge – to develop such a way of the domain modelling in which the discretization is completely eliminated, even for problems that require integration over the area.

In the case of two-dimensional problems, the domain can be defined using flat surfaces (Farin, 1990; Foley *et al.*, 2001) very popular in computer graphics. In relation to considered tasks (defined on polygonal domains), the rectangular Bézier surfaces are proposed. An example of the modelling of the polygonal domain using the rectangular Bézier surface of the first degree is presented in Fig. 2. For comparison, the authors also present the same problem defined in BEM.

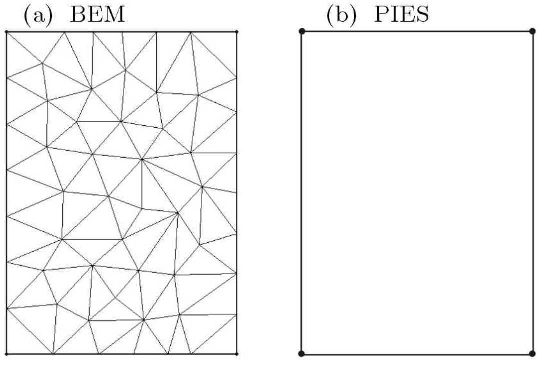

Fig. 2. Modelling of polygon domains in: (a) BEM – division of the domain into cells, (b) PIES – 1 rectangular Bézier surface of the first degree, 4 corner points

There are two main advantages of the presented in Fig. 2b way of the modelling of the area for integration in PIES. The first of them concerns the simplicity and effectiveness of the approach. It turns out that when considering polygonal problems of elasticity with body forces, only corner points were posed, the same way as in the case of issues without body forces (Fig. 1c). In the case of BEM, the area should be divided into cells, as shown in Fig. 2a. The second advantage concerns the modification of the defined domain by means of corner points. It is done very efficiently – by moving the position of a selected corner point (or selected points), without the necessity of performing any additional steps (as was in the case of re-discretization in BEM). Modification of the rectangular domain (Fig. 3a) using one (Fig. 3b,c) and three (Fig. 3d) corner points is shown in Fig. 3.

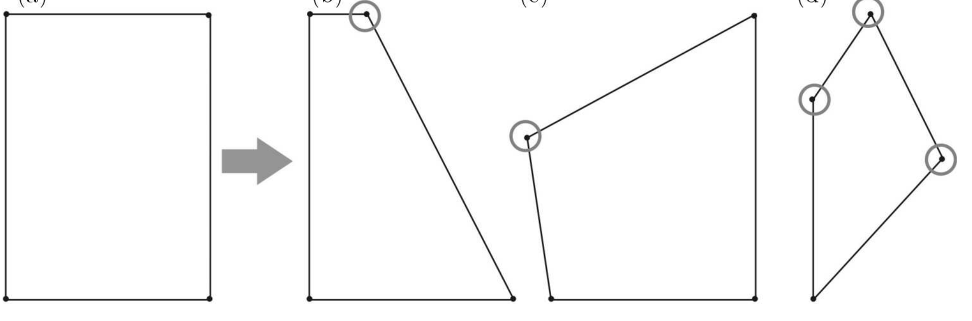

Fig. 3. Modification of the domain shape in PIES

In PIES (2.2), the boundary geometry is directly defined by means of Bézier curves $\boldsymbol{\Gamma}(s_1)$, whilst the domain Ω by the surface $\boldsymbol{B}(\boldsymbol{x})$ included into integrand (2.3). Therefore, an automatic adaptation of the mathematical formulas describing Bézier surface to new values of coordinates of corner points is performed. It is the process infinitely more efficient compared to the re-division of the modified area into cells in BEM. This is particularly important in problems of the boundary shape identification, where the problem of analysis is solved many times.

4. Integration over a domain and numerical solution of PIES

The next step in solving boundary value problems using the presented approach, after defining the shape of the boundary and posing boundary conditions, is numerical solution of PIES. It reduces to finding unknown functions on the boundary which fulfill the boundary problem. Such issues without body forces were considered in Zieniuk and Bołtuć (2006b, 2008).

For problems with body forces considered in this work is also required integration over the domain. In the proposed approach, the domain is defined using the flat Bézier surface. Such an approach gives the possibility of global modelling of the whole domain without the necessity of dividing it into sub-areas (of course if one surface will be enough for accurate projection of the desired shape). Integration over large areas, however, requires a different strategy than in BEM.

In the case of BEM, this procedure consists in dividing the area of integration into smaller sub-areas where the low order quadrature was applied to the integration, and then the results from individual cells were summed. The sum is the final value of the integral for the whole area. In this paper, such a strategy was replaced by another one, consisting in the use of the Gauss-Legendre quadrature (Stroud and Secrest, 1966) of high order for global integration over the whole domain defined by the surface and without the cell division.

Such a strategy after testing the examples presented below, has proved to be right and produces satisfactory results. The only issues that needed to be clarified are: the number of coefficients in the applied quadrature and the effectiveness of the proposed approach in cases of modified (in relation to the base rectangular) shapes. The second question comes from the fact that the Gauss-Legendre quadrature is designed for rectangular areas. Both issues are discussed and reviewed in detail in the next Section.

5. Analysis of the stability and accuracy of solutions

5.1. Presentation of the idea on the example of a body under the gravitational force

The first example deals with the problem of an elastic soil body due to its own weight (Fig. 4a). This problem has been reduced to the two-dimensional task, for which the shape, size and boundary conditions are presented in Fig. 4b. The considered square area was modelled taking into account one rectangular Bézier surface of the first degree defined using only four corner points.

The gravitational force acts in the y direction and is equal to $b_y = \rho g = 1.0$. The material parameters are taken to be $E = 1.0$, $\nu = 0.3$.

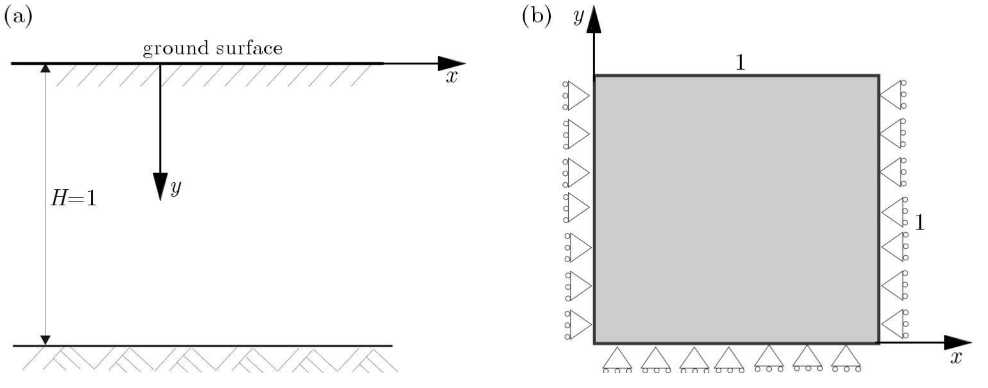

Fig. 4. Considered problem: (a) idea, (b) shape, size, boundary conditions

Table 1. Solutions at selected boundary points obtained by BEM and PIES

Points	Exact	BEM				PIES
		$4^{(a)}$	$4+3^{(b)}$	$8^{(a)}$	$8+3^{(b)}$	
v, $y=0$	0.371	0.409	0.382	0.405	0.383	0.371
sy, $y=1$	1.000	1.105	1.028	1.094	1.029	1.000
sx, $y=1$	0.429	0.473	0.441	0.461	0.443	0.429

$^{(a)}$ number of quadratic boundary elements,
$^{(b)}$ number of quadratic boundary elements + number of interior points

Analytical solutions and numerical results obtained using BEM at three boundary points were taken from Park (2002). These solutions were obtained taking into account various options regarding the number of elements and internal points. Comparison of these results with the results obtained using the discussed idea and PIES is presented in Table 1.

Analysing solutions presented in Table 1, it can be concluded that results obtained by the proposed approach for the considered example are very accurate compared to analytical results. The question arises if in other examples the solutions will be satisfactory and their accuracy will depend on the global modelling of the domain and global integration over the domain.

5.2. Global modelling vs. the accuracy of solutions

Another considered issue is related to the body force (centrifugal) arising as a result of body rotation (Fig. 5) around the x-axis. Material parameters and values of the angular velocity and density are: $E = 10\,\mathrm{MPa}$, $\nu = 0.3$, $\omega = 10\,\mathrm{s}^{-1}$, $\rho = 10\,\mathrm{kg/m}^3$.

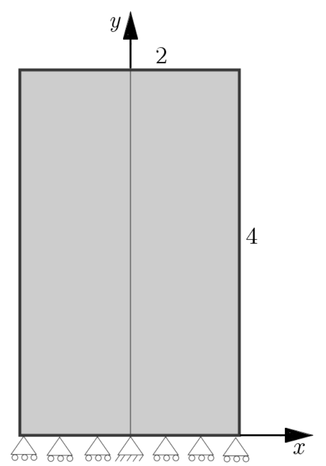

Fig. 5. Rectangle plate

Analytical solutions (Neves and Brebbia, 1991; Yan *et al.*, 2008) are presented by the following expressions

$$\nu = \frac{\rho\omega^2}{2E} y\Big(16 - \frac{y^2}{3}\Big) \qquad\qquad \sigma_y = 8\rho\omega^2 - \frac{y^2}{2}\rho\omega^2 \tag{5.1}$$

In the paper, the technique of global modelling and integration was applied, which makes the discretization of the area not required. Therefore, for an unambiguous and accurate definition of the domain considered in the example, only one rectangular Bézier surface of the first degree was used. Solutions obtained by PIES in the cross-section $x = 0$ were compared with analytical ones and are presented in Fig. 6.

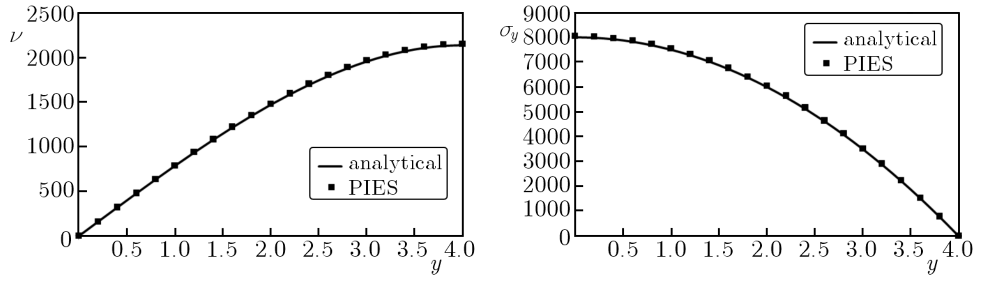

Fig. 6. Results obtained using PIES compared to analytical solutions

For the purposes of research, the authors made a decision to perform the "simulation" of discretization and to check whether the number of used surfaces for modelling has influence on the accuracy of obtained solutions. For that reason, the division into sub-areas was made (two and three), and each of them was modelled using the Bézier surface. It is a process which imitates the discretization of the area using so-called cells in BEM. Average relative errors in the considered cross-section taking into account different variants of the modelling are presented in Table 2.

Table 2. Average relative errors for displacements and stresses in the y direction in the cross-section $x = 0$

	1 surface	2 surfaces	3 surfaces
σ_y	0.79631	0.800997	0.782097
ν	0.704463	0.70418	0.704368

As can be seen in Table 2, the number of surfaces used for the modelling of the domain does not influence the accuracy of solutions. It can be concluded that the introduction of more than one surface is meaningful only when required by the complexity of the shape of the projected area. Improving the accuracy of results is done by one parameter which is the number of coefficients in the quadrature applied to integration over the domain.

5.3. Influence of the number of coefficients in the Gauss-Legendre quadrature on the accuracy of results

The problem of the square plate with the length of the edge $L = 100\,\mathrm{mm}$ rotating about the x-axis, as shown in Fig. 7, is considered. The density distribution is given by

$$\rho(y) = \rho_0\Big[1 + \Big(\frac{y}{L}\Big)^2\Big] \tag{5.2}$$

where $\rho_0 = 10^{-6}\,\mathrm{kg/mm^3}$, whilst $\omega = 100\,\mathrm{rad/s}$, $E = 210\,\mathrm{GPa}$ and $\nu = 0.3$.

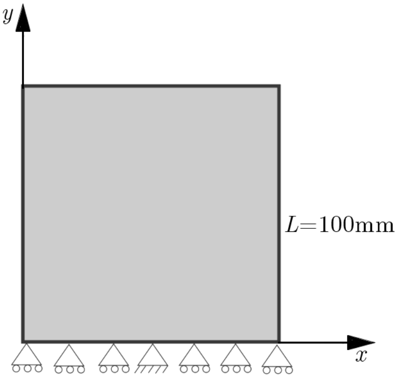

Fig. 7. Square plate rotating about the x-axis

The analytical solution (Ochiai, 2009) is known and is given by

$$\sigma_y = \frac{\rho_0 \omega^2 (L-y)}{4L^2} [8L^3 - 8L^2(L-y) + 4L(L-y)^2 - (L-y)^3] \tag{5.3}$$

Solutions in the cross-section $x = 50$ mm were generated taking into account different numbers of nodes in the Gauss-Legendre quadrature. The obtained results were compared with exact solutions, and effects of that comparison in the form of average relative errors were presented in Fig. 8.

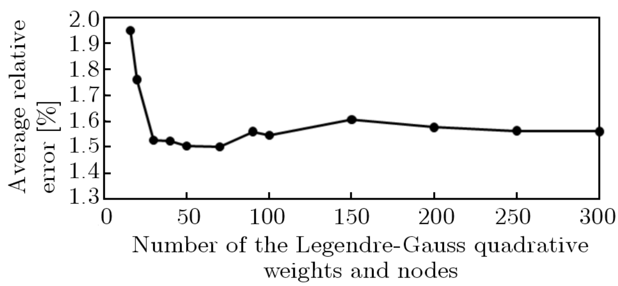

Fig. 8. Average relative errors obtained using PIES depending on the number of nodes in the Gauss-Legendre quadrature

Analysing the results presented in Fig. 8, it can be stated that most accurate solutions were obtained using the quadrature for integration with the number of nodes exceeding 30. Then the average relative error of solutions is more or less equal to 1.5%. It should be emphasised that taking into account even the smaller number of nodes, the solutions were characterised by a satisfactory accuracy (the average relative error of the solutions is smaller than 2%).

The accuracy of obtained solutions in comparison with other numerical methods was also examined. The work by Ochiai (2009) derived the solution obtained through the triple-reciprocity boundary element method with two different variants of the number of internal points. These solutions were compared with the results obtained using PIES and were presented in Table 3.

Table 3. Stress σ_y distribution in the cross-section $x = 50\,\text{mm}$

y	Exact	BEM		PIES
		(80/81)*	(80/64)*	
10	74.498	75.510	75.529	76.004
20	72.960	73.491	73.509	74.470
30	70.298	70.780	70.796	71.784
40	66.360	66.825	66.841	67.756
50	60.938	61.336	61.351	62.137
60	53.760	54.002	54.017	54.647
70	44.498	44.473	44.486	45.000
80	32.760	32.367	32.379	32.907
90	18.098	17.319	17.331	18.002
Average relative error [%]		1.126	1.128	1.559

(*) number of constant boundary elements/number of internal points

As shown in Table 3, the solutions obtained using the proposed approach are comparable in accuracy to these obtained by BEM. It should be noted that in the case of BEM, the integrals over the domain were transformed into integrals along the boundary. The same procedure can be performed also in PIES, but the purpose of this study was to present the approach which in a universal, effective and accurate way solves issues with the necessity of integration over the area. There is only one question if that algorithm can be applied also in the case of the modified shape of the considered area and the coefficients of the quadrature intended for the rectangular area will be proper in such a situation.

5.4. Usability of the method considering modified shapes

The last example concerns a dam subjected to the hydrostatic pressure on its upstream side (Yan *et al.*, 2008) (Fig. 9). Gravity is also considered. The parameters necessary to solve the problem are: $E = 100M\,\text{Pa}$, $\nu = 0.3$, density of water $\rho_w = 1000\,\text{kg/m}^3$, density of the dam material $\rho_m = 2400\,\text{kg/m}^3$.

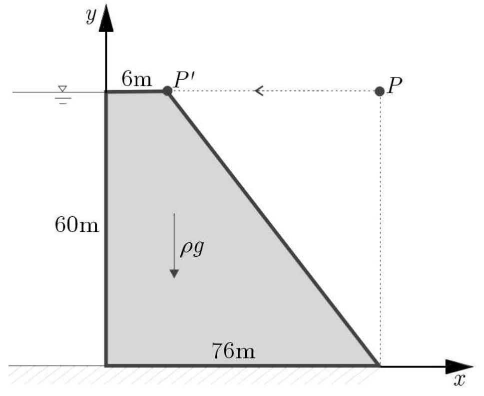

Fig. 9. Dam subjected to water pressure and gravity

The cross-section of the dam was defined by one rectangular Bézier surface of the first degree. It was done by moving the position of one corner point P (as shown in Fig. 9) of the rectangular surface, which is very efficient in terms of comparison to the discretization with cells in the BEM. The question arises if in such a case one can use the same quadrature for integration, which is designed for rectangular areas.

There is no analytical solution to the problem defined, therefore, solutions obtained using PIES were compared with solutions obtained using the other numerical method – BEM. In order to do that, software BEASY was used. Results of the comparison are presented in Fig. 10.

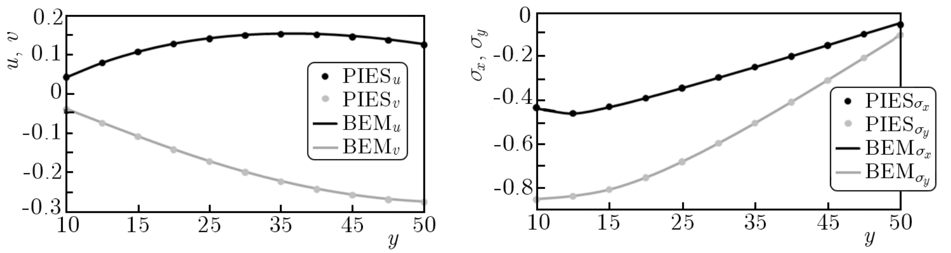

Fig. 10. Normal stresses and displacements in the cross-section $x = 5\,\text{m}$ obtained by PIES compared with BEM

Results presented in Fig. 10 confirm the reliability of obtained solutions and the efficiency of the applied method because they are very close to the solutions obtained by BEASY which bases on BEM. It should be mentioned that in the case of BEM, the integral over the area was replaced by an integral along the boundary, and it leads to better quality of solutions (than those that would have been obtained using integration over the area).

6. Conclusions

The paper presents the application of Bézier surfaces to the modelling of 2D domains on which boundary problems were defined. The application of surfaces and quadratures with a large number of coefficients gives the possibility of elimination of cells required for numerical integration over the domain (as in the classic BEM). It can be treated that the integration is made globally over the whole domain. The modelling of the domain is very effective, because is analogical as in issues without the necessity of calculation of the integral over the domain. In practice, one poses only corner points. The very important advantage is also the possibility of effective modelling and modification of the domain by changing positions of these points.

The analysis of the accuracy and stability of solutions obtained by PIES using the mentioned global modelling and integration over the domain was performed. The obtained results confirmed the efficiency of the proposed approach, they were characterised by high accuracy in comparison with analytical solutions and numerical results obtained using other popular methods. There was also positive verification of the approach on problems with modified shape.

The presented results are encouraging enough to test the method on problems with the two-dimensional curved edge and more complex shapes. In these cases, it may be necessary to apply triangular patches to model the domains. Further studies could be connected with other problems for which there is the necessity of integration over the domain (e.g. non-linear problems of mechanics or other). An interesting task is generalization of the strategy applied to 3D issues.

Acknowledgements

The scientific work is partially funded by resources for science in the years 2010-2013 as a research project.

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PURC w dwuwymiarowych problemach teorii sprężystości z siłami masowymi na wielokątnych obszarach

Streszczenie

W pracy zaprezentowano i gruntownie zweryfikowano efektywny sposób rozwiązywania zagadnień z zakresu płaskiej teorii sprężystości z siłami masowymi różnego typu. Zaproponowany sposób polega na uogólnieniu parametrycznego układu równań całkowych (PURC), wcześniej z sukcesem stosowanego do rozwiązywania zagadnień brzegowych bez sił masowych. Celem uogólnienia było zastosowane takiego podejścia, które charakteryzowałoby się brakiem konieczności fizycznej dyskretyzacji obszaru czy dzielenia go na komórki, jak jest to stosowane w klasycznej metodzie elementów brzegowych (MEB). W pracy w pierwszej kolejności ograniczono się do zagadnień zdefiniowanych na obszarach wielokątnych. W pracy dokonano analizy dokładności otrzymywanych rozwiązań w porównaniu do wyników analitycznych oraz numerycznych.

Manuscript received June 17, 2010; accepted for print October 25, 2010