ACTION-REACTION BASED SYNTHESIS OF ACOUSTIC WAVEFIELD EQUATIONS

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The analysis of acoustic fields is usually based on the well-known mathematics of second order partial differential equations called wave equations. The author explores the duality and symmetry of linear fluid mechanics and develops two distinct equations of acoustics on the basis of a causal approach to local small-scale phenomena. Wavefields that are solutions of these equations have different composition, the spherical pressure wave being only a specific components of one of them. A new perspective on the phenomena, presented in this paper, endeavours to establish the right equilibrium between mathematical formalism and physical phenomenology in the perception of acoustic wavefields.

INTRODUCTION

Merely the scalar aspect of acoustic fields is taken into account in most analyses. Particle velocity representing the vector aspect of acoustics is almost absent as an autonomous field quantity. It is treated as if it were merely a derivative of either the acoustic pressure or the velocity potential. In other words, the velocity is significantly marginalized by the two scalar quantities.

Recognizing that it is an important task to establish an equilibrium and formal symmetry in the physical, phenomenological image of acoustics, the author starts from the basic mechanical properties of fluid continuum, then brings both Euler's conservation equations into the most symmetric form, and, finally, derives the two best known second order differential equations of acoustic field, in an original way, on a causal basis, involving an action-reaction chain. The solutions of these equations each describe a pair of coupled pressure-velocity fields of a wave character. Some particular features of these scalar-vector fields turn out to be exclusively source-dependent. Fluid itself carries two kinds of mechanical linear wavefields as they have been impressed by the primary source-fluid impact, without modifications other than magnitude decrease with distance and time delay.

The present approach proves clear distinction and equivalence of both the aspects of linear acoustics.

1. EULER'S RELATIONS OF LINEAR FLUIDMECHANICS

Compared to classical dynamics of fluid continua, acoustics is a branch of fluid mechanics limited to small-scale effects, without, however, assumption of fluid incompressibility. In fact, acoustic effects are based on the simultaneous movement and deformation of fluid particles. It is worth stressing here a fundamental importance of the phenomenological notion of particles in material continua, ingeniously introduced by Euler [1]. Both inertia and elasticity of particles count when regarding mechanical phenomena from either a dynamic or energetic point of view. Similarly, both particle velocity and pressure created by deformed particles should be seen as physical quantities playing distinct roles of equal importance.

Physical requirements, derived from Newton's third law, on local, instantaneous dynamic balance at any place and time, are described by Euler's relations that in the same time have a sense of conservation laws. Their formulation concerns, separately, dynamics of fluid particles in pressure-disequilibrated conditions, on the one hand, and elastic reaction of particles in nonuniform-flow conditions, on the other. The formulation of the problem of dynamics neglects fluid elasticity, whereas that of the elasticity concerns a gentle flow justifying omission of fluid inertia. The first assumption means fluid incompressibility, and is commonly made use of in dynamics of liquids and gases, the second one is a natural assumption in problems of hydro- and aero-statics.

Below, the two Euler's formula are given in the most symmetric form that stresses an interesting duality of mechanical phenomena [2]. Although both terms of each equations concern exactly the same moment, we are able to distinguish which side of the equation is earlier and which later. As the cause must precede the effect (surely that is exactly what it MUST do??!!), it is clear that the acting term precedes the reacting one. It is worth noting here that this advance is not only immeasurable, it is clearly null as it is absent in mathematical formulations. Nevertheless, from now on, we will explicitly indicate the orientation of the cause-effect chain, placing the effects as first terms of equations, and the causes as their second terms, as follows:

$$a)\quad -\rho\frac{\partial}{\partial t}\vec{v}(\vec{x},t) = grad\ p(\vec{x},t), \qquad b)\quad -\kappa\frac{\partial}{\partial t}p(\vec{x},t) = div\ \vec{v}(x,t), \tag{1}$$

$$\textit{dynamic reaction} \Leftarrow \textit{deformation action}, \qquad \textit{elastic reaction} \Leftarrow \textit{flow action}.$$

In the above equations, acoustic pressure p and particle velocity $\vec{v}$ represent, respectively, the elastic deformation, being a scalar effect, and the inertial movement, being a vector one. At any time t and place $\vec{x}$, both quantities are linked to each other, equations (1) being two fundamental laws formulated by Euler for a fluid continuum with two characteristic parameters: (1) mass density ρ and (2) elastic compressibility κ (the latter parameter is the inverse of the fluid bulk modulus *Y*). The first one (1a) is the momentum conservation law stating that the local momentum density rate of change is forced by the spatial disequilibrium of the pressure, viz. its gradient. The second law (1b) concerns the mass conservation and states that the local density in an unbalanced flow changes at the rate proportional to the mass flow divergence. As acoustic pressure is proportional to the density variations ($\partial\rho = \rho\kappa\,\partial p$), the latter law means also that the flow disequilibrium imposes the change of local pressure.

The first equation is in fact a generalisation of Newton's second law, dealing with merely punctual masses, on material continua. The second one expresses the law of stream continuity in the case of an elastic matter. The differential form of the equations points to their applicability at any and every point of fluid space, at any time moment.

2. SECOND-ORDER RELATIONS OF ACOUSTICS

As acoustics concerns simultaneous action of inertia and elasticity, the two Euler's relations (1a) and (1b) must be fulfilled in parallel. From the mathematical point of view, we deal with a system of two first-order partial differential equations with two variables. One of them, pressure, is a scalar, the other one, velocity, is a vector. Derivatives of the variables count in both the equations: time derivative of one of the variables is balanced by a space derivative of the other one, be it either a gradient where a scalar variable is concerned, or a divergence where it is a vector. In both cases a suitable multiplication factor is involved, namely mass density or elastic compressibility.

In the common approach, the two equations are easily reduced to one of the second-order partial differential equations, either a scalar or a vector one, of merely one variable, be it either pressure or velocity, respectively. Mathematical manipulations look to be the same in both cases. The transformation procedure is as follows. One of the equations is subject to time derivation, and the other one to space derivation, be it gradient or divergence, depending on the scalar or vector character of Euler's equation concerned.

We can guess that it has been the above mentioned similitude of both derivations that has led to the prevailing custom of treating the two resulting equations as equivalent to such a point, that one of them can be, and almost always is, abandoned in subsequent steps of analysis, as less convenient to use.

Below, a clearly distinct character of the two equations and their equivalent importance will be proved on a phenomenological basis. The two second-order equations will be derived, keeping in mind that local, instantaneous, cause-effect relations are of primary importance. First, two situations will be considered, both assuming at any site in fluid space, a preexisting nonuniform distribution of, respectively, either pressure or velocity, as a primitive cause of imbalance initiating wave-like disturbances crossing the space.. Second, scalar- and vector-type sources will be assumed in fluid, as explicit primitive causes of the disturbances.

3. SELF-AFFECTING COMPRESSION IMBALANCE

Let us first consider a situation where there is a locally nonuniform pressure distribution resulting from some previous cause. The pressure gradient is the measure of local space imbalance that will modify local flow of the matter according to the equation of dynamics (1a). The resulting velocity will be calculated from the following relation being an integral form of the dynamics equation:

$$\vec{v} = -\frac{1}{\rho}\int grad\ p\, dt, \qquad (2)$$

$$\textit{first reaction} \Leftarrow \textit{primary action.}$$

The above nonzero pressure gradient should be seen as the primary reason of violation of the local balance, causing, as the first fluid reaction, a flow modification being a manifestation of momentum density change. In turn, the flow distribution, even if previously spatially balanced, becomes imbalanced, the velocity divergence being the measure of the new imbalance. At the same time the local matter distribution is being modified and, finally, the new distribution manifests itself in the pressure change that will be calculated from the following integral form of the equation of elasticity (1b):

$$p = -\frac{1}{\kappa}\int div\ \vec{v}\, dt, \qquad (3)$$

$$\textit{final reaction} \Leftarrow \textit{secondary action.}$$

The final pressure distribution resulting from the above action-reaction, cause-effect chain of the pressure distribution in the elastic matter, self-modified by a primary imbalance of the local pressure, can be written in the following form:

$$p = -\frac{1}{\kappa}\int div\left(-\frac{1}{\rho}\int grad\, p\, dt\right)dt \tag{4}$$

Here, the time and space operations are mutually independent and can be performed in an arbitrary order, hence Eq. (4) can be put in the following double time-integral form:

$$p = \frac{1}{\kappa\rho}\iint div\, grad\, p\, dt^2 \tag{5}$$

Eq. (5) expresses certain physically coupled elasto-dynamic feedback conditions reigning at any space-and-time location. It can also be put into an equivalent double time-differential form, as follows:

$$\frac{\partial^2 p}{\partial t^2} = \frac{1}{\kappa\rho} div\, grad\ p \tag{6}$$

This second order partial differential equation relating to acoustic pressure, is easily recognisable as an homogeneous wave equation in scalar form.

4. SELF-AFFECTING FLOW IMBALANCE

Let now consider a dual situation where there is a locally nonuniform flow velocity distribution resulting from a previous cause. The space imbalance of this distribution expressed by its divergence, will now modify local distribution of the elastic matter, as described by the following relation being an integral form of the equation of elasticity (1b):

$$p = -\frac{1}{\kappa}\int div\ \vec{v}\, dt, \qquad \textit{first reaction} \Leftarrow \textit{primary action.} \tag{7}$$

Formally, Eq. (7) does not differ from Eq. (3). However, it is worth rewriting as its role in the cause-effect chain is here not the same, as indicated by the in-the-equation commentaries.

The velocity divergence should here be seen as the primary reason of violation of the local balance, causing, as the first reaction, a pressure modification being a measurable manifestation of the density change of an elastic matter. In turn, the matter distribution, even if previously spatially balanced, becomes imbalanced, the pressure gradient expressing the new imbalance. The local matter flow is being modified at the same time and its new distribution manifests itself in the velocity change. The latter will be calculated from the integral form of the equation of dynamics (1a), formally identical with Eq. (2):

$$\vec{v} = -\frac{1}{\rho}\int grad\ p\, dt, \qquad \textit{final reaction} \Leftarrow \textit{secondary action.} \tag{8}$$

The final velocity distribution resulting from the above action-reaction, cause-effect chain of the elastic matter flow self-modified by a primary imbalance of the local flow, can be written in the following form:

$$\vec{v} = -\frac{1}{\rho}\int grad\left(-\frac{1}{\kappa}\int div\,\vec{v}\,dt\right)dt \tag{9}$$

For the reasons indicated previously in the dual case, Eq. (9) can be put in the following form of a double time integral:

$$\vec{v} = \frac{1}{\rho\kappa}\iint grad\;\;div\,\vec{v}\,dt^2 \tag{10}$$

Eq. (10) expresses the physically coupled feedback conditions reigning at any space-and-time location. In this case the conditions are dual to the previous ones. They should be named in a distinctive manner as dynamo-elastic ones. The above equation can also be presented in the equivalent double time-differential form:

$$\frac{\partial^2\vec{v}}{\partial t^2} = \frac{1}{\rho\kappa}grad\;\;div\,\vec{v} \tag{11}$$

This equation relating to particle velocity, is dual to Eq. (6). Here it is an homogeneous wave equation in a vector form. A specific symmetry relation exists between two wave equations (6) and (11), very similar to that observed between the first-order Euler equations:

In textbooks that mention the velocity equation, the combined, double space operators $div\,grad$ and $grad\,div$ in Eqs. (6) and (11), are usually put both in the form of laplacians, the scalar one and the vector one, respectively, and presented as being equivalent in a field void of vorticity [3]. In fact, only plane wave solutions of the two equations turn out to be similar. On the contrary, it can be shown that in the presence of point sources, the two operators lead to significantly different results [4].

The above acoustic wave homogeneous equations have myriad solutions. In fact, any reasonably regular function is a solution with argument invariant to time and space shift with a constant rate equal $t - x/c$, where c is the wave celerity in the fluid, equal to the inverse of the geometric mean of the two fluid parameters, i.e. $c = 1/\sqrt{\kappa\rho}$. However, there is no reason in the present study to discuss the solutions to these equations, as our goal is to indicate clear differences between them and to reveal their equal importance.

5. EXTERNAL MATTER INFLOW-INDUCED IMBALANCE

In the previous sections, no sources external to the fluid were considered. Below, two kinds of sources will be added for generating impacts that can violate the local balance. First, inertia-less scalar sources will act, in a quasi-static manner, on the elasticity of directly neighbouring particles and second, vector sources will act dynamically on the surrounding particles as if they were rigid ones, i.e. without deforming them directly.

Let an additional matter be injected into fluid in a gentle way, the area and time evolution of the matter volume flow being given as $q(\vec{x},t)$ in [s^{-1}], meaning a volume density of the flow [$m^3/s \cdot m^{-3}$]. The former equilibrium will be disturbed at this area, according to the following differential relation that affects fluid particle elasticity:

$$\kappa\frac{\partial}{\partial t}p(\vec{x},t) = q(\vec{x},t), \tag{12}$$

$$\textit{fluid initial reaction} \Leftarrow \textit{source action.}$$

The presence of the additional matter directly modifies local fluid density in the vicinity of the source area. It manifests itself as a pressure reaction that can be described in the following integral form:

$$p(\vec{x},t) = \frac{1}{\kappa}\int q(\vec{x},t)\,dt, \qquad (13)$$
$$\textit{fluid initial reaction} \Leftarrow \textit{source action.}$$

In the same time and place, the modified pressure induces an instantaneous modification the flow, in exactly the same manner as it was in the source-less case of Section 3. From this place on, the action-reaction causal chain acts in the same way, and the subsequent detailed discussion could be shortened, if the derivation of the acoustic field equations was simply a matter of mathematical transformations. However, in the author's opinion, it is worth continuing the step-by-step phenomenological synthesis and copying, each time, the integral relations of dynamics and elasticity with a comment regarding the actual step. So, the velocity will be calculated from the relation of dynamics:

$$\vec{v}(\vec{x},t) = -\frac{1}{\rho}\int grad\; p(\vec{x},t)\,dt, \qquad (14)$$
$$\textit{first reaction} \Leftarrow \textit{primary in.fluid action.}$$

In turn, the density (pressure) distribution is being altered as a result of the secondary action of the previously induced flow:

$$p(\vec{x},t) = -\frac{1}{\kappa}\int div\,\vec{v}(\vec{x},t)\,dt, \qquad (15)$$
$$\textit{final in.fluid reaction} \Leftarrow \textit{secondary action.}$$

Finally, the resulting condition of the dynamic equilibrium in the case of scalar, volume-flow distributed sources, can be written as:

$$p(\vec{x},t) = \frac{1}{\kappa\rho}\iint div\,grad\,p(\vec{x},t)\,dt^2 + \frac{1}{\kappa}\int q(\vec{x},t)dt, \qquad (16)$$
$$\textit{total reaction} \Leftarrow \textit{fluid double action} + \textit{source action.}$$

The above integral acoustic equation can be put in the following equivalent time-differential form:

$$\frac{\partial^2}{\partial t^2}\,p(\vec{x},t) - \frac{1}{\kappa\rho}\,div\,grad\;p(\vec{x},t) = \frac{1}{\kappa}\frac{\partial}{\partial t}q(\vec{x},t) \qquad (17)$$

Eq. (17) is easily recognisable as an inhomogeneous scalar wave equation, the field in fluid medium being explicitly separated from its source. One could say that the last steps of derivation, i.e. Eq. (16) could be equivalently presented in the following form looking more concise:

$$p(\vec{x},t) = \frac{1}{\kappa\rho}\iint\left(div\,grad\;p(\vec{x},t) + \rho\frac{\partial}{\partial t}q(\vec{x},t)\right)dt^2, \qquad (18)$$
$$\textit{total reaction} \Leftarrow \textit{combined initiating action}$$

that would directly lead to the following differential form:

$$\kappa\rho \frac{\partial^2}{\partial t^2} p(\vec{x},t) - div\, grad\; p(\vec{x},t) = \rho \frac{\partial}{\partial t} q(\vec{x},t) \tag{19}$$

which is the most spread. It is worth stressing, however, that although mathematically identical with Eq. (17), we must be aware that when analysed from the phenomenological point of view, Eq. (19) can erroneously suggest an inertial inflow of extra matter as the primary cause of disturbance, whereas the disturbance reason is the non-inertial, deforming impact of the extra volume of fluid matter onto the surrounding particles of elastic fluid.

6. EXTERNAL FORCE-INDUCED IMBALANCE

Now let an external force field be applied to fluid. The field of an arbitrary space distribution and time evolution, given as $\vec{f}(\vec{x},t)$ in [kg· m^{-2}·s^{-2}] = [N·m^{-3}] representing the volume density of forces, is related to the so-called body forces that are external to the inherently fluid mechanical surface ones being related to acoustic pressure. The former equilibrium will be violated according to the following differential relation that illustrates dynamic action of the force field:

$$\rho \frac{\partial}{\partial t} \vec{v}(\vec{x},t) = \vec{f}(\vec{x},t), \tag{20}$$

$$\textit{fluid initial reaction} \Leftarrow \textit{source action.}$$

The application of the external forces results in the direct modification of the local flow dynamics. It manifests itself as a movement described in the following integral form:

$$\vec{v}(\vec{x},t) = \frac{1}{\rho} \int \vec{f}(\vec{x},t)\, dt, \tag{21}$$

$$\textit{fluid initial reaction} \Leftarrow \textit{source action.}$$

The modified flow induces, in the same time and place, a change in the instantaneous distribution of local density, and thus of acoustic pressure, in the same manner as in the source-less case of Section 4. The pressure will be calculated from the integral form of the equation of elastic continuity (3):

$$p(\vec{x},t) = -\frac{1}{\kappa} \int div\, \vec{v}(x,t)\, dt, \tag{22}$$

$$\textit{first reaction} \Leftarrow \textit{primary in.fluid action.}$$

In turn, the dynamic flow distribution is being altered as a result of the secondary action of the previously induced density distribution, according to the integral form of the equation of dynamics (2):

$$\vec{v}(\vec{x},t) = -\frac{1}{\rho} \int grad\; p(\vec{x},t)\, dt, \tag{23}$$

$$\textit{final in.fluid reaction} \Leftarrow \textit{secondary action.}$$

Finally, we can write the resulting condition of dynamic equilibrium in the case of a vector source field:

$$\vec{v}(\vec{x},t) = \frac{1}{\rho\kappa} \iint grad\;\; div\, \vec{v}(\vec{x},t)\, dt^2 + \frac{1}{\rho} \int \vec{f}(\vec{x},t)\, dt, \tag{24}$$

$$\textit{total reaction} \Leftarrow \textit{fluid double action} + \textit{source action.}$$

The above integral acoustic equation can be put in the following equivalent time-differential form of inhomogeneous vector wave equation with the field in fluid medium separated explicitly from its source:

$$\frac{\partial^2}{\partial t^2}\vec{v}(\vec{x},t) - \frac{1}{\rho\kappa}grad\ \ div\ \vec{v}(\vec{x},t) = \frac{1}{\rho}\frac{\partial}{\partial t}\vec{f}(\vec{x},t) \tag{25}$$

Also in this case, one could propose to present the last step of derivation, eq. (24) in the form looking more concise:

$$\vec{v}(\vec{x},t) = \frac{1}{\rho\kappa}\iint\left(grad\ \ div\ \vec{v}(\vec{x},t) + \kappa\frac{\partial}{\partial t}\vec{f}(\vec{x},t)\right) dt^2, \tag{26}$$

$$\textit{total reaction} \Leftarrow \textit{combined initiating action}$$

that would directly lead to the following differential form:

$$\rho\kappa\frac{\partial^2}{\partial t^2}\vec{v}(\vec{x},t) - grad\ \ div\ \vec{v}(\vec{x},t) = \kappa\frac{\partial}{\partial t}\vec{f}(\vec{x},t) \tag{27}$$

However, also in this case, although equations (25) and (27) are mathematically identical, phenomenological analysis of the latter can erroneously suggest a deforming action of the force field as the cause of disturbance, whereas this time the disturbance reason is the inertial, non-deforming impact of body forces onto the surrounding massive fluid.

The distributed body forces above do not present a specific problem, as they can act on the fluid via surfaces of contact with fluid. However, a problem arises when point forces are in action. In this case a surface coupler is needed between the source force and fluid. But this is another story, developed in an associated paper [5].

7. CONCLUSIONS – TWO KINDS OF CAUSAL COUPLING

The causal approach to the synthesis of wavefield equations of linear acoustics, presented in the paper, reveals new aspects of a source-field-fluid cooperation the author has never met in textbooks. First, a wavefield, and hence the form of its equation, depends strictly on its source. Second, the general form of the field is induced by the source in its immediate vicinity and remains unchanged throughout the wavefield propagation in fluid; this also concerns wavefields described by homogeneous equations, meaning that fluid is able to transmit both scalar-induced and vector-induced waves. In other words, two kinds of causal coupling between mechanical effects are active in linear fluid. One is the pressure-velocity-pressure chain of action, the other is the velocity-pressure-velocity chain of action, both of equal importance, each chain being initiated by respective type of sources.

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