JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
49, 2, pp. 355-367, Warsaw 2011

SPLINE DESCRIPTION OF NON-TYPICAL GEARS FOR BELT TRANSMISSIONS

Piotr Krawiec
Poznan University of Technology, Chair of Basics of Machine Design, Poznań, Poland
e-mail: piotr.krawiec@put.poznan.pl

Adam Marlewski
Poznan University of Technology, Institute of Mathematics, Poznań, Poland
e-mail: adam.marlewski@put.poznan.pl

The usage of noncircular gears makes possible to get better characteristics of the transmission. This problem is widely studied in the case of regular noncircularity, e.g., when a gear is of an elliptical or cycloidal shape. In this study, there are taken into account non-typical irregular gears. The design of such gears involves more advanced techniques, in particular there has to be applied a numerical treatment to get a mathematical description of the gear profile. In the paper, we discuss a parametric spline interpolation of the third degree and we give an example of such approximation. We also present a prototype drive making that the belt transmission has changeable kinematical features.

Key words: spline, noncircular wheel, belt transmission, computer aided design

1. Introduction

The looking for exact and cheap manufacturing methods becames one of tasks whose solution would enable more common usage of uneven strand transmissions in controlling and driving mechanisms. These transmissions enable obtaining changeable kinematical features such as the gear ratio and velocity. Modern drives of CNC (computer numerical control) machine tools allow to control the movement of the tool on the path which is described mathematically by, e.g., a Bézier curve or a spline. The application of these curves to description of turned curves allows one to shorten the preparation time and simplify the software for the control of forming motion of non-typical belt pulleys and gears.

The design of noncircular gears to work in teeth transmissions is widely described in literature, see, e.g. Bair (2002, 2009), Bair *et al.* (2007), Chen and Tsay (2004), Danieli and Mundo (2005), Hasse (2000), Li *et al.* (2007), Tsay and Fong (2005). In most cases, there are considered regular noncircular wheels, i.e., the gears of the elliptical and cycloidal profile (and only such a plane curve is discussed by Litvin and Fuentes (2004), a bible for gear designers). In the last years, practical experiments and theoretical considerations on noncircular wheels in belt/chain drivers were carried out. Till now, the literature dedicated to this relatively new concept of toothing is rather modest, e.g. Krawiec (2005), Li and Liu (2002).

The paper concerns the description of irregular noncircular wheels by splines. The use of splines in describing complicated contours is a widely applied technique, mostly in the above mentioned cases, i.e., when regular noncircular wheels are considered (to get the idea see, e.g., Korzeniowski (2006)). In this paper, we deal with irregular wheels (and let us say that we did not find any paper treating such irregular profiles via splines). First, we give a concise introduction to spline curves, in particular to parametric cubic splines. Next, we present an example of the spline interpolation determined by knots laying on a prototype noncircular wheel. At last, we report the industrial application of this wheel in an uneven strand transmission with cog-belt. In the conlusions, we underline the usefulness of the considered approach in an engineer practice and in design of non-typical belt drives.

2. Spline interpolation

In brief, a spline, or spline function, is a smooth piecewise polynomial function of the real variable. This graphically means that a spline is composed of fragments of polynomials and adjacent polynomials are joint smoothly, i.e., their derivatives are continuous, including the derivative of the order 0. To give the formal definition of a real spline, we start with given $n+1$ distinct real numbers t_k, $k = 1, 2, \ldots, n+1$, increasingly ordered, $t_j < t_{j+1}$; in the considered context t_k is called k-th knot. The knots determine n disjoint intervals $I_j := \langle t_j, t_{j+1} \rangle$, $j \in \{1, 2, \ldots, n\}$. Let d be a natural number. We say that g is a spline (in the variable t, and of degree d) determined by these $(n+1)$ knots, if g is a polynomial g_j of the degree d in every interval I_j and

$$g_k^{(r)}(t_{k+1}) = g_{k+1}^{(r)}(t_{k+1}) \tag{2.1}$$

at every $t = t_k$, $k = 1, 2, \ldots, n$, and for $r = 0, 1, \ldots, d$.

If, moreover, there are $n+1$ real numbers y_k given, $k \in \{1, 2, \ldots, n+1\}$, and

$$g_k(t_k) = y_k \tag{2.2}$$

for all k then we have a collocative spline defined by the given $n+1$ points $P_k = (t_k, y_k)$. Obviously, for every $k = 1, 2, \ldots, n$ the graph of k-th piece, g_k, of the spline g passes through points P_k and P_{k+1}. More frequently we say that g is an interpolatory spline because, in general, it is used to interpolate between neighbouring points P_k; in this context the point P_k is called a k-th knot of collocation, knot of interpolation.

The above definition was proposed by Isaac Jacob Schoenberg. In his first paper dealing with splines, Schoenberg (1946) says that the concept of the spline was known to Pierre Simon Laplace. Schumaker (1980) in his book, which is recognised as a bible for splinists, says that probably the first systematic considerations on splines were undertaken by Popoviciu (1934). Since Schoenberg and Schumaker, there has been still noticed interest in splines. A nice survey was given by Demongeot (2007). Most frequently used splines are Ferguson/natural splines, Bézier curves, B-splines and NURBS (non-uniform rational B-splines). There are very widely applied interpolatory cubic splines, i.e. those of 3 degree, and their particular form called "cubic natural splines". They were originally introduced by Ferguson (1964) and they have to fulfill some additional conditions which are perceived as natural ones; we will tell about them later on.

An interpolatory cubic spline, g, defined by points P_k, where $k = 1, 2, \ldots, n$, is such that

$$g(t) = g_j(t) \tag{2.3}$$

for $t \in I_j$, where $j \in \{1, 2, \ldots, n\}$ and g_j is a polynomial of the third degree. In the Stevin (a.k.a. standard, natural) base, it is

$$g_j(t) = a_j + b_j t + c_j t^2 + d_j t^3 \tag{2.4}$$

and a_j, b_j, c_j, d_j are coefficients to be determined from the following system of equations

$$\begin{aligned}
&g_k(t_k) = y_k && k = 1, 2, \ldots, n \\
&g_k(t_{k+1}) = y_{k+1} && k = 1, 2, \ldots, n \\
&g_k(t_k) = g_{k+1}(t_k) && k = 1, 2, \ldots, n-1 \\
&g'_k(t_k) = g'_{k+1}(t_k) && k = 1, 2, \ldots, n-1 \\
&g''_k(t_k) = g''_{k+1}(t_k) && k = 1, 2, \ldots, n-1
\end{aligned} \tag{2.5}$$

where, as usually, g' denotes the derivative of g (for details see, e.g., Capra and Canale (1990)). As we see, there are $4n - 2$ equations and $4n$ coefficients. Imposing two more restrictions, we can form the Cramer system and, consequently, obtain the unique solution.

In 1973, Herriot and Reinsch (1973) gave the method to produce the required spline much more quickly than via direct solving of the Cramer system mentioned above. This method, since then called a Herriot-Reinsch algorithm (HeRA), does not work with the Stevin polynomials, Eq. (2.4), but it deals with the polynomials of the form

$$g_j(t) = a_j + b_j(t - t_j) + c_j(t - t_j)^2 + d_j(t - t_j)^3 \tag{2.6}$$

Thanks to it, in HeRA we do not solve the system in $4n$ unknowns (a_j, b_j, c_j, d_j for $j = 1, 2, \dots, n$), but we find n unknowns, $c_1, c_2, \dots, c_n$, satysfying the system of linear algebraic equations with a sparse matrix (for details see, e.g., de Boor and Schoenberg (1976), Herriot and Reinsch (1973)).

This matrix is tridiagonal, if we deal with a natural spline, i.e., we put $c_1 = c_n = 0$; it makes that the curvature of the spline vanishes at extreme points t_1 and t_n. Having $y_{n+1} = y_1$ and imposing appropriate conditions, we get a periodic spline, i.e., a spline g such that $g(t + mp) = g(t)$ for all real t and for every integer m, where the difference $p = t_{n+1} - t_1$ is the period; now the matrix of the system differs from the mentioned tridiagonal one at two entries: the lowest left one and the most upper right one. The periodic splines were investigated in Friedrich Krinzessa's doctoral dissertation entitled *Zur periodischen Spline-interpolation* (Bochum, 1969), see also Gajda *et al.* (2008), Krawiec (2009), Krinzessa (2006).

Below we apply the HeRA to obtain the periodic spline collocated at $n = 24$ points $P_k = (x_k, y_k)$, $k = 1, 2, \dots, 24$, the closeness is forced by the smooth passage between the functions g_{24} and g_1. All points lay on the oval curve, see Figs. 1-4, and, in fact, we calculate two sets of coefficients (a_j, b_j, c_j, d_j), one for the points (t_k, x_k), one for the points (t_k, y_k), in both cases with $t_k = k$. Therefore, both systems of linear equations (with unknowns c_k) have the same matrix: its every its diagonal element is equal to 4, other non-zero entries are equal to 1. We obtain 24 vector equations covering every function g_k, the k-th piece of this curve is described parametrically, the parameter denoted as t runs the interval $\langle k, k+1 \rangle$. This fragment joins the nodes P_k and P_{k+1}; since now it is denoted by A_k and called a k-th arc of the spline.

To give some details, let us describe the situation concerning functions no. 9 and no. 10 (see Figs. 1-4). These functions are covered by the equations in the

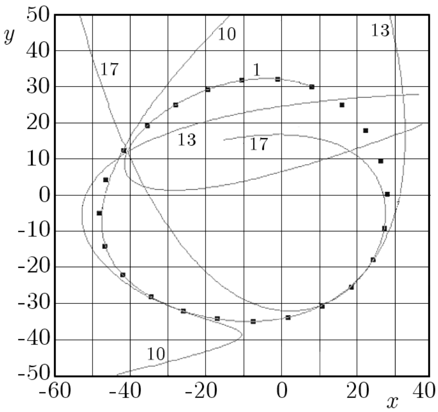

Fig. 1. Curves no. 1,10,13,17 drawn when t runs from 0 to 24

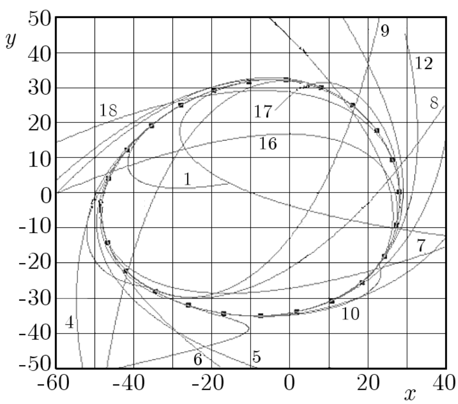

Fig. 2. Arcs of all 24 curves, $(x, y) = g_k(t)$, drawn for $t \in \langle k, k+10 \rangle$

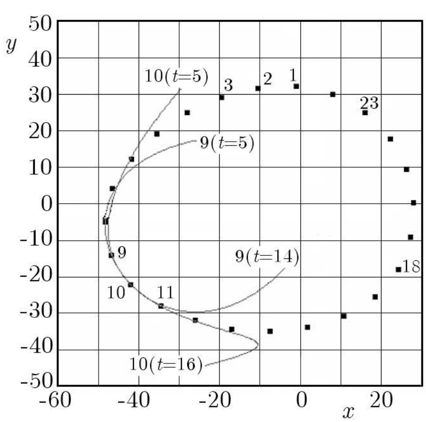

Fig. 3. Curves no. 9 drawn when t runs from 9 to 14, and no. 10 drawn for $t \in \langle 5, 16 \rangle$

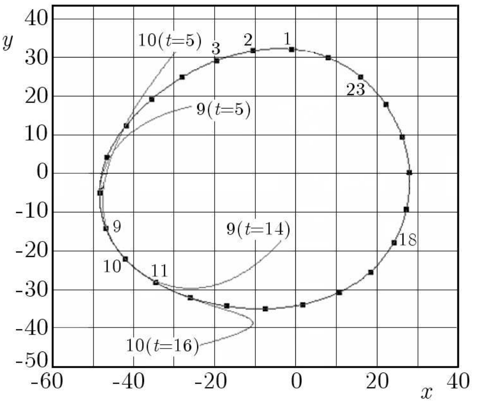

Fig. 4. Spline passing through given 24 points (some of them are numbered) and arcs seen in Fig. 3. The appropriate pieces of these arcs, namely the arc A_9 joining nodes 9 and 10, and the arc A_{10} going from node 10 to node 11, make part of the spline and they are joint smoothly at point 10

parameter t and their parts contribute in the final spline when $t \in \langle 9, 10\rangle$ and $t \in \langle 10, 11\rangle$, respectively. These equations are obtained in form (2.6) and may be written in standard form (2.4). Let, as above, A_j denote the j-th arc extended between the points $P_j = (x_j, y_j)$ and $P_{j+1} = (x_{j+1}, y_{j+1})$. Then the 9th arc, A_9, joins the point $P_9 = (-46.7562, -14.1921)$ with the point $P_{10} = (-41.8987, -22.2177)$ and its parametric description is

$$x(t) = a_{x,9} + b_{x,9}(t-9) + c_{x,9}(t-9)^2 + d_{x,9}(t-9)^3$$
$$y(t) = a_{y,9} + b_{y,9}(t-9) + c_{y,9}(t-9)^2 + d_{y,9}(t-9)^3$$

where

$a_{x,9} = x_9 = -46.7562$	$b_{x,9} = 3.31931$	$c_{x,9} = 1.65761$
$d_{x,9} = -0.119423$	$a_{y,9} = y9 = -14.1921$	$b_{y,9} = -8.83033$
$c_{y,9} = 0.596595$	$d_{y,9} = 0.208135$	

That is why the 9-th part of the spline, A_9, is governed by the equation

$$x(t) = -46.7562 + 3.31931(t-9) + 1.65761(t-9)^2 - 0.119423(t-9)^3 =$$
$$= -0.119423?t3 + 4.88203t^2 - 55.5374t + 144.695$$
$$y(t) = -14.1921 - 8.83033(t-9) + 0.596595(t-9)^2 + 0.208135(t-9)^3 =$$
$$= 0.208135t^3 - 5.02305t^2 + 31.0077t - 38.1253$$

where $t \in \langle 9, 10\rangle$.

Analogously, the 10th arc, A_{10}, is determined by the coefficients

$$
\begin{array}{ll}
a_{x,10} = x_{10} = -41.8987 & b_{x,10} = 6.27626 \\
c_{x,10} = 1.29934 & d_{x,10} = -0.260208 \\
a_{y,10} = y10 = -22.2177 & b_{y,10} = -7.01272 \\
c_{y,10} = 1.22100 & d_{y,10} = -0.095979
\end{array}
$$

(all calculations and figures are produced in the computer algebra system Derive for Windows 5 from Texas Instruments, Inc.; the coordinates of points are the centers of notches in the wheel photographed in Fig. 6a)

3. Multiwheelness and belt linkage

In two-wheel transmission systems, the average transmission ratio must be equal to 1:2, 2:1, 2:3 etc. It essentially impedes the use of transmission systems of that type. Thus, it has become necessary to conduct a feasibility analysis whether it would be possible to design and construct multi-wheel transmission systems producing any transmission ratio.

In order to ensure the cyclicity of the four-wheel transmissions system, there must be taken an appropriate shape of the noncircular wheel. Circumferences of all wheels must be equal to the multiplicity of the belt pitch. To avoid the belt skipping on teeth of the wheels, the belt has to be properly pre-tightened.

In most cases, the shape of the rim of the noncircular wheel had to be computed iteratively, in a way to ensure the constant tension of the belt and the cyclicity of the transmission system. Noncircular gears are mostly applied as the driving mechanism for a linkage to modify displacements and/or velocities (e.g., in conveyors driven by the Maltese cross; Litvin and Fuentes (2004) calls it a Geneva mechanism). Noncircular gear toothing is well described and used, e.g., in cycle hydraulic engines.

An example of the considered linkage is an uneven strand transmission, see Fig. 5. There are four wheels in it: driving wheel, driven wheel and two wheels used to ensure the fixed length of the envelope of the system (Fig. 6). Two of these wheels are circular, one wheel is circular and eccentrically, the other is noncircular.

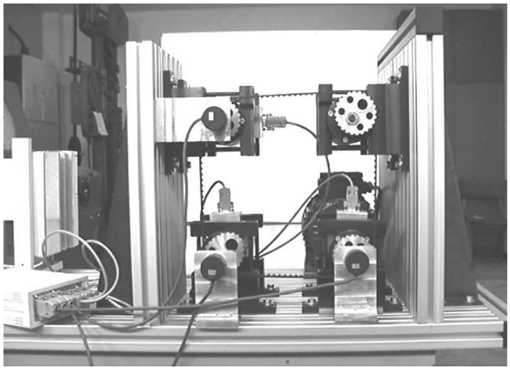

Fig. 5. Four-wheel transmission systems installed in LEDM (Laboratory for Experimental Design and Manufacturing at the Faculty of Machines and Transport, Poznan University of Technology); the system has two circular wheels, one noncircular wheel and one eccentrically mounted wheel

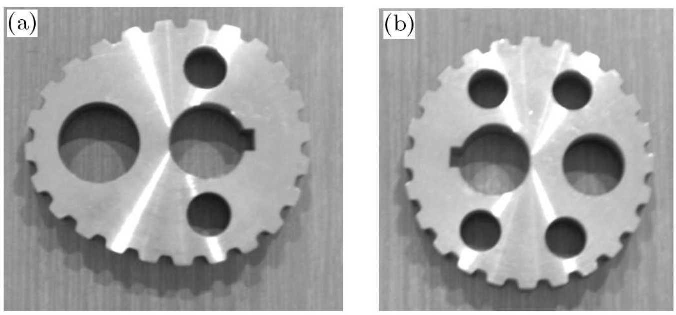

Fig. 6. Wheels of the variable speed transmission system: (a) noncircular wheel, (b) circular eccentrically mounted wheel

4. Design, forming and verification

In the transmission system with noncircular gears, the process of shape-frictional contact is characterised by essential differences, which result from different geometrical features of gears (elliptical, oval, cycloidal and, in general, irregular) and different kinematic features. In each case, in the manufacturing process it is necessary to have the description of rolling lines by appropriate equations. These equations are of various forms and there exist some techniques to obtain them. In this paper, we describe noncircular gears via splines. The advantage of this approach is the direct consequence of the properties of the mathematical representation: the arc A_j is covered by the parametric equation

$$x = x(t) \qquad y = y(t) \qquad j \leqslant t \leqslant j+1$$

Hence, at the endpoints, there hold the equalities

$$(x_j(j), y_j(j)) = P_j \qquad\qquad (x(j+1), y(j+1)) = P_{j+1}$$

an arbitrary point $P(t)$ laying on the arc A_j is identified by the certain value $t \in \langle j, j+1 \rangle$ and, in the considered case (when the length $|A_j|$ of the j-th arc is relatively small, $|A_j = 9.525\,\mathrm{mm}$), this identification is practically linear. In consequence, it is easy to indicate, up to a certain accuracy (and it is big enough in manufacturing practice), the value t which produces the desired point $P(t)$. Clearly, it is trivial to say exactly which values of t produce the endpoints of the arc A_j. In consequence, the use of spline curves reduces the setup time of software driven on numerically controlled machines.

Noncircular gears may be formed, e.g., on profile cutters, on CNC gas laser cutting machines, by cutting out on CNC electrical discharge machines, by water cutting with abrasive and sintered metal powders, and the precision of their manufacturing was analysed by Krawiec (2009).

Our irregular wheel, represented by the parametric spline (see Fig. 4), was manufactured on CNC machine with the set of end mills (see Fig. 7). The accuracy of forming geometrical parameters and stereometry of the wheel surface was experimentally verified on CMM (coordinate measuring machine) Contura from Zeiss and the profilograph from Taylor Hobson (see Fig. 8). Till now, there are no regulations concerning the systems composed of noncircular wheels and timing belts, so we can only refer to the requirements which are obligatory to circular wheels. This examination shows that the manufactured wheel has desired values of the parameters (such as tolerance of the outside rim, tolerance of the tooth shape with respect to the axle, errors on the pitch).

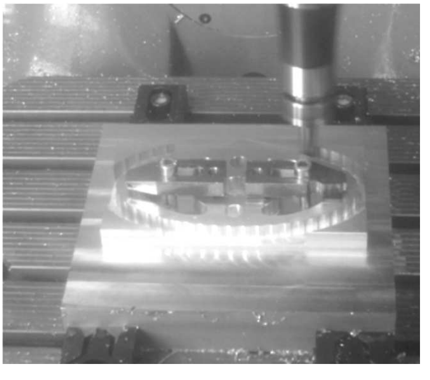

Fig. 7. Forming noncircular pulleys

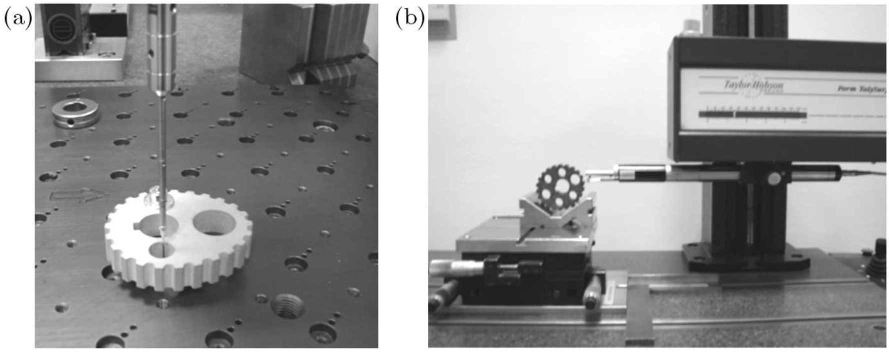

Fig. 8. Measurement of geometrical features (a) and roughness features (b) of noncircular wheels executed in LEDM

Our irregular wheel has been mounted in the prototype stand shown in Fig. 5. In this stand there are pulleys with the noncircular envelope in the uneven transmission with cog-belt. It makes possible to change kinematical features such as the gear ratio and velocity. For the stand presented in Fig. 5, the changes in the transmission ratio are shown in Fig. 9.

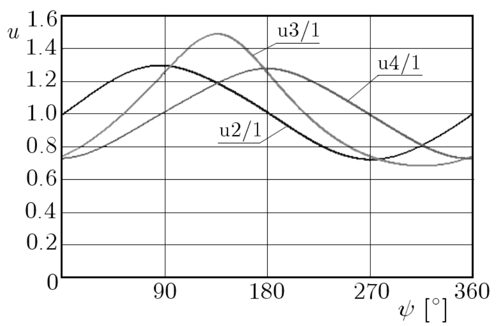

Fig. 9. Changes in the transmission ratio of: wheel 2 in relation to wheel 1 (u2/1), wheel 3 in relation to wheel 1 (u3/1), wheel 4 in relation to wheel 1 (u4/1). Along the horizontal axis there are marked values of the angle (ψ, measured in degrees) of rotation of the driving wheel

5. Conclusions

The paper deals with the cubic spline interpolation of non-typical irregular wheels. The discussion concerns theoretical and practical aspects of this approximation, in both mathematical and technical issues of the problem, and reveals the advantages of this approach in both design and manufacturing stages. An example of a prototype 4-wheel drive installed in a stand at LEDM (see Fig. 5) is also presented. Drives like that have a changeable kinematical characteristic of the velocity of the element driven by the motor working with a constant speed. The feature is obtained by the use of an appropriately shaped noncircular toothed-wheel rims when the motion is transmitted by a timing belt.

The importance of the presented problem for industrial practice is determined by kinematic features of uneven transmission drives. In the designing of a drive having required parameters there can be applied various techniques. In the case of regular noncircular gears, we can refer to well-know formulas such as the equation of an ellipse. When we study irregular shapes (like that presented in this paper), we have to find their mathematical description. An example of such an investigation is given by Gajda *et al.* (2008), where the Bézier approximation is analysed. Another attempt is discussed in this paper. Here we deal with the spline interpolation. In relation to the Bézier approach, the method of splines is more effective, because we can indicate more easily the points which are crucial in the design and forming processes (e.g., the points at which a designer has to get the perpendicular lines to the profile at hand) and to determine the angles at which the projected gear has to be situated with respect to the cutting tool, e.g. WEDM (wire electrical discharge machine), CNC machine. Another positive feature of the spline approximation is its friendliness as seen from the numerical point of view. Indeed, the spline is a piecewise polynomial function, the equation of its every arc is composed of two formulas of the form given by expression (2.6). This form is stable and fast in computation (when rewritten in the Horner scheme). All the features discussed above let to conclude that the spline approximation is highly recommended to be used in the mathematical description of wheels exhibiting significant irregularity of shape.

Acknowledgments

Authors express their thanks to an anonymous Reviewer for all remarks which improved this paper in its substantial and linguistic aspects.

References

1. BAIR B.W., 2002, Computer aided design of elliptical gears, *ASME J. Mech. Des.*, **124**, 12, 787-793

2. Bair B.W., 2009, Tooth profile generation and analysis of crowned elliptical gears, *ASME J. Mech. Des.*, **131**, 6, 074503-1-6

3. Bair B.W., Chen C., Chen S., Chou C., 2007, Mathematical model and characteristic analysis of elliptical gears manufactured by circular-arc shaper cutters, *ASME, J. Mech. Des.*, **129**, 2, 210-217

4. de Boor C., Schoenberg I.J., 1976, *Cardinal Interpolation and Spline Functions VIII. The Budan-Fourier Theorem for Splines and Applications*, Springer Lecture Notes in Mathematics, **501**, 1-79

5. Burden R.L., Faires J.D., 1985, *Numerical Analysis*, PWS Publishers, Boston

6. Chapra S.C., Canale R.P., 1990, *Numerical Methods for Engineers*, McGraw Hill International Editions

7. Chen C.-F., Tsay C.-B., 2004, Computerized tooth profile generation and analysis of characteristics of elliptical gears with circular-arc teeth, *J. Materials Processing Technology*, **148**, 2, 226-234

8. Danieli G.A., Mundo D., 2005, New developments in variable radius gears using constant pressure angle teeth, *Mech. Mach. Theory*, **40**, 203-217

9. Demongeot J., 2007, A brief history about analytic tools in medical imaging: splines, wavelets, singularities and partial differential equations, *Proc. 29th Annual Int. Conf. IEEE EMBS*, Cité Internationale, Lyon, France, 3474-3480

10. Ferguson J., 1964, Multivariable curve interpolation, *J. Assoc. Comp. Machinery*, **11**, 2, 221-228

11. Gajda K., Krawiec P., Marlewski A., 2008, Bezier fitting to almost oval gear devices, *Proc. 7th Int. Workshop on Modelling and Applied Simulation*, Campora S. Giovanni (Edit.), Italia, 210-216

12. Hasse T., 2000, Unrundgetriebe mit deckungsgleichen unrunden Zahnrädern für tipische Getriebeaufgaben, *Konstruktion*, **52**, 11/12, 64-69

13. Herriot J.G., Reinsch C.H., 1973, Algorithm 472: procedures for natural spline interpolation, *Communications of the Association for Computing Machinery*, **16**, 12, 763-768

14. Korzeniewski H., 2006, Uzębienie ewolwentowe kół trochoidalnych, *Archive of Mechanical Technology and Automation*, **26**, 2, 150-157

15. Krawiec P., 2009, Experimental method for measurements of geometrical features and surface stereometry of non-circular belt pulleys, *Measurements Automation and Monitoring*, **55**, 6, 364-367

16. Krawiec P., 2005, A multi-wheel toothed belt transmission system with a periodically variable transmission ratio, *Archive of Mechanical Technology and Automation*, **25**, 2, 125-132

17. Krinzessa F., 2006, *Fast Construction and Evaluation of Interpolatory Periodic Spline Curves*, Grin Verlag, Munich

18. Li J.-G., Wu X., Mao S.-M., 2007, Numerical computing method of noncircular gear tooth profiles generated by shaper cutters, *Int.J. Adv. Manuf. Technology*, **33**, 1098-1105

19. Li X.-C., Liu J., 2002, Research on the mapping models and designing methods of variable-ratio chain/belt drives, *Mech. Mach. Theory*, **37**, 9, 955-970

20. Litvin F.L., Fuentes A., 2004, *Gear Geometry and Applied Theory*, Cambridge University Press

21. Popoviciu T., 1934, Sur le prolognement des fonctions convexes d'ordre superieur, *Bull. Math. Soc. Roumanie des Sc.*, **36**, 75-108

22. Schoenberg I.J., 1946, Contributions to the problem of approximation of equidistant data by analytic functions. Part A – on the problem of smoothing or graduation. A first class of analytic approximation formulae, *Quart. Appl. Math.*, **4**, 45-99

23. Schumaker L., 1980, *Spline Functions: Basic Theory*, Wiley, New York

24. Tsay M.-F., Fong Z.-H., 2005, Study on the generalized mathematical model of noncircular gears, *Mathematical and Computer Modelling*, **41**, 555-569

Opis krzywymi sklejanymi nietypowych kół zębatych stosowanych w przekładniach cięgnowych

Streszczenie

Zastosowanie nieokrągłych kół zębatych powala uzyskać lepsze charakterystyki przełożenia. Przypadki, gdy koła nieokrągłe mają kształty regularne, np. eliptyczny lub cykloidalny, zostały już szeroko przebadane. W tej pracy są poddane analizie nietypowe nieregularne koła zębate. Projektowanie takich kół wymaga bardziej zaawansowanych technik, w szczególności podejścia numerycznego w celu uzyskania matematycznego opisu profilu koła. W pracy omawiamy interpolację parametryczną za pomocą funkcji sklejanych stopnia trzeciego i stosujemy ją do uzyskania odpowiedniego opisu. Ponadto prezentujemy prototypowy napęd, w którym zostało zastosowane przedmiotowe nieregularne koło nieokrągłe. Zastosowanie tego koła w przekładni cięgnowej sprawia, że ma ona zmienne cechy kinematyczne.

Manuscript received December 17, 2009; accepted for print October 15, 2010