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Failure criteria of viscoelastic materials under multiaxial loads

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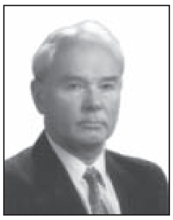
ABSTRACT



The load capacity of homogeneous, isotropic viscoelastic materials subjected to multiaxial stresses is considered. For this purpose uniaxial equivalent stresses in selected load cases are determined and relevant criteria applied. It is shown that in the case of constant load the yield criterion does not differ from that for perfectly elastic materials. Similar conclusion has been drawn for the fatigue and yield criteria at in-phase and out-of-phase stresses. On the contrary, the criteria derived for viscoelastic materials subjected to periodic loads do not coincide with those for perfectly elastic materials.

Keywords: viscoelastic material, multiaxial stress, fatigue criteria, yield criterion, static load, vibratory load

INTRODUCTION

Even in the region below the limit of proportionality, metals are not perfectly elastic, and their deformation is accompanied by internal friction. As a result, their strain response to static loads is time-dependent, and when the load varies in time, the strain lags behind the stress. In particular, after the rapid shortening of a bar that occurs when its axial tensile load is removed, the bar continues to shorten gradually until the initial length has been reached. This gradual return to the initial length following unloading is called anelastic behaviour or viscoelasticity [1, 2]. Anelastic strain differs from plastic strain because it is recoverable rather than permanent, and it differs from elastic strain because it is recoverable at a rate which is slow in comparison to the rate of recovery of elastic strain. The elastic behaviour of metals and structural steels is much more significant than their anelastic properties, which enables the strength criteria to be formulated with the aid of the Hooke's law for perfectly elastic materials. However, as shown in the present paper, dissipative properties of engineering materials may be also taken into account in design considerations, especially in the case of periodic load when an equivalent reduced stress is to be determined in calculations of a structural limit state.

During the last two decades, the emphasis in structural design has been moving from the allowable stress design to the limit state design [2]. Generally, four types of limit states may be specified:
- serviceability limit state,
- ultimate or yield limit state,
- fatigue limit state,
- accidental limit state.

This paper concerns the yield and fatigue limit states of viscoelastic materials under combined static and vibratory loads.

YIELD CRITERION OF VISCOELASTIC MATERIALS UNDER MULTIAXIAL STATIC LOADS

The stress and strain at which a material either begins to yield or fractures due to a uniaxial load can be measured relatively easily. But for an arbitrarily shaped body under arbitrary loads, the prediction of yield or fracture is very difficult. Some criterion is needed to make predictions without testing every material under every possible loading. An ideal criterion would be one that is based on a simple uniaxial test. Then the normal stress, normal strain, shear stress, the strain energy, or the distortion energy, among other possibilities, could be taken into account. Each of these reaches its failure value at the same load in a uniaxial test, but this is no longer true if the state of stress is either two- or three-dimensional. Therefore various theories regarding the initiation of yielding have been developed. The earliest, and the simplest, relation describing the conditions for initiating plastic flow under static or quasi-static load is the shear stress law. This law states that the metal will yield when the largest shear stress reaches a critical value, irrespective of the stress state. A somewhat more accurate law is the "energy-of-distortion" criterion, also called the Huber-Mises-Hencky (HMH) strength theory [2-4]. It embodies the physical hypothesis that yielding occurs when a certain critical value of distortion energy is reached, which corresponds to that where the equivalent reduced stress, σ_{eo}, reaches the yield strength, R_e, determined from the uniaxial tension test. Experiments show that this is an excellent criterion for the yielding of ductile materials [2]. This criterion is also useful for ductile materials under multiaxial proportional loading and high-cycle fatigue [5]. Therefore it will be also used in the present paper.

To begin with, let us consider a perfectly elastic solid under general state of static load resulting in normal and shear stress and strain components σ_{jo}, ε_{jo}, τ_{ko}, γ_{ko}, (j = x, y, z; k = xy, yz, zx).

The elastic strain energy per unit volume is [4]:

$$\psi_o = \frac{1}{2}\left(\sum_j \sigma_{jo}\varepsilon_{jo} + \sum_k \tau_{ko}\gamma_{ko}\right) \tag{1}$$

where:

$$\begin{aligned} \varepsilon_{xo} &= \frac{1}{E}\left[\sigma_{xo} - \upsilon\left(\sigma_{yo} + \sigma_{zo}\right)\right] \\ \varepsilon_{yo} &= \frac{1}{E}\left[\sigma_{yo} - \upsilon\left(\sigma_{xo} + \sigma_{zo}\right)\right] \\ \varepsilon_{zo} &= \frac{1}{E}\left[\sigma_{zo} - \upsilon\left(\sigma_{xo} + \sigma_{yo}\right)\right] \\ \gamma_{ko} &= \frac{1}{G}\tau_{ko} \end{aligned} \tag{2}$$

In Eqs (2), expressing the Hooke's law for multiaxial stress in elastic solids, E is the Young modulus, υ is the Poisson's ratio, and:

$$G = \frac{E}{2(1+\upsilon)} \tag{3}$$

is the shear modulus. Substitution of Eqs (2) and (3) into Eq. (1) results in:

$$\psi_o = \frac{1}{E}\left[\begin{array}{c} \frac{1}{2}\left(\sigma_{xo} + \sigma_{yo} + \sigma_{zo}\right)^2 + \\ +\left(1+\upsilon\right)\left(\begin{array}{c}\tau_{xyo}^2 + \tau_{yzo}^2 + \tau_{zxo}^2 - \sigma_{xo}\sigma_{yo} + \\ -\sigma_{yo}\sigma_{zo} - \sigma_{zo}\sigma_{xo}\end{array}\right)\end{array}\right] \tag{4}$$

which can be rewritten as [4]:

$$\psi_o = \psi_{do} + \psi_{vo} \tag{5}$$

where:

$$\psi_{do} = \frac{1+\upsilon}{6E}\left[\begin{array}{c}\left(\sigma_{xo} - \sigma_{yo}\right)^2 + \\ +\left(\sigma_{yo} - \sigma_{zo}\right)^2 + \left(\sigma_{zo} - \sigma_{xo}\right)^2 + \\ +6\left(\tau_{xyo}^2 + \tau_{yzo}^2 + \tau_{zxo}^2\right)\end{array}\right] \tag{6}$$

is the strain energy of distortion, and:

$$\psi_{vo} = \frac{1-2\upsilon}{6E}\left(\sigma_{xo} + \sigma_{yo} + \sigma_{zo}\right)^2 \tag{7}$$

is the strain energy of volume change. According to the HMH theory, at yield, the distortion energy at a point in a three-dimensional state of stress is equal to the distortion energy at yield in the uniaxial case. Thus:

$$\psi_{deo} = \psi_{do} \tag{8}$$

where:

$$\psi_{deo} = \frac{1+\upsilon}{3E}\sigma_{eo}^2 \tag{9}$$

is the strain energy of distortion per unit volume under reduced stress σ_{eo} in tension. Hence [2, 4, 5]:

$$\sigma_{eo} = \left[\begin{array}{c}\sigma_{xo}^2 + \sigma_{yo}^2 + \sigma_{zo}^2 - \sigma_{xo}\sigma_{yo} - \sigma_{yo}\sigma_{zo} + \\ -\sigma_{zo}\sigma_{xo} + 3\left(\tau_{xyo}^2 + \tau_{yzo}^2 + \tau_{zxo}^2\right)\end{array}\right]^{1/2} \tag{10}$$

and the criterion in question reads:

$$\sigma_{eo} < R_e \tag{11}$$

Now suppose that the considered load is applied to a viscoelastic solid at the time t = 0 and the stress components σ_{jo} and τ_{ko} are maintained constant. Then the constitutive equations for strains are [6]:

$$\begin{aligned} \varepsilon_j(t) &= \varepsilon_{jo}\left(1 - e^{-\frac{E}{\eta}t}\right) \\ \gamma_k(t) &= \gamma_{ko}\left(1 - e^{-\frac{E}{\eta}t}\right) \end{aligned} \tag{12}$$

and the elastic strain energy becomes time-dependent:

$$\psi(t) = \frac{1}{2}\left[\sum_j \sigma_{jo}\varepsilon_j(t) + \sum_k \tau_{ko}\gamma_k(t)\right] \tag{13}$$

In Eqs (12), η is the coefficient of viscous damping of normal strain in the Kelvin-Voigt's model of the material [2, 6]. Substitution of Eqs (2) and (12) into Eq. (13) gives:

$$\psi(t) = \psi_o\left(1 - e^{-\frac{E}{\eta}t}\right) \tag{14}$$

with:

$$\lim_{t\to\infty}\psi(t) = \psi_o \tag{15}$$

Hence it is clear that Eq. (10) and the criterion (11) are applicable also to viscoelastic materials under constant loads.

FAILURE CRITERIA OF VISCOELASTIC MATERIALS UNDER MULTIAXIAL HARMONIC LOADS

Under dynamic loading conditions, the strength of a structural element is degraded due to the cyclic application of load or strain which may lead to fatigue damage. It is why in this paper not only the yield criterion but also fatigue criteria are considered.

The fatigue load that a structure can withstand is often significantly less than the load which it would be capable of if the load were applied only once. For the case of combined dynamic loading, where torsion, bending and/or tension loads vary in time, no one strength theory is universally accepted, and all existing multiaxial fatigue criteria can demonstrate large scatter [7]. Only multiaxial in-phase stresses with constant principal directions can be treated fairly well using the conventional strength theories [8]. In [9] an attempt was made to extend the application range of the HMH theory to other stress states, in particular to those with variable principal directions. In what follows, the results obtained in [6, 9] are taken into account.

If a multiaxial harmonic load is producing in a perfectly elastic solid the zero mean stress with in-phase components:

$$\begin{aligned} \sigma_j &= \sigma_{ja}\sin\omega t \;\; ; \quad j = x, y, z \\ \tau_k &= \tau_{ka}\sin\omega t \;\; ; \quad k = xy, yz, zx \end{aligned} \tag{16}$$

the equivalent reduced stress in tension-compression takes the form:

$$\sigma_e = \sigma_{ea} \sin\omega t \tag{17}$$

where:

σ_{ja}, τ_{ka} – amplitudes of the stress components ; ω – circular frequency ; σ_{ea} – amplitude of the reduced stress given by [8]:

$$\sigma_{ea} = \begin{bmatrix} \sigma_{xa}^2 + \sigma_{ya}^2 + \sigma_{za}^2 - \sigma_{xa}\sigma_{ya} - \sigma_{ya}\sigma_{za} + \\ -\sigma_{za}\sigma_{xa} + 3\left(\tau_{xya}^2 + \tau_{yza}^2 + \tau_{zxa}^2\right) \end{bmatrix}^{1/2} \tag{18}$$

Consequently, the criterion of an infinite fatigue life and the yield criterion read:

$$\sigma_{ea} < Z_{rc} \tag{19}$$

$$\sigma_{ea} < R_e \tag{20}$$

where:

Z_{rc} – the fatigue limit of the material under fully reversed tension-compression.

In the high-cycle fatigue regime, the criterion of a finite fatigue life based on the S-N curve (Wöhler's curve) is commonly accepted [10]. According to this approach, in the case of fully reversed tension-compression under the stress (17) one gets:

$$\frac{N_d}{K}\sigma_{ea}^m < 1 \tag{21}$$

where:

N_d – required number of stress cycles to achieve a given design life
K – fatigue strength coefficient
m – fatigue strength exponent.

If, however, the in-phase stress (16) is applied to a viscoelastic material, its strain response is [6]:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{xa} - \upsilon\left(\sigma_{ya} + \sigma_{za}\right)\right]\sin\left(\omega t - \alpha\right) \\
\varepsilon_y &= \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{ya} - \upsilon\left(\sigma_{xa} + \sigma_{za}\right)\right]\sin\left(\omega t - \alpha\right) \\
\varepsilon_z &= \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{za} - \upsilon\left(\sigma_{xa} + \sigma_{ya}\right)\right]\sin\left(\omega t - \alpha\right) \\
\gamma_k &= \frac{2\left(1+\upsilon\right)}{\sqrt{E^2 + \eta^2\omega^2}}\tau_{ka}\sin\left(\omega t - \alpha\right), \quad \alpha = \mathrm{arctg}\frac{\eta\omega}{E}
\end{aligned} \tag{22}$$

So, the elastic strain energy per unit volume:

$$\psi = \frac{1}{2}\left(\sum_j \sigma_j\varepsilon_j + \sum_k \tau_k\gamma_k\right) \tag{23}$$

becomes:

$$\begin{aligned}
\psi &= \frac{1}{2\sqrt{E^2 + \eta^2\omega^2}}\Big\{\sigma_{xa}\left[\sigma_{xa} - \upsilon\left(\sigma_{ya} + \sigma_{za}\right)\right] + \sigma_{ya}\left[\sigma_{ya} - \upsilon\left(\sigma_{xa} + \sigma_{za}\right)\right] + \sigma_{za}\left[\sigma_{za} - \upsilon\left(\sigma_{xa} + \sigma_{ya}\right)\right] + \\
&+ 2\left(1+\upsilon\right)\sum_k \tau_{ka}^2\Big\}\sin\omega t\sin\left(\omega t - \alpha\right) = \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\Big[\frac{1}{2}\left(\sigma_{xa} + \sigma_{ya} + \sigma_{za}\right)^2 + \\
&+ \left(1+\upsilon\right)\left(\tau_{xya}^2 + \tau_{yza}^2 + \tau_{zxa}^2 - \sigma_{xa}\sigma_{ya} - \sigma_{ya}\sigma_{za} - \sigma_{za}\sigma_{xa}\right)\Big]\sin\omega t\sin\left(\omega t - \alpha\right)
\end{aligned} \tag{24}$$

On the other hand, for the reduced stress (17) and corresponding reduced strain:

$$\varepsilon_e = \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\sigma_{ea}\sin\left(\omega t - \alpha\right) \tag{25}$$

the elastic strain energy per unit volume is given by:

$$\psi_e = \frac{1}{2}\sigma_e\varepsilon_e = \frac{1}{2\sqrt{E^2 + \eta^2\omega^2}}\sigma_{ea}^2\sin\omega t\sin\left(\omega t - \alpha\right) \tag{26}$$

Consequently, the relationships for distortion energies per unit volume in the actual and reduced stress states read:

$$\psi_d = \frac{1+\upsilon}{6\sqrt{E^2+\eta^2\omega^2}}\left[\left(\sigma_{xa}-\sigma_{ya}\right)^2+\left(\sigma_{ya}-\sigma_{za}\right)^2+\left(\sigma_{za}-\sigma_{xa}\right)^2+6\left(\tau_{xya}^2+\tau_{yza}^2+\tau_{zxa}^2\right)\right]\sin\omega t\sin(\omega t-\alpha) \quad (27)$$

$$\psi_{de} = \frac{1+\upsilon}{3\sqrt{E^2+\eta^2\omega^2}}\sigma_{ea}^2\sin\omega t\sin(\omega t-\alpha) \quad (28)$$

Equating the right-hand sides of Eqs (27) and (28) yields again Eq. (18) so that the criteria (19) through (21) remain valid.

In the case of non-zero mean stress with in-phase components:

$$\sigma_j = \sigma_{jo} + \sigma_{ja}\sin\omega t \quad ; \quad \tau_k = \tau_{ko} + \tau_{ka}\sin\omega t \quad (29)$$

the equivalent reduced stress can be expressed as:

$$\sigma_e = \sigma_{eo} + \sigma_{ea}\sin\omega t \quad (30)$$

where: σ_{eo} is the mean value of the reduced stress. In Eq. (30), there are two unknown parameters of the reduced stress (σ_{eo} and σ_{ea}) which cannot be determined from the single equation, corresponding to the HMH theory, without additional assumptions. On the other hand, since this theory can be applied to the stress (29) when:

$$\sigma_{jm} \neq 0 \quad ; \quad \tau_{km} \neq 0 \quad ; \quad \sigma_{ja} = \tau_{ka} = 0$$

and when:

$$\sigma_{jm} = \tau_{km} = 0 \quad ; \quad \sigma_{ja} \neq 0 \quad ; \quad \tau_{ka} \neq 0$$

it should be also applicable to the stress (29) when:

$$\sigma_{jm} \neq 0 \quad ; \quad \tau_{km} \neq 0 \quad ; \quad \sigma_{ja} \neq 0 \quad ; \quad \tau_{ka} \neq 0$$

Therefore the following hypothesis ("average-distortion-energy strength hypothesis") was formulated [11].

The reduced stress (30) *is equivalent in terms of static and dynamic effort of a material to the stress* (29) *if:*
(i) *the time-independent parts of distortion energies per unit volume in these both stress states are equal*
(ii) *the reduced stress and the stress components (29) have the same frequency*
(iii) *the integral time averages of instantaneous values of distortion energies per unit volume in these both stress states are equal.*

The distortion energy per unit volume in the general state of stress is given by [4]:

$$\psi_d = \frac{1}{2}\left[\sigma_x\left(\varepsilon_x-\varepsilon_m\right)+\sigma_y\left(\varepsilon_y-\varepsilon_m\right)+\sigma_z\left(\varepsilon_z-\varepsilon_m\right)+\sum_k \tau_k\gamma_k\right] \quad (31)$$

where:

$$\varepsilon_m = \frac{1}{3}\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right) \quad (32)$$

When the stress (29) is applied to a viscoelastic material, according to Eqs (2), (12) and (22) one gets:

$$\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_{xo}-\upsilon\left(\sigma_{yo}+\sigma_{zo}\right)\right]\left(1-e^{-\frac{E}{\eta}t}\right)+\frac{1}{\sqrt{E^2+\eta^2\omega^2}}\left[\sigma_{xa}-\upsilon\left(\sigma_{ya}+\sigma_{za}\right)\right]\sin(\omega t-\alpha)\\
\varepsilon_y &= \frac{1}{E}\left[\sigma_{yo}-\upsilon\left(\sigma_{xo}+\sigma_{zo}\right)\right]\left(1-e^{-\frac{E}{\eta}t}\right)+\frac{1}{\sqrt{E^2+\eta^2\omega^2}}\left[\sigma_{ya}-\upsilon\left(\sigma_{xa}+\sigma_{za}\right)\right]\sin(\omega t-\alpha)\\
\varepsilon_z &= \frac{1}{E}\left[\sigma_{zo}-\upsilon\left(\sigma_{xo}+\sigma_{yo}\right)\right]\left(1-e^{-\frac{E}{\eta}t}\right)+\frac{1}{\sqrt{E^2+\eta^2\omega^2}}\left[\sigma_{za}-\upsilon\left(\sigma_{xa}+\sigma_{ya}\right)\right]\sin(\omega t-\alpha)\\
\gamma_k &= \frac{1}{G}\tau_{ko}\left(1-e^{-\frac{E}{\eta}t}\right)+\frac{2(1+\upsilon)}{\sqrt{E^2+\eta^2\omega^2}}\tau_{ka}\sin(\omega t-\alpha), \quad \alpha = \mathrm{arctg}\frac{\eta\omega}{E}
\end{aligned} \quad (33)$$

or, after sufficiently long time:

$$
\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_{xo} - \upsilon\left(\sigma_{yo} + \sigma_{zo}\right)\right] + \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{xa} - \upsilon\left(\sigma_{ya} + \sigma_{za}\right)\right]\sin\left(\omega t - \alpha\right) \\
\varepsilon_y &= \frac{1}{E}\left[\sigma_{yo} - \upsilon\left(\sigma_{xo} + \sigma_{zo}\right)\right] + \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{ya} - \upsilon\left(\sigma_{xa} + \sigma_{za}\right)\right]\sin\left(\omega t - \alpha\right) \\
\varepsilon_z &= \frac{1}{E}\left[\sigma_{zo} - \upsilon\left(\sigma_{xo} + \sigma_{yo}\right)\right] + \frac{1}{\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{za} - \upsilon\left(\sigma_{xa} + \sigma_{ya}\right)\right]\sin\left(\omega t - \alpha\right) \\
\gamma_k &= \frac{1}{G}\tau_{ko} + \frac{2\left(1+\upsilon\right)}{\sqrt{E^2 + \eta^2\omega^2}}\tau_{ka}\sin\left(\omega t - \alpha\right), \quad \alpha = \mathrm{arctg}\frac{\eta\omega}{E}
\end{aligned}
\tag{34}
$$

Substitution of Eqs (3), (32) and (34) into Eq. (31) results in:

$$
\begin{aligned}
\psi_d = {} & \frac{1+\upsilon}{3E}\left(\sigma_{xo}^2 + \sigma_{yo}^2 + \sigma_{zo}^2 - \sigma_{xo}\sigma_{yo} - \sigma_{yo}\sigma_{zo} - \sigma_{zo}\sigma_{xo} + 3\sum_k \tau_{ko}^2\right) + \\
& + \frac{1+\upsilon}{3\sqrt{E^2 + \eta^2\omega^2}}\left(\sigma_{xa}^2 + \sigma_{ya}^2 + \sigma_{za}^2 - \sigma_{xa}\sigma_{ya} - \sigma_{ya}\sigma_{za} - \sigma_{za}\sigma_{xa} + 3\sum_k \tau_{ka}^2\right)\sin\omega t \sin\left(\omega t - \alpha\right) + \\
& + \frac{1+\upsilon}{6E}\left[\sigma_{xa}\left(2\sigma_{xo} - \sigma_{yo} - \sigma_{zo}\right) + \sigma_{ya}\left(2\sigma_{yo} - \sigma_{xo} - \sigma_{zo}\right) + \sigma_{za}\left(2\sigma_{zo} - \sigma_{xo} - \sigma_{yo}\right) + \right. \\
& \left. + 6\sum_k \tau_{ka}\tau_{ko}\right]\sin\omega t + \frac{1+\upsilon}{6\sqrt{E^2 + \eta^2\omega^2}}\left[\sigma_{xo}\left(2\sigma_{xa} - \sigma_{ya} - \sigma_{za}\right) + \sigma_{yo}\left(2\sigma_{ya} - \sigma_{xa} - \sigma_{za}\right) + \right. \\
& \left. + \sigma_{zo}\left(2\sigma_{za} - \sigma_{xa} - \sigma_{ya}\right) + 6\sum_k \tau_{ko}\tau_{ka}\right]\sin\left(\omega t - \alpha\right)
\end{aligned}
\tag{35}
$$

Hence the distortion energy per unit volume of the viscoelastic solid under the reduced stress (30) is:

$$
\begin{aligned}
\psi_{de} = {} & \frac{1+\upsilon}{3E}\sigma_{eo}^2 + \frac{1+\upsilon}{3\sqrt{E^2 + \eta^2\omega^2}}\sigma_{ea}^2 \sin\omega t\left(\omega t - \alpha\right) + \\
& + \frac{1+\upsilon}{3E}\sigma_{ea}\sigma_{eo}\sin\omega t + \frac{1+\upsilon}{3\sqrt{E^2 + \eta^2\omega^2}}\sigma_{eo}\sigma_{ea}\sin\left(\omega t - \alpha\right)
\end{aligned}
\tag{36}
$$

The condition (i) gives Eq. (10), and the condition (**iii**), i.e.,

$$
\frac{1}{T}\int_0^T \psi_{de}\,dt = \frac{1}{T}\int_0^T \psi_d\,dt \quad ; \quad T = \frac{2\pi}{\omega}
\tag{37}
$$

leads to Eq. (18). Then the criterion of an infinite fatigue life reads [2, 10]:

$$
\frac{\sigma_{eo}}{R_e} + \frac{\sigma_{ea}}{Z_{rc}} < 1
\tag{38}
$$

and the yield criterion is:

$$
\frac{\sigma_{eo}}{R_e} + \frac{\sigma_{ea}}{R_e} < 1
\tag{39}
$$

At the stress (30), the criterion of a finite fatigue life of ductile materials becomes [10]:

$$
\frac{N_d}{K}\left(\frac{\sigma_{ea}}{1 - \dfrac{\sigma_{eo}}{R_e}}\right)^m < 1
\tag{40}
$$

Eq. (40) is valid if:

$$Z_{rc} < \frac{\sigma_{ea}}{1 - \frac{\sigma_{eo}}{R_e}} \leq L \tag{41}$$

where:
L – the maximum stress amplitude under fully reversed tension-compression in the high-cycle fatigue regime (above which the low-cycle fatigue may occur).

Another stress state which can be dealed with the aid of the aforementioned hypothesis is that with non-zero mean out-of-phase components:

$$\sigma_j = \sigma_{jo} + \sigma_{ja} \sin(\omega t + \varphi_j) \quad ; \quad \tau_k = \tau_{ko} + \tau_{ka} \sin(\omega t + \varphi_k) \tag{42}$$

where: φ_j and φ_k are the phase angles. With the known strain response of viscoelastic materials to the stress (42) [6] and with the reduced stress in the form (30), it is easy to prove that the mean value of the reduced stress is given again by Eq. (10), but now the formula for its amplitude reads:

$$\sigma_{ea} = \begin{bmatrix} \sigma_{xa}^2 + \sigma_{ya}^2 + \sigma_{za}^2 - \sigma_{xa}\sigma_{ya}\cos(\varphi_x - \varphi_y) - \sigma_{ya}\sigma_{za}\cos(\varphi_y - \varphi_z) + \\ - \sigma_{za}\sigma_{xa}\cos(\varphi_z - \varphi_x) + 3(\tau_{xya}^2 + \tau_{yza}^2 + \tau_{zxa}^2) \end{bmatrix}^{1/2} \tag{43}$$

It means that in the considered above load cases the fatigue and yield criteria for elastic and viscoelastic solids coincide. Of course, to be on the safe side, in Eqs (38) through (40) arbitrary safety margins may be introduced.

FAILURE CRITERIA OF VISCOELASTIC MATERIALS UNDER MULTIAXIAL PERIODIC LOADS

In general, structural elements and machinery details are simultaneously subjected to static and dynamic loads. In addition to the time-varying stress, the steady stress resulting from static load, and the mean stress (the average of the maximum and minimum of the cyclic stress) influence the strength of an element. It is important to recognize that the total strength of the element is altered if residual stresses (caused by cold forming, heat treatment, welding, etc.) exist. Since residual stresses have a similar influence on the fatigue behaviour of materials as do mechanically imposed constant stresses of the same magnitude [12], in what follows no distinction will be made between any kind of static stresses.

Among the types of dynamic loads encountered in practice, one of the most important is periodic load. In the general state of periodic stress, its components can be expanded in Fourier series:

$$\begin{aligned} \sigma_j &= \sigma_{jo} + \sum_n \sigma_{jn} \sin(n\omega t + \varphi_{jn}) \quad ; \quad j = x, y, z \\ \tau_k &= \tau_{ko} + \sum_n \tau_{kn} \sin(n\omega t + \varphi_{kn}) \quad ; \quad k = xy, yz, zx \end{aligned} \tag{44}$$

where:
σ_{jo}, τ_{ko} – mean values ; φ_{jn}, φ_{kn} – phase angles of n-th terms ; $\omega = 2\pi/T$ – fundamental circular frequency ; T – stress period.

Our aim is to determine such a reduced stress:

$$\sigma_e = \sigma_{eo} + \sigma_{ea} \sin(\omega_e t) \tag{45}$$

of mean value σ_{eo}, amplitude σ_{ea} and circular frequency ω_e, that would be equivalent to the original stress in terms of fatigue and yield strengths of viscoelastic materials. For this purpose the theory of energy transformation systems [13] can be used, which links the dissipated energy with the breakdown time of a system. According to this theory, two stress states are equivalent in terms of time to failure if the energies dissipated internally and externally in both these states are respectively equal. Apparently, the dissipated energy can be estimated by evaluation of certain symptoms, e.g., the externally dissipated energy by the vibration, and the internally dissipated energy by the temperature [13]. For viscoelastic materials the use can be made of the following relationship derived in [14] for the internally dissipated energy per unit volume under periodic stress with the components (44) during the period T:

$$\begin{aligned} \phi(T) = \frac{1}{2}\omega T \sum_n \frac{n \sin\alpha_n}{\sqrt{E^2 + (\eta n\omega)^2}} \Big\{ & \sigma_{xn}^2 + \sigma_{yn}^2 + \sigma_{zn}^2 - 2\upsilon \big[\sigma_{xn}\sigma_{yn}\cos(\varphi_{xn} - \varphi_{yn}) + \\ & + \sigma_{yn}\sigma_{zn}\cos(\varphi_{yn} - \varphi_{zn}) + \sigma_{zn}\sigma_{xn}\cos(\varphi_{zn} - \varphi_{xn}) \big] + 2(1+\upsilon)(\tau_{xyn}^2 + \tau_{yzn}^2 + \tau_{zxn}^2) \Big\} \end{aligned} \tag{46}$$

where:

$$\alpha_n = \operatorname{arct}\frac{\eta n\omega}{E} \tag{47}$$

Consequently, the following equivalency condition can be postulated:

$$\varphi_e(T) = \varphi(T) \tag{48}$$

where:

$\varphi_e(T)$ – the energy dissipated in the viscoelastic material per unit volume under the reduced stress (45) during T seconds. Under assumption that:

$$\omega_e = r\omega \tag{49}$$

where:

r – natural number to be determined, on the basis of Eqs (46) and (47) one can write:

$$\phi_e(T) = \frac{1}{2} r\omega T \frac{\sin\alpha_e}{\sqrt{E^2 + (\eta r\omega)^2}} \sigma_{ea}^2 \tag{50}$$

$$\alpha_e = \operatorname{arct} \frac{\eta r\omega}{E} \tag{51}$$

Thus:

$$\frac{r\sin\alpha_e}{\sqrt{E^2 + (\eta r\omega)^2}} \sigma_{ea}^2 = A \tag{52}$$

where:

$$A = \sum_n \frac{n\sin\alpha_n}{\sqrt{E^2 + (\eta n\omega)^2}} \Big\{\sigma_{xn}^2 + \sigma_{yn}^2 + \sigma_{zn}^2 - 2\upsilon\big[\sigma_{xn}\sigma_{yn}\cos(\varphi_{xn} - \varphi_{yn}) + \\ + \sigma_{yn}\sigma_{zn}\cos(\varphi_{yn} - \varphi_{zn}) + \sigma_{zn}\sigma_{xn}\cos(\varphi_{zn} - \varphi_{xn})\big] + 2(1+\upsilon)\big(\tau_{xyn}^2 + \tau_{yzn}^2 + \tau_{zxn}^2\big)\Big\} \tag{53}$$

As for the remaining equations, necessary for determination of the reduced stress (45), in [9] it was assumed that the energy dissipated externally by materials subjected to multiaxial loads is proportional to the average strain energy of distortion per unit volume. Bearing in mind the role of distortion energy in evaluation of effort of ductile materials under multiaxial stresses, in the present paper the same assumption is retained. Derivation of similar equations based on the strain energy of volume change or total strain energy is analogous.

In conformity with the aforesaid, we have:

$$\frac{1}{T}\int_0^T \psi_{de} dt = \frac{1}{T}\int_0^T \psi_d \ , \ T = \frac{2\pi}{\omega} \tag{54}$$

where:

$$\psi_{de} = \frac{1+\upsilon}{3E}\sigma_{eo}^2 + \frac{1+\upsilon}{3\sqrt{E^2 + (\eta r\omega)^2}}\sigma_{ea}^2 \sin r\omega t \sin(r\omega t - \alpha_e) + \\ + \frac{1+\upsilon}{3E}\sigma_{ea}\sigma_{eo}\sin r\omega t + \frac{1+\upsilon}{3\sqrt{E^2 + (\eta r\omega)^2}}\sigma_{eo}\sigma_{ea}\sin(r\omega t - \alpha_e) \tag{55}$$

is the strain energy of distortion per unit volume under reduced stress (45) and ψ_d is that under periodic stress (44). To determine the latter from Eqs (31) and (32), the strain response of a viscoelastic material to periodic stress must be known. This problem was solved in [6] to give:

$$\boldsymbol{\varepsilon} = \varepsilon_o + \sum_n H_n \operatorname{Im}\left[\overline{\sigma}_n e^{i(n\omega t - \alpha_n)}\right] \tag{56}$$

where:

$$\boldsymbol{\varepsilon_o} = \left[\varepsilon_{xo}\ \varepsilon_{yo}\ \varepsilon_{zo}\ \gamma_{xyo}\ \gamma_{yzo}\ \gamma_{zxo}\right]^T$$

is the vector of mean strain components given by Eqs (2),

$$\mathbf{H_n} = \frac{1}{\sqrt{E^2 + (\eta n\omega)^2}} \begin{bmatrix} 1 & -v & -v & 0 & 0 & 0 \\ -v & 1 & -v & 0 & 0 & 0 \\ -v & -v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+v) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+v) \end{bmatrix}$$

is the matrix of dynamical flexibility of the viscoelastic material at the load circular frequency nω,

$$\overline{\boldsymbol{\sigma}}_{\mathbf{n}} = \left[\overline{\sigma}_{xn}\,\overline{\sigma}_{yn}\,\overline{\sigma}_{zn}\,\overline{\tau}_{xyn}\,\overline{\tau}_{yzn}\,\overline{\tau}_{zxn}\right]^{T}$$

is the vector of complex amplitudes of n-th terms of the stress components, defined as:

$$\overline{\sigma}_{jn} = \sigma_{jn} e^{i\varphi_{jn}} \quad ; \quad \overline{\tau}_{kn} = \tau_{kn} e^{i\varphi_{kn}}$$

i is the imaginary unity, Im is the imaginary part, and:

$$\alpha_n = \operatorname{arctg}\frac{\eta n\omega}{E}$$

is the phase angle of *n*-th terms of the strain components.

Eqs (31), (32) and (54) through (56) lead to:

$$\frac{1+v}{3E}\sigma_{eo}^2 + \frac{(1+v)\cos\alpha_e}{6\sqrt{E^2+(\eta r\omega)^2}}\sigma_{ea}^2 = \frac{1+v}{3E}\Big[\sigma_{xo}^2+\sigma_{yo}^2+\sigma_{zo}^2-\sigma_{xo}\sigma_{yo}-\sigma_{yo}\sigma_{zo}+$$

$$-\sigma_{zo}\sigma_{xo}+3\left(\tau_{xyo}^2+\tau_{yzo}^2+\tau_{zxo}^2\right)\Big]+\frac{1+v}{6}\sum_n\frac{\cos\alpha_n}{\sqrt{E^2+(\eta n\omega)^2}}\Big[\sigma_{xn}^2+\sigma_{yn}^2+\sigma_{zn}^2-\sigma_{xn}\sigma_{yn}\cos\left(\varphi_{xn}-\varphi_{yn}\right)+ \tag{57}$$

$$-\sigma_{yn}\sigma_{zn}\cos\left(\varphi_{yn}-\varphi_{zn}\right)-\sigma_{zn}\sigma_{xn}\cos\left(\varphi_{zn}-\varphi_{xn}\right)+3\left(\tau_{xyn}^2+\tau_{yzn}^2+\tau_{zxn}^2\right)\Big]$$

Hence:

$$\sigma_{eo}^2 = \sigma_{xo}^2+\sigma_{yo}^2+\sigma_{zo}^2-\sigma_{xo}\sigma_{yo}-\sigma_{yo}\sigma_{zo}-\sigma_{zo}\sigma_{xo}+3\left(\tau_{xyo}^2+\tau_{yzo}^2+\tau_{zxo}^2\right) \tag{58}$$

which results in Eq. (10) for the reduced mean stress, and:

$$\frac{\cos\alpha_e}{\sqrt{E\;\;+(\eta r\omega)^2}}\sigma_{ea}^2 = B \tag{59}$$

where:

$$B = \sum_n\frac{\cos\alpha_n}{\sqrt{E^2+(\eta n\omega)^2}}\Big[\sigma_{xn}^2+\sigma_{yn}^2+\sigma_{zn}^2-\sigma_{xn}\sigma_{yn}\cos\left(\varphi_{xn}-\varphi_{yn}\right)-\sigma_{yn}\sigma_{zn}\cos\left(\varphi_{yn}-\varphi_{zn}\right)+$$

$$-\sigma_{zn}\sigma_{xn}\cos\left(\varphi_{zn}-\varphi_{xn}\right)+3\left(\tau_{xyn}^2+\tau_{yzn}^2+\tau_{zxn}^2\right)\Big] \tag{60}$$

From Eqs (52) and (59) one obtains:

$$\sigma_{ea} = \left[\frac{E^2+(\eta r\omega)^2}{r^2}\left(A^2+r^2B^2\right)\right]^{1/4} \tag{61}$$

and:

$$r\,\mathrm{tg}\alpha_e = \frac{A}{B} \tag{62}$$

that is:

$$r^2\frac{\eta\omega}{E} = \frac{A}{B} \tag{63}$$

Hence the real number ρ , close to the natural number r,

$$r = \mathrm{Round}(\rho) \tag{64}$$

is:

$$\rho = \left(\frac{AE}{\eta\omega B}\right)^{1/2} \tag{65}$$

With these results, the criterion of an infinite fatigue life and the yield criterion can be expressed by Eqs (38) and (39), whereas the criterion of a finite fatigue life becomes:

$$\frac{r\omega T_d}{2\pi K}\left(\frac{\sigma_{ea}}{1-\dfrac{\sigma_{eo}}{R_e}}\right)^m < 1 \tag{66}$$

where: T_d – the required design life.

CONCLUSIONS

- On the basis of the Huber-Mises-Hencky (HMH) theory and constitutive equations for strains in viscoelastic solids subjected to multiaxial static loads, it is shown that the yield criterion does not differ from that for perfectly elastic solids.
- On the basis of the HMH theory and constitutive equations for strains in viscoelastic solids subjected to multiaxial zero mean in-phase loads, it is shown that the criterion of an infinite fatigue life, the criterion of a finite fatigue life and the yield criterion do not differ from those for perfectly elastic solids.
- With the aid of the average-distortion-energy strength hypothesis and constitutive equations for strains in viscoelastic solids subjected to non-zero mean in-phase loads, as well as subjected to non-zero mean out-of-phase loads, it is shown that the aforementioned criteria do not differ from those for perfectly elastic solids.
- On the basis of the theory of energy transformation systems and constitutive equations for strains in viscoelastic solids subjected to periodic loads, it is shown that the aforementioned criteria do not coincide with those for perfectly elastic solids.
- The load capacity of a homogeneous, isotropic viscoelastic material at a given temperature is completely defined by the Young modulus, tensile yield strength, S-N curve for tension-compression, Poisson's ratio and coefficient of viscous damping of normal strain.

NOMENCLATURE

A, B – quantities defined by Eqs (53) and (60)
E – Young modulus
G – shear modulus
H_n – matrix of dynamical flexibility of the viscoelastic material at the load circular frequency $n\omega$
i – imaginary unity
Im – imaginary part
K – fatigue strength coefficient in equation of the S-N curve for tension-compression
L – maximum stress amplitude under fully reversed tension-compression in the high-cycle fatigue regime (above which the low-cycle fatigue may occur)
m – fatigue strength exponent in equation of the S-N curve for tension-compression
n – natural number
N_d – required number of stress cycles to achieve a given design life
r – natural number given by Eq. (64)
R_e – tensile yield strength
t – time
T – stress period
T_d – required design life
Z_{rc} – fatigue limit under fully reversed tension-compression
α – phase angle of the strain components under in-phase loads
α_e – phase angle defined by Eq. (51)
α_n – phase angle defined by Eq. (47)
γ_k – k-th shear strain component (k = xy, yz, zx)
γ_{ko} – k-th strain component under static load, mean value of k-th strain component
$\boldsymbol{\varepsilon}$ – vector of the strain components
ε_e – reduced strain
ε_j – j-th normal strain component (j = x, y, z)
ε_{jo} – j-th strain component under static load, mean value of j-th strain component
ε_m – quantity defined by Eq. (32)
$\boldsymbol{\varepsilon_o}$ – vector of the mean values of strain components
η – coefficient of viscous damping of normal strain
ν – Poisson's ratio,
ρ – number given by Eq. (65)
σ_e – reduced stress
σ_{ea} – amplitude of the reduced stress
σ_{eo} – reduced stress under static load, mean value of the reduced stress
σ_j – j-th stress component
σ_{ja} – amplitude of j-th stress component
σ_{jn} – amplitude of n-th term in Fourier expansion of σ_j
$\overline{\sigma_{jn}}$ – complex amplitude of n-th term in Fourier expansion of σ_j
σ_{jo} – j-th stress component under static load, mean value of j-th stress component
$\overline{\sigma_n}$ – vector of complex amplitudes of n-th terms in Fourier expansion of the stress components
τ_k – k-th stress component
τ_{ka} – amplitude of k-th stress component
τ_{ko} – k-th stress component under static load, mean value of k-th stress component
τ_{kn} – amplitude of n-th term in Fourier expansion of τ_k
$\overline{\tau_{kn}}$ – complex amplitude of n-th term in Fourier expansion of τ_k
φ_j – phase angle of j-th stress component
φ_{jn} – phase angle of n-th term in Fourier expansion of σ_j
φ_k – phase angle of k-th stress component
φ_{kn} – phase angle of n-th term in Fourier expansion of τ_k
φ – dissipation energy per unit volume
φ_e – dissipation energy per unit volume under the reduced stress
ψ – strain energy per unit volume
ψ_d – strain energy of distortion per unit volume
ψ_{de} – strain energy of distortion per unit volume under the reduced stress
ψ_{deo} – strain energy of distortion per unit volume under the reduced static stress
ψ_{do} – strain energy of distortion per unit volume of perfectly elastic solid under static load
ψ_e – strain energy per unit volume under the reduced stress
ψ_o – strain energy per unit volume of perfectly elastic solid under static load
ψ_{vo} – strain energy of volume change per unit volume of perfectly elastic solid under static load
ω – circular frequency, fundamental circular frequency
ω_e – circular frequency of the reduced stress at periodic loads

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