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A NEW INCREMENTAL FORMULATION FOR LINEAR VISCOELASTIC ANALYSIS: CREEP DIFFERENTIAL APPROACH

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This paper presents a new incremental formulation in the time domain for linear, non-aging viscoelastic materials undergoing mechanical deformation. The formulation is derived from linear differential equations based on a discrete spectrum representation for the creep tensor. The incremental constitutive equations are then obtained by finite difference integration. Thus the difficulty of retaining the stress history in computer solutions is avoided. A complete general formulation of linear viscoelastic stress analysis is developed in terms of increments of strains and stresses. Numerical results of good accuracy are achieved. The analytical and numerical solutions are compared using the energy release rate in pure mode I and pure mode II.

Key words: incremental constitutive law, strain history, discrete creep spectrum

1. Introduction

The important use of viscoelastic materials in civil engineering structures requires understanding of the behaviour of time dependent mechanical fields which can lead to collapse of such structures. The main problem in computation mechanics is to know the response of a viscoelastic material taking into account its complete past history of stress and strain. Most of analytical solutions proposed in the literature can not deal with real and complex problems because these methods require the retaining of the complete past history of stress and strain in the memory of a digital computer.

In this context, a number of theories have been proposed in the past in order to formulate incremental constitutive equations for the linear viscoelastic behaviour. Among them, Kim and Sung Lee (2007), Ghazlan *et al.* (1995), Chazal and Dubois (2001), Klasztorny (2008), Dubois *et al.* (1999a) proposed the incremental formulation and constitutive equations in the finite element context. In fracture of viscoelastic mechanics, Dubois *et al.* (1999b, 2002), Dubois and Petit (2005) and Moutou Pitti *et al.* (2007, 2008) applied the incremental formulation in order to evaluate the crack growth process in wood. However, the formulation used was based on the spectral decomposition using a generalized Kelvin Voigt model.

To avoid the use of the generalized Kelvin Voigt model, we develop in this paper a new incremental formulation based on a discrete creep spectrum and the finite difference method using generalized differential equations in the time domain. The incremental stress-strain constitutive equation is not restricted to isotropic materials and can be used to resolve complex viscoelastic problems without retaining the past history of the material.

The first section recalls the discrete creep spectrum representation and its use in Boltzmann's superposition principle (Boltzmann, 1878). The one-dimensional linear viscoelastic behaviour is used to reduce the three-dimensional response.

The second section contains the development of the generalized differential equations in terms of one-dimensional stress and strain components.

In the third section, the solution of the differential equations is proposed using the finite difference method and the new constitutive stress-strain relations are then obtained.

Finally, the constitutive law is implemented in finite element software CASTEM (Charvet-Quemin *et al.* 1986) and the numerical results are compared to the analytical solution given by Moutou Pitti *et al.* (2007).

2. Creep spectrum representation

In this work, we consider only small strains. According to the results obtained by Mandel (1978), Ghazlan *et al.* (1995), Chazal and Dubois (2001), Moutou Pitti *et al.* (2007, 2008) and Dubois and Petit (2005), the components of the creep tensor $\mathbf{J}(t)$ can be represented in terms of an exponential series

$$J_{ijk\ell}(t) = \Big[J^{(0)}_{ijk\ell} + \sum_{m=1}^{M} J^{(m)}_{ijk\ell}\Big(1 - \mathrm{e}^{-t\lambda^{(m)}_{ijk\ell}}\Big)\Big]H(t) \tag{2.1}$$

Where $\lambda_{ijk\ell}^{(m)}$, $m = 1, \dots, M$, are strictly positive scalars, and the repeated indices do not imply summation convention. $J_{ijk\ell}^{(0)}$ and $J_{ijk\ell}^{(m)}$ represent the instantaneous and the differed creep tensor, respectively, and $H(t)$ is the Heaviside unit step function.

According to Boltzmann's principle (Boltzmann, 1878), the constitutive equations between the components $\sigma_{ij}(t)$ of the stress tensor and the components of the strain tensor $\varepsilon_{ij}(t)$ for linear viscoelastic materials can be expressed in the time domain by the hereditary integral

$$\varepsilon_{ij}(t) = \sum_k \sum_\ell \int_{-\infty}^{t} J_{ijk\ell}(t-\tau) \frac{\partial \sigma_{k\ell}(\tau)}{\partial \tau}\, d\tau \tag{2.2}$$

Let us consider the fourth order tensor $\mathbf{\Pi}(t)$ of the components $\mathit{\Pi}_{ijk\ell}(t)$ defined by

$$\mathit{\Pi}_{ijk\ell}(t) = \int_{-\infty}^{t} J_{ijk\ell}(t-\tau) \frac{\partial \sigma_{k\ell}(\tau)}{\partial \tau}\, d\tau \qquad \forall i,j,k,\ell \in [1,2,3], \quad \forall t \in \Re \tag{2.3}$$

The components $\mathit{\Pi}_{ijk\ell}(t)$ can be interpreted as the contribution of the stress history $\{\sigma_{k\ell}(\tau), \tau \leqslant t\}$ of the components $\sigma_{k\ell}(t)$ of the stress tensor $\boldsymbol{\sigma}(t)$ to the strain components $\varepsilon_{ij}(t)$.

Introducing equation (2.3) into (2.2), we obtain

$$\varepsilon_{ij}(t) = \sum_k \sum_\ell \mathit{\Pi}_{ijk\ell}(t) \qquad \forall i,j,k,\ell \in [1,2,3], \quad \forall t \in \Re \tag{2.4}$$

Each equation of relation (2.4) represents a one-dimensional non-aging linear viscoelastic material defined by its creep function $\mathbf{J}(t)$.

3. Formulation of differential equations

When we apply the mechanical stress defined by the stress history $\{\sigma_{k\ell}(\tau), \tau \in \Re\}$, the response of the material is then given by the history of strains $\{\mathit{\Pi}_{ijk\ell}(t), t \in \Re\}$ defined by behaviour equation (2.3) in which the creep function is given by equation (2.1).

We note by $\sigma_{k\ell}(t)$ the stress applied to the material at the time t and by $\mathit{\Pi}_{ijk\ell}(t)$ the total strain at the same time t. Then the response in strains can be obtained using the finite creep spectrum representation given by equation (2.1)

$$\Pi_{ijk\ell}(t) = \int\limits_{-\infty}^{t} \Big[J_{ijk\ell}^{(0)} + \sum_{m=1}^{M} J_{ijk\ell}^{(m)}\Big(1 - \mathrm{e}^{-\lambda_{ijk\ell}^{(m)}(t-\tau)}\Big)\Big] \frac{\partial\sigma_{k\ell}(\tau)}{\partial\tau}\, d\tau \tag{3.1}$$

This equation can be rewritten in the following form

$$\Pi_{ijk\ell}(t) = \Pi_{ijk\ell}^{(0)}(t) + \sum_{m=1}^{M} \Pi_{ijk\ell}^{(m)}(t) \tag{3.2}$$

with

$$\begin{aligned} \Pi_{ijk\ell}^{(0)}(t) &= \int\limits_{-\infty}^{t} J_{ijk\ell}^{(0)} \frac{\partial\sigma_{k\ell}(\tau)}{\partial\tau}\, d\tau = J_{ijk\ell}^{(0)}\sigma_{k\ell}(t) \\ \Pi_{ijk\ell}^{(m)}(t) &= \int\limits_{-\infty}^{t} J_{ijk\ell}^{(m)}\Big(1 - \mathrm{e}^{-\lambda_{ijk\ell}^{(m)}(t-\tau)}\Big) \frac{\partial\sigma_{k\ell}(\tau)}{\partial\tau}\, d\tau \end{aligned} \tag{3.3}$$

In these equations, $\Pi_{ijk\ell}^{(0)}(t)$ and $\Pi_{ijk\ell}^{(m)}(t)$ represent the instantaneous and the differed part of the one-dimensional strain of the material.

Using equation (2.4), the rate of the total strain is determined by

$$\frac{\partial\varepsilon_{ij}(t)}{\partial t} = \sum_k \sum_\ell \frac{\partial\Pi_{ijk\ell}(t)}{\partial t} = \sum_k \sum_\ell \Big(\frac{\partial\Pi_{ijk\ell}^{(0)}(t)}{\partial t} + \sum_{m=1}^{M} \frac{\partial\Pi_{ijk\ell}^{(m)}(t)}{\partial t}\Big) \tag{3.4}$$

According to equation $(3.3)_1$, the rate of the instantaneous part of the one-dimensional strain $\Pi_{ijk\ell}^{(0)}(t)$ is given by

$$\frac{\partial\Pi_{ijk\ell}^{(0)}(t)}{\partial t} = J_{ijk\ell}^{(0)} \frac{\partial\sigma_{k\ell}(t)}{\partial t} \tag{3.5}$$

However, the rate of the differed part of the one-dimensional strain $\Pi_{ijk\ell}^{(m)}(t)$ is more complicated to be determined. Using equation $(3.3)_2$, we can write

$$\begin{aligned} \frac{\partial\Pi_{ijk\ell}^{(m)}(t)}{\partial t} &= J_{ijk\ell}^{(m)}\Big(1 - \mathrm{e}^{-\lambda_{ijk\ell}^{(m)}(t-t)}\Big) \frac{\partial\sigma_{k\ell}(\tau)}{\partial\tau} + \\ &+ \int\limits_{-\infty}^{t} J_{ijk\ell}^{(m)}\Big(0 + \lambda_{ijk\ell}^{(m)}\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}(t-\tau)}\Big) \frac{\partial\sigma_{k\ell}(\tau)}{\partial\tau}\, d\tau \end{aligned} \tag{3.6}$$

or

$$\frac{\partial \Pi^{(m)}_{ijk\ell}(t)}{\partial t} = J^{(m)}_{ijk\ell}\lambda^{(m)}_{ijk\ell}\int\limits_{-\infty}^{t} \mathrm{e}^{-\lambda^{(m)}_{ijk\ell}(t-\tau)}\frac{\partial \sigma_{k\ell}(\tau)}{\partial \tau}\,d\tau \tag{3.7}$$

knowing that

$$\begin{aligned}\lambda^{(m)}_{ijk\ell}\Pi^{(m)}_{ijk\ell}(t) &= J^{(m)}_{ijk\ell}\lambda^{(m)}_{ijk\ell}\int\limits_{-\infty}^{t}\Big(1-\mathrm{e}^{-\lambda^{(m)}_{ijk\ell}(t-\tau)}\Big)\frac{\partial \sigma_{k\ell}(\tau)}{\partial \tau}\,d\tau = \\ &= J^{(m)}_{ijk\ell}\lambda^{(m)}_{ijk\ell}\sigma_{k\ell}(t) - \lambda^{(m)}_{ijk\ell}J^{(m)}_{ijk\ell}\int\limits_{-\infty}^{t} \mathrm{e}^{-\lambda^{(m)}_{ijk\ell}(t-\tau)}\frac{\partial \sigma_{k\ell}(\tau)}{\partial \tau}\,d\tau = \\ &= J^{(m)}_{ijk\ell}\lambda^{(m)}_{ijk\ell}\sigma_{k\ell}(t) - \frac{\partial \Pi^{(m)}_{ijk\ell}(t)}{\partial t}\end{aligned} \tag{3.8}$$

The last equation can be written as a linear differential equation and can be integrated analytically

$$\frac{\partial \Pi^{(m)}_{ijk\ell}(t)}{\partial t} + \lambda^{(m)}_{ijk\ell}\Pi^{(m)}_{ijk\ell}(t) = J^{(m)}_{ijk\ell}\lambda^{(m)}_{ijk\ell}\sigma_{k\ell}(t) \tag{3.9}$$

The solution to this differential equation gives the rate of the one-dimensional strain $\Pi^{(m)}_{ijk\ell}(t)$.

Finally, the general differential equations governing the non-aging linear viscoelastic behaviour can be obtained from equation (3.4) after summation on k and ℓ indices. One finds

$$\frac{\partial \varepsilon_{ij}(t)}{\partial t} = \sum_{k}\sum_{\ell} J^{(0)}_{ijk\ell}\frac{\partial \sigma_{k\ell}(t)}{\partial t} + \sum_{m=1}^{M}\frac{\partial \Lambda^{(m)}_{ij}(t)}{\partial t} \tag{3.10}$$

where $\Lambda^{(m)}_{ij}(t)$, $i,j \in \{1,2,3\}$, $m \in \{1,\dots,M\}$ are the solutions to the following equations

$$\Lambda^{(m)}_{ij}(t) = \sum_{k=1}^{3}\sum_{\ell=1}^{3}\Pi^{(m)}_{ijk\ell}(t) \tag{3.11}$$

with

$$\frac{\partial \Pi^{(m)}_{ijk\ell}(t)}{\partial t} + \lambda^{(m)}_{ijk\ell}\Pi^{(m)}_{ijk\ell}(t) = J^{(m)}_{ijk\ell}\lambda^{(m)}_{ijk\ell}\sigma_{k\ell}(t) \tag{3.12}$$

The non-aging linear viscoelastic behaviour is completely defined by differential equations (3.10), (3.11) and (3.12).

We note that this formulation, written in terms of strain and stress rates, is easily adapted to temporal discretisation methods such as finite difference ones.

4. Finite difference integration

Here we describe the solution process of a step-by-step nature in which the loads are applied stepwise at various time intervals. Let us consider the time step $\Delta t_n = t_n - t_{n-1}$. The subscript $n-1$ and n refer to the values at the beginning and end of the time step, respectively. We assume that the time derivative during each time increment is constant and is expressed by

$$\frac{\partial \zeta_{ij}}{\partial t} = \frac{\zeta_{ij}(t_n) - \zeta_{ij}(t_{n-1})}{\Delta t_n} = \frac{\Delta(\zeta_{ij})_n}{\Delta t_n} \tag{4.1}$$

where ζ_{ij} represent strains or stresses. The following expressions can be then written

$$\begin{aligned}
\frac{\partial \Lambda_{ij}^{(m)}(t_n)}{\partial t} &= \frac{\Lambda_{ij}^{(m)}(t_{n+1}) - \Lambda_{ij}^{(m)}(t_n)}{\Delta t_n} = \frac{\Delta\Lambda_{ij}^{(m)}(t_n)}{\Delta t_n} \\
\frac{\partial \sigma_{ij}(t_n)}{\partial t} &= \frac{\sigma_{ij}(t_{n+1}) - \sigma_{ij}(t_n)}{\Delta t_n} = \frac{\Delta\sigma_{ij}(t_n)}{\Delta t_n}
\end{aligned} \tag{4.2}$$

A linear approximation is used for stresses, and is expressed by

$$\sigma_{k\ell}(\tau) = \sigma_{k\ell}(t_n) + \frac{\tau - t_n}{\Delta t_n}[\sigma_{k\ell}(t_{n+1}) - \sigma_{k\ell}(t_n)]H(\tau - t_n) \tag{4.3}$$

By integrating equation (3.10) between t_n and t_{n+1}, it can be written in the form

$$\Delta\varepsilon_{ij}(t_n) = \sum_k \sum_\ell J_{ijk\ell}^{(0)} \Delta\sigma_{k\ell}(t_n) + \sum_{m=1}^{M} \Delta\Lambda_{ij}^{(m)}(t_n) \tag{4.4}$$

In order to determine the strain increments from this equation, we have to determine the strain increments $\Delta\Lambda_{ij}^{(m)}(t_n)$.

First, let us consider differential equation (3.12). The analytical solution to this differential equation can be expressed as

$$\begin{aligned}
&\Pi_{ijk\ell}^{(m)}(t_{n+1}) - \Pi_{ijk\ell}^{(m)}(t_n) = \left(\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n} - 1\right)\Pi_{ijk\ell}^{(m)}(t_n) + \\
&\quad + J_{ijk\ell}^{(m)}\Big\{\left(1 - \mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\right)\sigma_{k\ell}(t_n) + \Delta\sigma_{k\ell}(t_n)\Big[1 - \frac{1}{\Delta t_n \lambda_{ijk\ell}^{(m)}}\left(1 - \mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\right)\Big]\Big\}
\end{aligned} \tag{4.5}$$

Consequently, when we substitute equation (4.5) into equation (3.11), we obtain the strain increments $\Delta\Lambda_{ij}^{(m)}(t_n)$

$$\sum_{m=1}^{M} \Delta\Lambda_{ij}^{(m)}(t_n) = \sum_{m=1}^{M} \sum_{k=1}^{3} \sum_{\ell=1}^{3} [\Pi_{ijk\ell}^{(m)}(t_{n+1}) - \Pi_{ijk\ell}^{(m)}(t_n)] \tag{4.6}$$

or

$$\sum_{m=1}^{M} \Delta\Lambda_{ij}^{(m)} = \sum_{m=1}^{M}\sum_{k=1}^{3}\sum_{\ell=1}^{3}\Big(\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n} - 1\Big)\Pi_{ijk\ell}^{(m)}(t_n) + \\ + J_{ijk\ell}^{(m)}\Big\{\Big(1-\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\Big)\sigma_{k\ell}(t_n) + \Delta\sigma_{k\ell}(t_n)\Big[1 - \frac{1}{\Delta t_n \lambda_{ijk\ell}^{(m)}}\Big(1-\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\Big)\Big]\Big\} \tag{4.7}$$

5. Incremental viscoelastic constitutive equations

In this section, the incremental constitutive equations can now be obtained from equation (4.4). Substituting equation (4.7) into (4.4), we find

$$\Delta\varepsilon_{ij}(t_n) = \sum_k\sum_\ell D_{ijk\ell}(\Delta t_n)\Delta\sigma_{k\ell}(t_n) + \widetilde{\varepsilon}_{ij}(t_n) \tag{5.1}$$

where $D_{ijk\ell}(\Delta t_n)$ is a fourth order tensor which can be interpreted as a compliance tensor, it is given by

$$D_{ijk\ell}(\Delta t_n) = J_{ijk\ell}^{(0)} + \sum_{m=1}^{M} J_{ijk\ell}^{(m)}\Big[1 - \frac{1}{\Delta t_n\lambda_{ijk\ell}^{(m)}}\Big(1-\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\Big)\Big] \tag{5.2}$$

and $\widetilde{\varepsilon}_{ij}(t_n)$ is a pseudo-strain tensor which represents the influence of the complete past history of stresses. It is given by

$$\widetilde{\varepsilon}_{ij}(t_n) = -\sum_{k=1}^{3}\sum_{\ell=1}^{3}\sum_{m=1}^{M}\Big(1-\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\Big)\Pi_{ijk\ell}^{(m)}(t_n) + \\ +\sum_{k=1}^{3}\sum_{\ell=1}^{3}\sigma_{k\ell}(t_n)\Big[\sum_{m=1}^{M} J_{ijk\ell}^{(m)}\Big(1-\mathrm{e}^{-\lambda_{ijk\ell}^{(m)}\Delta t_n}\Big)\Big] \tag{5.3}$$

Finally, the incremental constitutive law given by equation (5.1) can now be inverted to obtain

$$\Delta\sigma_{ij}(t_n) = \sum_k\sum_\ell C_{ijk\ell}(\Delta t_n)\Delta\varepsilon_{k\ell}(t_n) - \widetilde{\sigma}_{ij}(t_n) \tag{5.4}$$

where $C_{ijk\ell} = (D_{ijk\ell})^{-1}$ is the inverse of the compliance tensor and $\widetilde{\sigma}_{ij}(t_n)$ is a pseudo-stress tensor which represents the influence of the complete past history of strain. It is given by

$$\widetilde{\sigma}_{ij}(t_n) = \sum_{k=1}^{3}\sum_{\ell=1}^{3} C_{ijk\ell}(\Delta t_n)\widetilde{\varepsilon}_{ij}(t_n) \tag{5.5}$$

The incremental constitutive law represented by equation (5.4) can be introduced in a finite element discretisation in order to obtain solutions to complex viscoelastic problems.

6. Numerical results

The finite element computation is compared with an analytical solution. The incremental constitutive viscoelastic law given by equation (5.4) is implemented in Finite Element software CASTEM (Charvet-Quemin *et al.*, 1986). In order to validate our method, we employ a timber specimen of side 50 mm. The crack length chosen is 25 mm. The external load is a unit loading applied to steel Arcan as seen in Fig. 1 (Moutou Pitti *et al.*, 2008).

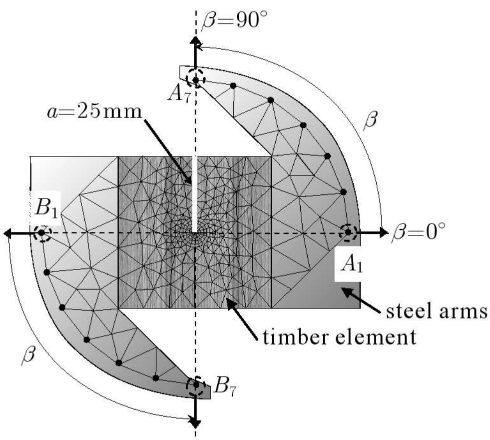

Fig. 1. CTS specimen (Moutou Pitti *et al.*, 2008)

This specimen has similar properties of CTS (Compact Tension Shear) specimens used by Zhang *et al.* (2006), Ma *et al.* (2006) and developed by Richard and Benitz (1983) for an isotropic material. The points A_α and B_α with $\alpha \in (1, \ldots, 7)$ are holes where unspecified forces can be applied with the angle β directed according to the crack in the trigonometrical direction. Pure mode I (opening mode) is obtained by using opposite forces in A_1 and B_1 with $\beta = 0°$. A loading with $\beta = 90°$, in A_7 and B_7, corresponds to the case of pure mode II (shear mode). Intermediary positions induce different mixed mode configurations. The timber element is framed with perfectly rigid steel Arcan.

In order to simplify the analytic development, a time proportionality for the creep tensor is chosen

$$J(t) = \frac{1}{E(t)} C_0 \tag{6.1}$$

in which C_0 is a constant compliance tensor composed by a unit elastic modulus and a constant Poisson's coefficient $\nu = 0.4$, and $E(t)$ represents the tangent modulus for the longitudinal direction. In this context, the creep properties are given in terms of the creep function as given in equation (2.1)

$$\begin{aligned} \frac{1}{E(t)} = \frac{1}{E_X} \Big[1 &+ \frac{1}{74.3}\Big(1 - \mathrm{e}^{-\frac{74.3}{3.37}t}\Big) + \frac{1}{74.4}\Big(1 - \mathrm{e}^{-\frac{74.4}{33.37}t}\Big) + \\ &+ \frac{1}{22.9}\Big(1 - \mathrm{e}^{-\frac{22.9}{104.09}t}\Big) + \frac{1}{27.6}\Big(1 - \mathrm{e}^{-\frac{27.6}{1251}t}\Big) + \\ &+ \frac{1}{7.83}\Big(1 - \mathrm{e}^{-\frac{7.83}{3554}t}\Big) + \frac{1}{3.23}\Big(1 - \mathrm{e}^{-\frac{3.23}{14660}t}\Big)\Big] \end{aligned} \tag{6.2}$$

where E_X is the longitudinal modulus and is equal to 15000 MPa (Guitard, 1987). In this context, $\mathbf{C}_0$ admits the following definition for plane configurations

$$C_0 = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & \dfrac{E_X}{E_Y} & 0 \\ 0 & 0 & \dfrac{E_X}{G_{XY}} \end{bmatrix} \tag{6.3}$$

where E_Y and G_{XY} are the transverse and shear moduli, respectively. Their values are fixed to: $E_Y = 600$ MPa and $G_{XY} = 700$ MPa (elastic pine spruce properties, Guitard, 1987).

In this test, the numerical results are compared to the analytical solution given by the isothermal Helmholtz free energy (Staverman and Schwarzl, 1952). According to the last creep tensor form, the viscoelastic compliance takes the following form in pure mode I and pure mode II, respectively

$$\begin{aligned} C_1(t) &= C_1^0 f(t) = 7.35 \cdot 10^{-3} f(t) \\ C_2(t) &= C_2^0 f(t) = 1.47 \cdot 10^{-3} f(t) \end{aligned} \tag{6.4}$$

in which C_1^0 and C_2^0 are the reduced elastic compliances and

$$f(t) = \Big[1 + \frac{1}{74.3}\Big(1 - \mathrm{e}^{-\frac{74.3}{3.37}t}\Big) + \frac{1}{74.4}\Big(1 - \mathrm{e}^{-\frac{74.4}{33.37}t}\Big) + \frac{1}{22.9}\Big(1 - \mathrm{e}^{-\frac{22.9}{104.09}t}\Big) + \\ + \frac{1}{27.6}\Big(1 - \mathrm{e}^{-\frac{27.6}{1251}t}\Big) + \frac{1}{7.83}\Big(1 - \mathrm{e}^{-\frac{7.83}{3554}t}\Big) + \frac{1}{3.23}\Big(1 - \mathrm{e}^{-\frac{3.23}{14660}t}\Big)\Big] \tag{6.5}$$

In bi-dimensional analysis, we can express the energy release rate by the expression

$$\begin{aligned} {}^1G_v(t) &= \frac{1}{8}[2C_1(t) - C_1(2t)]({}^uK_1^0)^2 \\ {}^2G_v(t) &= \frac{1}{8}[2C_2(t) - C_2(2t)]({}^uK_2^0)^2 \end{aligned} \tag{6.6}$$

where ${}^uK_1^0$ and ${}^uK_2^0$ are the instantaneous stress intensity factors for mode I and mode II, respectively, computed with a classical elastic finite element process. ${}^1G_v$ and ${}^2G_v$ are viscoelastic energy release rates in mode I and mode II, respectively. Now, we present the comparison between the numerical results, given by incremental formulation (5.4), and the analytical solution given by expressions (6.6). The results are presented in Fig. 2 and Fig. 3 for pure mode I and pure mode II versus time. The average error observed in the numerical solution is less than 1% in pure modes I and II.

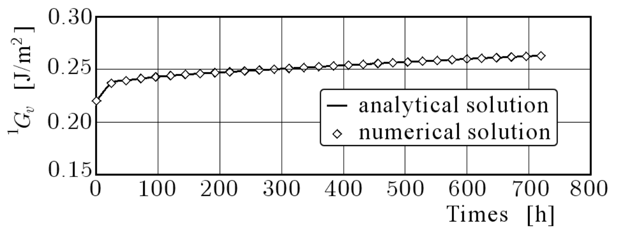

Fig. 2. Analytical and numerical solution in pure mode I for energy release rate ${}^1G_v$

7. Conclusions

A new linear incremental formulation in the time domain for non-aging viscoelastic materials undergoing mechanical deformation have been presented. The formulation is based on a differential approach using a discrete spectrum

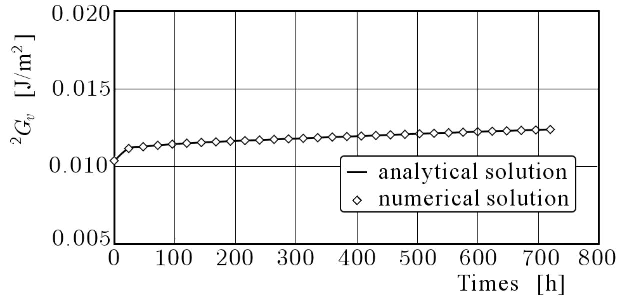

Fig. 3. Analytical and numerical solution in pure mode II for energy release rate ^2G_v

representation for the creep tensor. The governing equations are then obtained using a discretised form of Boltzmann's principle. The analytical solution to differential equations is then obtained using a finite difference discretisation in the time domain. In this way, the incremental constitutive equations for linear viscoelastic material use a pseudo fourth order rigidity tensor. The influence of the whole past history on the behaviour at the time t is given by a pseudo second order tensor. This formulation is introduced in a finite element discretisation. The numerical results obtained are compared with the analytical solution in terms of the energy release rate. The method can be easily extended to deal with ageing boundary viscoelastic problems.

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Nowe sformułowanie przyrostowe w liniowej analizie lepkosprężystości: różniczkowy opis pełzania

Streszczenie

Przedmiotem pracy jest prezentacja nowego przyrostowego opisu niestarzejących się materiałów lepkosprężystych poddanych deformacji w dziedzinie czasu. Sformułowanie wyprowadzono z równań różniczkowych opartych na dyskretnej reprezentacji widma tensora pełzania. Następnie, przyrostowe równania konstytutywne otrzymano w drodze całkowania różnicowego. W ten sposób uniknięto konieczności zachowywania w pamięci komputera historii naprężenia. Kompletna i ogólna liniowa analiza naprężeń lepkosprężystych została przedstawiona za pomocą przyrostów odkształceń i naprężeń. Otrzymane wyniki symulacji numerycznych uzyskano z dobrą dokładnością. Analityczne i numeryczne rozwiązania porównano poprzez zestawienie tempa uwalnianej energii dla czystej postaci I i II.

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