

A GENERALIZED VERSION OF THE PERTURBATION-BASED STOCHASTIC FINITE DIFFERENCE METHOD FOR ELASTIC BEAMS

MARCIN KAMIŃSKI

Technical University of Lodz, Chair of Mechanics of Materials, Lodz, Poland

e-mail: marcin.kaminski@p.lodz.pl

The main idea of this paper is to demonstrate a stochastic computational technique consisting of the generalized stochastic perturbation method using the Taylor expansions of random variables and the classical Finite Difference Method based on regular grids. As it is documented by computational illustrations, it is possible to determine, using this approach, also higher probabilistic moments for any random dispersion of input variables unlike in the second order second moment technique worked out before. A numerical algorithm is implemented here using straightforward partial differentiation of hierarchical equations with respect to the random input quantity and further symbolic computations of probabilistic moments and characteristics by the system MAPLE.

Key words: stochastic perturbation technique, finite difference method, response function method, elastic Euler-Bernoulli beam, Winkler foundation

1. Introduction

The Finite Difference Method (FDM) implementation (Liszka and Orkisz, 1980; Wasow and Forsythe, 1959) plays an important role in computational engineering in all those cases where additional differential equations (Collatz, 1960) (ordinary or partial) may not be solved straightforwardly – i.e. in the heat transfer (Minkowycz *et al.*, 1988), electro-magnetics (Taflove, 1998) and geodynamics, even in elasto-statics, as it is demonstrated here. So, it seems to be natural that this method is extended towards its new stochastic versions for some real systems with random parameters solved before using the traditional FDM in deterministic cases. One of such extension methods is the generalized perturbation-based stochastic technique, where the Taylor series

expansions of all random quantities lead to a system of equilibrium equations of the ascending order. This method was employed before for different stochastic extensions for the Finite Element Method (Kamiński, 2007; Kleiber and Hien, 1992), Boundary Element Method as well as even Finite Difference Method (Kamiński, 2001) (according to the second order second moment approach) but this implementation for the first time enables for (1) any order stochastic expansion, (2) any probability density function of the random input variable, (3) parametric study with respect to the perturbation parameter and coefficient of variation for the random input as well as (4) analytical derivations of most of discrete hierarchical equations implemented in the symbolic package MAPLE (other systems like freeware *Scilab* are also available of course).

The major difference in the comparison with stochastic versions of the FEM and BEM is the necessity of double differentiation with respect to the space variable discretized by Δx (4th order derivatives transformed to the finite differences) as well as with respect to the input random variables. Fortunately, since the application of symbolic calculus, this second differentiation is performed analytically for any available derivative orders, but in the case of a general computer program those derivatives must be implemented into it in form of the ready-to-use-formulas (or we need to assure the interoperability with the MAPLE environment). The remaining implementation issues are almost the same like in the case of the SFEM and the SBEM, but in a further perspective, a comparison with other stochastic methods like polynomial chaos expansions (Ghanem and Spanos, 1991) or Monte-Carlo simulations would be interesting (Hurtado and Barbat, 1998).

The stochastic implementation of the Finite Difference Method is displayed and discussed here on the example of the well-known fourth order ordinary differential equations relevant to elastic homogeneous and isotropic beams with and without an elastic single-parameter foundation. Although the general computer program is written in the internal language of the symbolic computing environment MAPLE, the algorithm has a general character and the grid applied to any beam may be essentially densified without any larger programming, where formation and solution of the ascending order hierarchical equations typical for the perturbation-based methodology will remain the same. Contrary to the previous second order second moment (SOSM) technique, now it is possible to compute any order moments (up to the fourth order here) for practically any value of the standard deviation for the random input. This methodology will be further extended towards 2 and 3-dimensional applications, also for transient problems with random coefficients.

2. Classical version of the Finite Difference Method for elastic beams

Let us consider the following ordinary fourth order differential equation for a linear elastic isotropic and statistically homogeneous beam exposed to the transversally distributed load $q(x)$

$$\frac{d^2}{dx^2} \left(E(x) J(x) \frac{d^2 w(x)}{dx^2} \right) = q(x) \tag{2.1}$$

fulfilling typical boundary conditions applicable in engineering theories of the elastic beams.

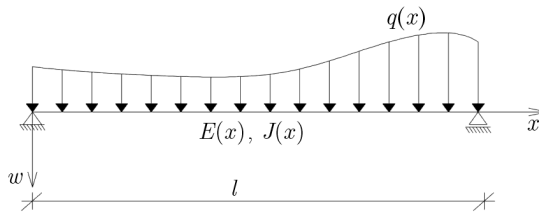


Fig. 1. Elastic beam subjected to transversal load $q(x)$

Let us also note that furtherly we neglect the influence of longitudinal and transversal forces as well as we reduce the analysis to small deflections, which significantly simplifies the final form of this equation. Let us divide the entire domain of the length l into n equidistant sub-domains with the length Δx . So that, for the i -th point of this discretization we adopt the following notation

$$E(i\Delta x) = E_i \qquad J(i\Delta x) = J_i \qquad q(i\Delta x) = q_i \tag{2.2}$$

On the other hand, it follows the series of approximations for derivatives of the ascending order with some finite differences as given below:

— first derivative

$$\left(\frac{\Delta w}{\Delta x} \right)_i = \frac{w_{i+1} - w_{i-1}}{2\Delta x} \tag{2.3}$$

— second derivative

$$\left(\frac{\Delta^2 w}{\Delta x^2} \right)_i = \frac{w_{i+1} - 2w_i + w_{i-1}}{\Delta x^2} \tag{2.4}$$

— third derivative

$$\left(\frac{\Delta^3 w}{\Delta x^3} \right)_i = \frac{-w_{i+2} + 2w_{i+1} - 2w_{i-1} + w_{i-2}}{2\Delta x^3} \tag{2.5}$$

— fourth derivative

$$\left(\frac{\Delta^4 w}{\Delta x^4}\right)_i = \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{\Delta x^4} \tag{2.6}$$

Introducing for the fourth order derivatives in Eq. (2.1), the above relation can be obtained for the i -th point of this grid

$$\begin{aligned} &E_{i-1}J_{i-1}w_{i-2} - 2(E_{i-1}J_{i-1} + E_iJ_i)w_{i-1} + \\ &+(E_{i-1}J_{i-1} + 4E_iJ_i + E_{i+1}J_{i+1})w_i + \\ &-2(E_iJ_i + E_{i+1}J_{i+1})w_{i+1} + E_{i+1}J_{i+1}w_{i+2} = q_i\Delta x^4 \end{aligned} \tag{2.7}$$

The particular case of $E(x)J(x) = \text{const} = EJ$ enables one to transform Eq. (2.7) into the formula

$$w_{i-2} - 4w_{i-1} + 6w_i - 4w_{i+1} + w_{i+2} = \frac{q_i\Delta x^4}{EJ} \tag{2.8}$$

The entire situation and FDM discretization is schematically presented in Fig. 2.

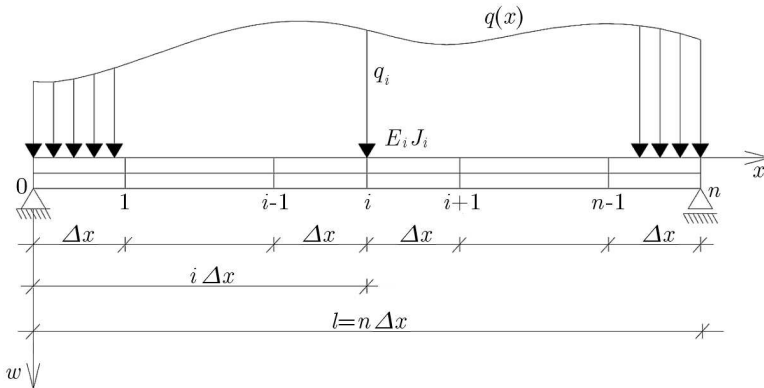


Fig. 2. Finite Difference Method discretization of the elastic beam

The situation changes only slightly when the elastic beam rests on a single parameter elastic foundation (known as the Winkler foundation). Let us assume that this foundation is homogeneous and characterised with the compliance coefficient k . Therefore, the equilibrium equation including the deflection $w(x)$ possesses a single extra component at the right-hand side. There holds

$$\frac{d^2}{dx^2}\left(E(x)J(x)\frac{d^2w(x)}{dx^2}\right) = -kw(x) + q(x) \tag{2.9}$$

Inserting, as previously, the additional formulas for the derivatives which include finite differences expressed by the discrete values of the function $w(x)$ in the neighborhood of the given i -th point of the grid, it is obtained that

$$\begin{aligned}
 & E_{i-1}J_{i-1}w_{i-2} - 2(E_{i-1}J_{i-1} + E_iJ_i)w_{i-1} + \\
 & +(E_{i-1}J_{i-1} + 4E_iJ_i + E_{i+1}J_{i+1} + k\Delta x^4)w_i + \\
 & -2(E_iJ_i + E_{i+1}J_{i+1})w_{i+1} + E_{i+1}J_{i+1}w_{i+2} = q_i\Delta x^4
 \end{aligned}
 \tag{2.10}$$

The reduction for the case $E(x)J(x) = \text{const} = EJ$ leads to the following equation

$$w_{i-2} - 4w_{i-1} + \left(6 + \frac{k\Delta x^4}{EJ}\right)w_i - 4w_{i+1} + w_{i+2} = \frac{q_i\Delta x^4}{EJ}
 \tag{2.11}$$

3. Perturbation-based n -th order approach to the finite difference analysis

Let us denote the corresponding random vector of the problem by $b(x)$, with the probability density function $p(b)$. Therefore, the m -th order central probabilistic moment of any random function of this parameter, namely $F(b)$, is defined as

$$\mu_m(F(b)) = \int_{-\infty}^{+\infty} (F(b) - E[F(b)])^m p(b) db
 \tag{3.1}$$

The basic idea of the stochastic perturbation approach follows the classical expansion idea and is based on the approximation of all input variables and state functions of the problem via truncated Taylor series about their spatial expectations in terms of the parameter $\varepsilon > 0$. For example, in the case of a random deflection, the n -th order truncated expansion may be written as

$$w(b) = w^0(b^0) + \sum_{k=1}^n \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k}
 \tag{3.2}$$

where

$$\varepsilon\Delta b = \varepsilon(b - b^0)
 \tag{3.3}$$

is the first variation of b about its expected value and, similarly,

$$\varepsilon^2(\Delta b)^2 = \varepsilon^2(b - b^0)^2
 \tag{3.4}$$

is the second variation of b around its expected value, where the n -th order variation can be expressed accordingly. Traditionally, the stochastic perturbation approach to all physical problems is entered by the respective perturbed equations of the zeroth, first and successively higher orders being a modification of the variational integral formulation. Hence, there holds — one zeroth order partial differential equation

$$E^0 J^0 \frac{d^4}{dx^4}(w^0(x)) = q^0 \tag{3.5}$$

— first order partial differential equation

$$\frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}\left(\frac{\partial w(x)}{\partial b}\right) = \frac{\partial q}{\partial b} \tag{3.6}$$

— second order partial differential equation

$$\begin{aligned} &\frac{\partial^2 E}{\partial b^2} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial^2 J}{\partial b^2} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}\left(\frac{\partial^2 w(x)}{\partial b^2}\right) + \\ &+ 2\left\{\frac{\partial E}{\partial b} \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}\left(\frac{\partial w(x)}{\partial b}\right) + \frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}\left(\frac{\partial w(x)}{\partial b}\right)\right\} = \frac{\partial^2 q}{\partial b^2} \end{aligned} \tag{3.7}$$

Quite similarly, one can derive the ascending order partial differential equations for the elastic beam on the elastic foundation starting from relation (2.9)

— one zeroth order partial differential equation

$$E^0 J^0 \frac{d^4}{dx^4}(w^0(x)) = -k^0 w^0(x) + q^0 \tag{3.8}$$

— first order partial differential equation

$$\begin{aligned} &\frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}\left(\frac{\partial w(x)}{\partial b}\right) = \\ &= \frac{\partial q}{\partial b} - \left(\frac{\partial k}{\partial b} w^0(x) + k^0 \frac{\partial w(x)}{\partial b}\right) \end{aligned} \tag{3.9}$$

as well as

— second order partial differential equation

$$\begin{aligned} &\frac{\partial^2 E}{\partial b^2} J^0 \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial^2 J}{\partial b^2} \frac{d^4}{dx^4}(w^0(x)) + E^0 J^0 \frac{d^4}{dx^4}\left(\frac{\partial^2 w(x)}{\partial b^2}\right) + \\ &+ 2\left\{\frac{\partial E}{\partial b} \frac{\partial J}{\partial b} \frac{d^4}{dx^4}(w^0(x)) + E^0 \frac{\partial J}{\partial b} \frac{d^4}{dx^4}\left(\frac{\partial w(x)}{\partial b}\right) + \frac{\partial E}{\partial b} J^0 \frac{d^4}{dx^4}\left(\frac{\partial w(x)}{\partial b}\right)\right\} = \\ &= \frac{\partial^2 q}{\partial b^2} - \left(\frac{\partial^2 k}{\partial b^2} w^0(x) + 2\frac{\partial k}{\partial b} \frac{\partial w(x)}{\partial b} + k^0 \frac{\partial^2 w(x)}{\partial b^2}\right) \end{aligned} \tag{3.10}$$

Derivation of higher order equations proceeds quite similarly – by systematic differentiation until the n -th order equation is recovered. Having solved those equations for $w^0(x)$ and their higher orders, respectively, (specifically its partial derivatives w.r.t. random input within all discrete points of the grid), we derive expressions for the expected values and other moments of elastic beam deflections. In order to calculate the expected values and higher order probabilistic moments of $w(x; b)$, the same Taylor expansion is employed for definitions of its probabilistic moments; there holds

$$E[w(b)] = \int_{-\infty}^{+\infty} w(b)p(b) db = \int_{-\infty}^{+\infty} \left(w^0(b^0) + \sum_{k=1}^n \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k} \right) p(b) db \quad (3.11)$$

If there is a high random dispersion in the input random variable and the symmetric probability density function is chosen, then the generalized expansion simplifies to

$$E[w(b)] = w^0(b^0) + \sum_{k=1}^n \frac{\varepsilon^{2k}}{(2k)!} \frac{\partial^{2k} w(b)}{\partial b^{2k}} \mu_{2k}(b) \quad (3.12)$$

where $\mu_{2k}(b)$ denotes $2k$ -th order probabilistic moment of the variable b . When the probability density function is defined as a Gaussian one with the standard deviation σ , we obtain additionally

$$\mu_{2k+1}(b) = 0 \qquad \mu_{2k}(b) = (2k - 1)! \sigma^{2k}(b) \quad (3.13)$$

Using such an extension of the random input, the desired efficiency of the expected values can be achieved by appropriate choice of the perturbation parameter and maximum order corresponding to the particular type of the input probability density function probabilistic moment interrelations, acceptable error of the computations, etc. This choice can be made reasonably by comparative studies with the Monte-Carlo simulations or theoretical results obtained by direct (i.e. symbolic) integration. Consequently, the m -th order probabilistic moment for the structural response function in the n -th order stochastic Taylor expansion is introduced as

$$\begin{aligned} \mu_m(w(b)) &= \int_{-\infty}^{+\infty} \left[\left(w^0(b^0) + \sum_{k=1}^n \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k} \right) - E[w(b)] \right]^m p(b) db = \\ &= \int_{-\infty}^{+\infty} \left(\sum_{k=1}^n \frac{\varepsilon^k}{k!} (\Delta b)^k \frac{\partial^k w(b)}{\partial b^k} \right)^m p(b) db \end{aligned} \quad (3.14)$$

Taking the first few components only, one can demonstrate that the relevant expansions for the 3-rd and 4-th order moments (Kamiński, 2007) equal to

$$\mu_3(w(b)) = \frac{3}{2}\varepsilon^4\mu_4(b)\left(\frac{\partial w}{\partial b}\right)^2\frac{\partial^2 w}{\partial b^2} + \frac{1}{8}\varepsilon^6\mu_6(b)\left(\frac{\partial^2 w}{\partial b^2}\right)^3 \quad (3.15)$$

and

$$\mu_4(w(b)) = \frac{3}{2}\varepsilon^4\mu_4(b)\left(\frac{\partial w}{\partial b}\right)^4 + \frac{3}{2}\varepsilon^6\mu_6(b)\left(\frac{\partial w}{\partial b}\frac{\partial^2 w}{\partial b^2}\right)^2 + \frac{1}{16}\varepsilon^8\mu_8(b)\left(\frac{\partial w}{\partial b}\right)^3\left(\frac{\partial^2 w}{\partial b^2}\right)^4 \quad (3.16)$$

The discrete equations for the Stochastic Finite Difference Method built upon the above equations for the perturbation-based analysis are essentially different in the case of an elastic beam with and without the foundation beneath it. Putting here $k = 0$ returns various order relations, as for example:

— zeroth order relation

$$w_{i-2}^0 - 4w_{i-1}^0 + 6w_i^0 - 4w_{i+1}^0 + w_{i+2}^0 = \frac{q_i^0(\Delta x^0)^4}{E^0 J^0} \quad (3.17)$$

— n -th order relation

$$\frac{\partial^n w_{i-2}}{\partial b^n} - 4\frac{\partial^n w_{i-1}}{\partial b^n} + 6\frac{\partial^n w_i}{\partial b^n} - 4\frac{\partial^n w_{i+1}}{\partial b^n} + \frac{\partial^n w_{i+2}}{\partial b^n} = \frac{\partial^n}{\partial b^n}\left(\frac{q_i \Delta x^4}{EJ}\right) \quad (3.18)$$

so that the uniformly distributed load gives here nonzero equations of up to the first order only, random length results in up to the fourth order equations, whereas the cross-sectional and/or material randomness brings here an infinite number of equations at the R.H.S. Looking for the perturbation-based SFDM equations for beams on the elastic foundation, one can receive:

— zeroth order equations

$$w_{i-2}^0 - 4w_{i-1}^0 + \left(6 + \frac{k^0(\Delta x^0)^4}{E^0 J^0}\right)w_i^0 - 4w_{i+1}^0 + w_{i+2}^0 = \frac{q_i^0(\Delta x^0)^4}{E^0 J^0} \quad (3.19)$$

— n -th order equations

$$\begin{aligned} &\frac{\partial^n w_{i-2}}{\partial b^n} - 4\frac{\partial^n w_{i-1}}{\partial b^n} + 6\frac{\partial^n w_i}{\partial b^n} + \sum_{p=1}^n \binom{n}{p} \frac{\partial^p}{\partial b^p}\left(\frac{k\Delta x^4}{EJ}\right) \frac{\partial^{n-p} w_i}{\partial b^{n-p}} + \\ &-4\frac{\partial^n w_{i+1}}{\partial b^n} + \frac{\partial^n w_{i+2}}{\partial b^n} = \frac{\partial^n}{\partial b^n}\left(\frac{q_i \Delta x^4}{EJ}\right) \end{aligned} \quad (3.20)$$

Let us also note that $n - 1$ of those equations may be generated automatically from the zeroth order formula using the symbolic computations systems like MAPLE in the computational illustrations below.

4. Computational illustrations

4.1. Deflection of the linear elastic beam with linearly varying cross-sectional area

Let us determine the first four probabilistic moments for the cantilever beam with linearly varying cross-sectional area under the constant distributed load $q = 1.0\text{kN/m}$. Young’s modulus of this beam is introduced as the input Gaussian random variable, where the expected value is given as $E[E] = 206.01\text{ GPa}$, the grid for this structures consisting of 6 elements with the constant length equal to $\Delta x = 0.1\text{ m}$ is proposed below (see Pietrzak *et al.*, 1986 for deterministic test).

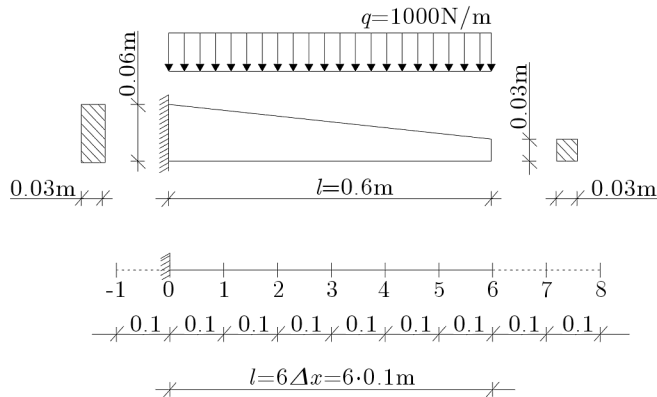


Fig. 3. A cantilever beam with linearly varying cross-sectional area

The following central finite difference equations hold true in this particular case

$$\begin{aligned}
 J_0 w_{-1} - 2(J_0 + J_1)w_0 + (J_0 + 4J_1 + J_2)w_1 - 2(J_1 + J_2)w_2 + J_2 w_3 &= \frac{q\Delta x^4}{E} \\
 J_1 w_0 - 2(J_1 + J_2)w_1 + (J_1 + 4J_2 + J_3)w_2 - 2(J_2 + J_3)w_3 + J_3 w_4 &= \frac{q\Delta x^4}{E} \\
 J_2 w_1 - 2(J_2 + J_3)w_2 + (J_2 + 4J_3 + J_4)w_3 - 2(J_3 + J_4)w_4 + J_4 w_5 &= \frac{q\Delta x^4}{E} \\
 J_3 w_2 - 2(J_3 + J_4)w_3 + (J_3 + 4J_4 + J_5)w_4 - 2(J_4 + J_5)w_5 + J_5 w_6 &= \frac{q\Delta x^4}{E} \\
 J_4 w_3 - 2(J_4 + J_5)w_4 + (J_4 + 4J_5 + J_6)w_5 - 2(J_5 + J_6)w_6 + J_6 w_7 &= \frac{q\Delta x^4}{E}
 \end{aligned}
 \tag{4.1}$$

$$J_5 w_4 - 2(J_5 + J_6)w_5 + (J_5 + 4J_6 + J_7)w_6 - 2(J_6 + J_7)w_7 + J_7 w_8 = \frac{q\Delta x^4}{E}$$

Introducing the fictitious nodes as well as using kinematic boundary conditions, we obtain

$$\begin{aligned} (2J_0 + 4J_1 + J_2)w_1 - 2(J_1 + J_2)w_2 + J_2 w_3 &= \frac{q\Delta x^4}{E} \\ -2(J_1 + J_2)w_1 + (J_1 + 4J_2 + J_3)w_2 - 2(J_2 + J_3)w_3 + J_3 w_4 &= \frac{q\Delta x^4}{E} \\ J_2 w_1 - 2(J_2 + J_3)w_2 + (J_2 + 4J_3 + J_4)w_3 - 2(J_3 + J_4)w_4 + J_4 w_5 &= \frac{q\Delta x^4}{E} \\ J_3 w_2 - 2(J_3 + J_4)w_3 + (J_3 + 4J_4 + J_5)w_4 - 2(J_4 + J_5)w_5 + J_5 w_6 &= \frac{q\Delta x^4}{E} \\ J_4 w_3 - 2(J_4 + J_5)w_4 + (J_4 + 4J_5)w_5 - 2J_5 w_6 &= \frac{q\Delta x^4}{E} \\ 2J_5 w_4 - 4J_5 w_5 + 2J_5 w_6 &= \frac{q\Delta x^4}{E} \end{aligned} \tag{4.2}$$

This system of linear equations may be used for further analytical partial differentiation with respect to the random input variable and formation of the higher order equations. The symbolic computations package MAPLE, v. 11 is employed for derivation of up to the tenth order equilibrium perturbation-based equations and determination of the expected values (each time for the maximum deflection at the cantilever end) according to the 2nd, 4th, 6th, 8th and 10th approaches (Fig. 4, the additional order results are marked with the corresponding number close to the right vertical axes), computations of standard deviations and variances in the framework of the 2nd, 4th and 6th order theories (Figs. 5 and 6, the successive orders notations as before) as well as, finally, derivation of the third (Fig. 7) and fourth central probabilistic moments (Fig. 8). All those computations are performed with respect to the perturbation parameter ε belonging to the interval $[0.8, 1.2]$ and, secondly, the coefficient of variation of the randomized Young's modulus $\alpha(b)$ – standard deviation vs. the expected value – for the beam taken from the interval $[0.0, 0.3]$; the probabilistic moments are given here as the decimal powers, respectively. Problems with this coefficient equal to 0 are adequate to deterministic tests, of course, so that they can be treated as evaluation tests being quite insensitive to the perturbation parameter at all. Generally, it is clear that the particular values of all probabilistic characteristics increase together with the order of the method, and, furthermore, that the difference between the results of the

ascending order methods systematically decrease. It is important to mention that for the clarity of the presentation, those differences are visualised using the largest perturbation parameter value – 1.2 remaining not so transparent for its value taken in most of the engineering computations as 1.0. Anyway, even for the most extreme combinations of the problem parameters, the differences between neighboring order results are smaller than a single percent of the computed deflection value.

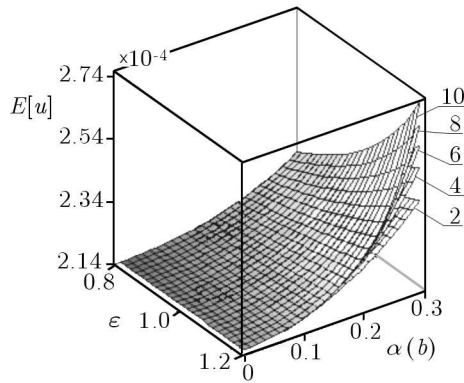


Fig. 4. Expected values as function of the perturbation parameter and input coefficient of variation (2nd, 4th, 6th, 8th and 10th orders)

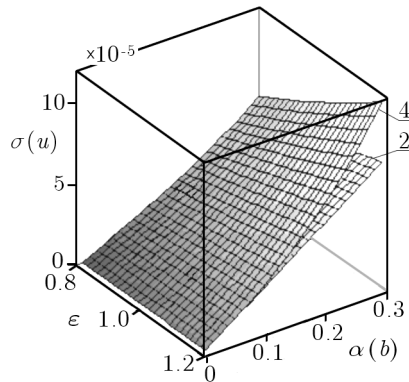


Fig. 5. Standard deviations as function of the perturbation parameter and input coefficient of variation (2nd and 4th orders)

Further, as one may recognize, an increase of the perturbation parameter and, at the same time, the additional increase of the coefficient of variation each time leads to the nonlinear increase of the probabilistic moment being computed. The impact of the perturbation parameter is very small in the se-

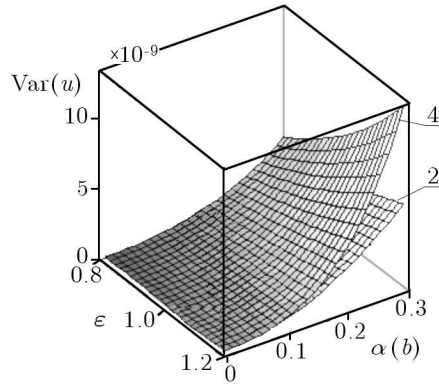


Fig. 6. Variances as function of the perturbation parameter and input coefficient of variation (2nd and 4th orders)

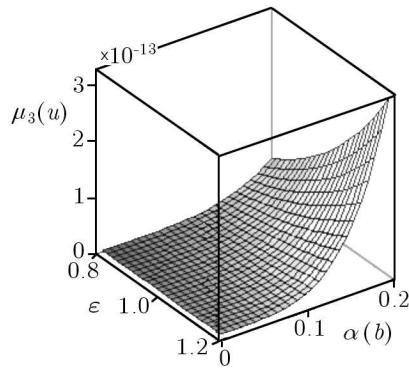


Fig. 7. 3rd central probabilistic moments as function of the perturbation parameter and input coefficient of variation

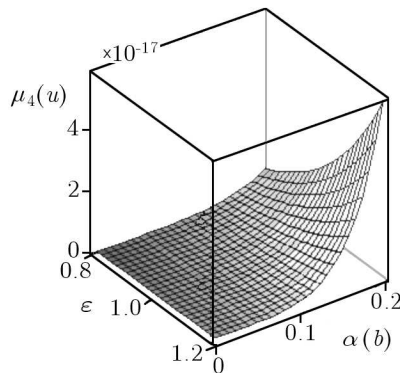


Fig. 8. 4th central probabilistic moments as function of the perturbation parameter and input coefficient of variation

cond order approach and becomes more and more important together with an increase of the theory order having similar importance as the variation coefficient for the tenth order technique. Neglecting the theory order, the coefficient of variation of the input Young modulus being randomized here is more decisive for the parameter variability of the probabilistic characteristics of deflections.

A comparison of the deflections in the deterministic case together with the expected values computed for the standard deviation of it equal to 10% of the Young modulus expected value is given below. As it is clear from this table, the expected values are larger than the corresponding deterministic values, which of course reflects the perturbation-based formula for expectations, see Eq. (3.2), where deterministic value is the first component, while the second one remains always positive.

Table 1. Deflections and their expected values for grid points in test No. 1

Node number	w [m]	$E[w]$ [m]
1	$0.828443 \cdot 10^{-5}$	$0.816330 \cdot 10^{-5}$
2	$0.311575 \cdot 10^{-4}$	$0.310456 \cdot 10^{-4}$
3	$0.664578 \cdot 10^{-4}$	$0.664655 \cdot 10^{-4}$
4	$0.111347 \cdot 10^{-3}$	$0.111559 \cdot 10^{-3}$
5	$0.162304 \cdot 10^{-3}$	$0.162775 \cdot 10^{-3}$
6	$0.215525 \cdot 10^{-3}$	$0.216275 \cdot 10^{-3}$

4.2. Deflection of the linear elastic beam resting on the elastic Winkler foundation

The second case deals with quite a similar cantilever beam with the constant cross-sectional area however, where $J = 1.71 \cdot 10^{-6} \text{ m}^4$, resting now on the elastic foundation characterised by the coefficient $k = 5 \cdot 10^7 \text{ N/m}^3$. The external distributed load is constant along this beam $q = 10000 \text{ N/m}$ for $l = 3.0 \text{ m}$; the structure is discretized with $\Delta x = 0.5 \text{ m}$, and, analogously to the previous case, the Young modulus is the input Gaussian random variable with the expectation equal to $E = 206.01 \text{ GPa}$ (see the deterministic test in Pietrzak *et al.*, 1986). The fundamental zeroth order SFDM equation has the following form

$$w_{i-2} - 4w_{i-1} + 14.8709w_i - 4w_{i+1} + w_{i+2} = 0.0177416 \tag{4.3}$$

with the following boundary conditions

$$\begin{aligned} w_0 &= 0 & w_{-1} &= w_1 \\ w_7 &= 2w_6 - w_5 & w_8 &= 4w_6 - 4w_5 + w_4 \end{aligned} \quad (4.4)$$

where the nodes 7 and 8 are fictitious, as before. A comparison of the deflections in the deterministic case together with the expected values computed for the standard deviation of it equal to 10% of the Young modulus expected value is given below. As it is clear from this table, the expected values are larger than the corresponding deterministic values, which of course once more reflects the perturbation-based formula for the expectations.

Table 2. Deflections and their expected values for grid points in test No. 2

Node number	w [m]	$E[w]$ [m]
1	$0.149565 \cdot 10^{-2}$	$0.151058 \cdot 10^{-2}$
2	$0.200862 \cdot 10^{-2}$	$0.202867 \cdot 10^{-2}$
3	$0.203892 \cdot 10^{-2}$	$0.205927 \cdot 10^{-2}$
4	$0.200995 \cdot 10^{-2}$	$0.203001 \cdot 10^{-2}$
5	$0.199978 \cdot 10^{-2}$	$0.201974 \cdot 10^{-2}$
6	$0.199808 \cdot 10^{-2}$	$0.201802 \cdot 10^{-2}$

The polynomial response function between the Young modulus of this elastic beam and its deflection at the right-hand side has been numerically determined using the polynomial interpolation option in the system MAPLE. This function is shown in Fig. 9. It was the basis for further symbolic derivation of higher order partial derivatives of the deflection with respect to the randomized Young modulus and probabilistic moments. The presentation of computational results is quite similar to that given in Sec. 4.1 – probabilistic moments for the maximum deflection at the end of the cantilever beam are collected in Figs. 10-14: the expected values, standard deviations, variances as well as third and fourth order central probabilistic moments. The elastic foundation for the elastic beam having exactly the same parameters reduces the maximum deflection by two orders of the magnitude (by comparison of Figs. 4 and 10), so that the standard deviations are reduced by two orders as well (the following numbers are slightly different) the variances – by four orders (as the second powers of the standard deviations), whereas the third and fourth central probabilistic moments – by six and eight orders, respectively. Finally, comparing the ascending order moment diagrams, it is clear

that the higher moment is computed, the bigger influence of the perturbation parameter on the overall result is noticed.

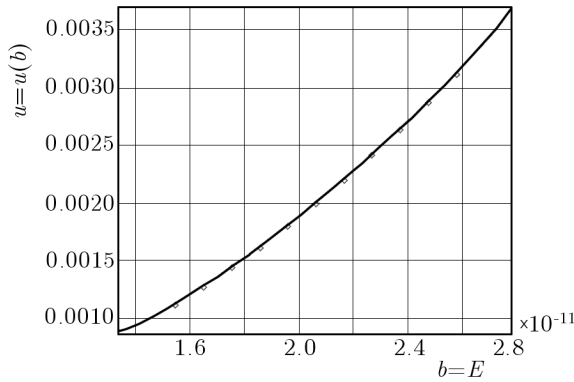


Fig. 9. The response function of the 6th node deflections of the beam

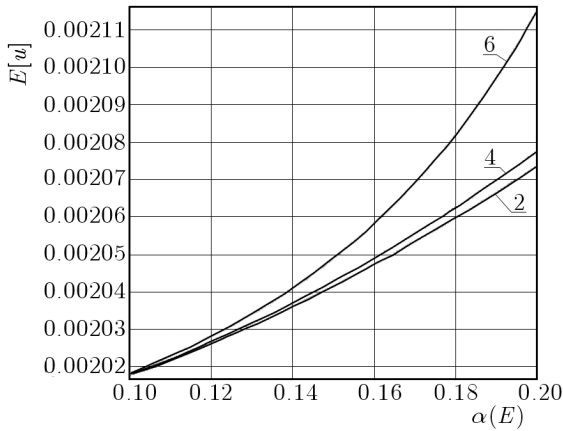


Fig. 10. Expected values as function of the perturbation parameter and input coefficient of variation (2nd, 4th and 6th orders)

This study brings also the following important remark – looking for the variability of 3rd and 4th order probabilistic moments in the range of applicability of the second order second moment technique within the interval $[0.0, 0.1]$ (Kleiber and Hien, 1992), one can observe that they are almost insensitive to any parameter of this study and practically do not differ much from 0. Their largest increase is noticed for the input coefficient of variation larger than 0.15 (the bigger coefficient – the larger moment increase).

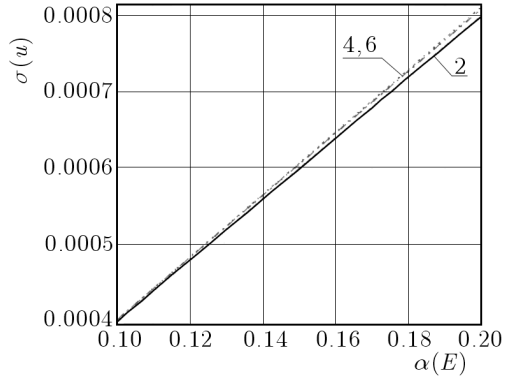


Fig. 11. Standard deviations as function of the perturbation parameter and input coefficient of variation (2nd, 4th and 6th orders)

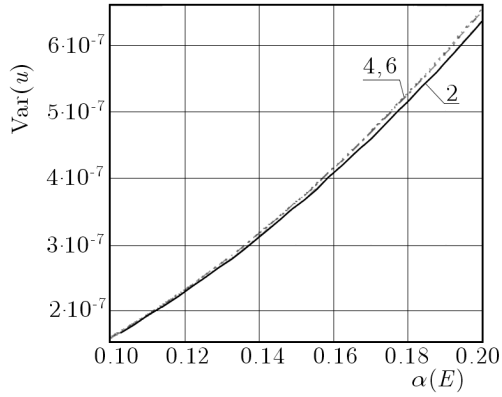


Fig. 12. Variances as function of the perturbation parameter and input coefficient of variation (2nd, 4th and 6th orders)

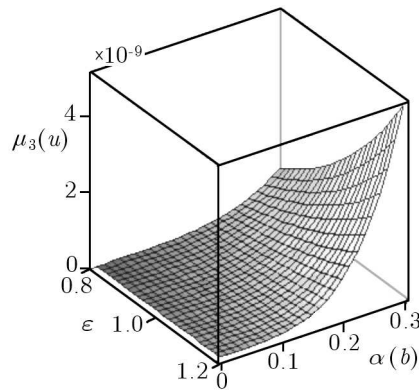


Fig. 13. 3rd central probabilistic moments as function of the perturbation parameter and input coefficient of variation

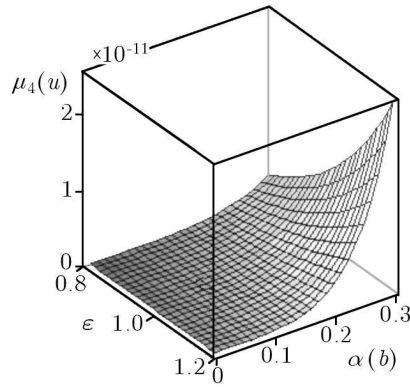


Fig. 14. 4th central probabilistic moments as function of the perturbation parameter and input coefficient of variation

5. Concluding remarks

- Theoretical considerations presented here and the additional computational implementation show how to make an efficient stochastic extension of any order for the well-known Finite Difference Method in the case when some parameters in the governing differential equations remain uncertain. This uncertainty is represented by the truncated Gaussian input random variable uniquely defined by the first two probabilistic moments and, by formation and straightforward solution of the ascending order equilibrium equations, results in probabilistic moments of the structural response. Although the entire methodology is displayed on the example of elastic statistically homogeneous and isotropic beams, it remains so general that any 2 and/or 3D problems may be solved quite analogously. The probabilistic moments of up to the fourth order are computed symbolically, using the system MAPLE, so that unlike most of the previous implementations (Kamiński, 2001; Kleiber and Hien, 1992), they include analytical functions with respect to the perturbation parameter as well as the input coefficient of variation for the random input. Furthermore, having in the mind the stochastic convergence of the entire method, we compare up to the tenth order perturbation theories in the case of expected values and up to sixth order approaches in the case of second order characteristics.
- As one may expect, the computations performed show that the coefficient of variation for the random input was determined as more impor-

tant than the perturbation parameter. Further, the differences between the neighboring order approaches decrease together with the increase of this order, which partially demonstrates the convergence of this methodology. Comparison of the expected values for the beam without and with the elastic foundation shows that the application of this foundation decreases the maximum deflection by more than two orders of magnitude. Finally, we need to point out that the Stochastic Finite Difference Method may be directly used for any order stochastic reliability studies since all moments and probabilistic characteristics are easily available. Although the computer code is not very large, it can be easily generalized towards larger space dimensions of structures, other probability density functions as well as to automatic formation of the ascending order equations typical not only for the SFDM but also for other discrete techniques implemented before.

Acknowledgment

The author would like to acknowledge the financial support of the research grant from the Ministry of Research and Higher Education NN 519 386 636.

References

1. COLLATZ L., 1960, *Numerical Methods for Partial Differential Equations*, PSP, Warsaw [in Polish]
2. GHANEM R.G., SPANOS P.D., 1991, *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, Berlin
3. HURTADO J.E., BARBAT A.H., 1998, Monte-Carlo techniques in computational stochastic mechanics, *Archives of Computer Methods in Engineering*, **5**, 3-30
4. KAMIŃSKI M., 2001, Stochastic perturbation approach in vibration analysis using finite difference method, *Journal of Sound and Vibration*, **251**, 4, 651-670
5. KAMIŃSKI M., 2007, Generalized perturbation-based stochastic finite element method in elastostatics, *Computers and Structures*, **85**, 586-594
6. KLEIBER M., HIEN T.D., 1992, *The Stochastic Finite Element Method*, Wiley, Chichester
7. LISZKA T., ORKISZ J., 1980, The finite difference method at arbitrary irregular grids and its applications in applied mechanics, *Computers and Structures*, **11**, 83-95

8. MINKOWYCZ W.J. ET AL., 1988, *Handbook of Numerical Heat Transfer*, Wiley-Interscience, New York
9. PIETRZAK J., RAKOWSKI G., WRZEŚNIEWSKI K., 1986, *Matrix Analysis of Structures*, PSP, Warsaw [in Polish]
10. TAFLOVE A., 1998, *Advances in Computational Electrodynamics: The Finite Difference Time Domain Method*, Artech House, Norwood
11. WASOW W.R., FORSYTHE G.E., 1959, *Finite Difference Methods for Partial Differential Equations*, Wiley & Sons, New York-London

O zastosowaniu uogólnionej Stochastycznej Metody Elementów Skończonych opartej na technice perturbacji do analizy belek sprężystych

Streszczenie

Głównym celem niniejszej pracy jest zastosowanie probabilistycznej analizy numerycznej składającej się z uogólnionej metody perturbacji opartej na szeregu Taylora ze współczynnikami losowymi oraz z Metody Różnic Skończonych dla siatek regularnych. Jak wykazano w przykładach numerycznych, używając tego podejścia można również wyznaczać momenty probabilistyczne wyższych rzędów dla dowolnych funkcji losowych, co było niemożliwe dla zastosowań metod drugiego rzędu znanych wcześniej. Zastosowany tutaj algorytm numeryczny jest oparty na różniczkowaniu bezpośrednim hierarchicznych równań równowagi względem przyjętych zmiennych i na symbolicznym wyznaczaniu momentów i charakterystyk losowych przy pomocy programu MAPLE.

Manuscript received December 30, 2008; accepted for print May 11, 2009