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ANALYSIS OF A PENNY-SHAPED CRACK IN A MAGNETO-ELASTIC MEDIUM

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The problem of a crack in a piezomagnetic material under magneto-mechanical loading is considered. The exact solution, obtained in this work, includes the unknown a priori normal component of the magnetic induction vector inside the crack. Several different physical assumptions associated with limited magnetic permeability of the crack are utilized to determine those unknown magnetic inductions through the crack boundaries. Analytical formulae for the stress and magnetic induction intensity factors are derived. The effects of magnetic boundary conditions (limited permeability) at the crack surface on the basic parameters of fracture mechanics are analysed and some features of the solution are discussed. If the permeability of the medium inside the crack tends to zero or is very large, extreme results i.e. impermeable or permeable crack solutions are obtained.

Key words: magnetoelasticity, limited permeable crack, stress and magnetic intensity factors

1. Introduction

Magneto-mechanical modelling of the piezomagnetic fracture is complicated by the fact that piezomagnetic materials exhibit magneto-elastic coupling behaviour as well as anisotropy.

The attractive property of piezomagnetic materials, that become strained when subjected to a magnetic field, is the underlying foundation for achieving numerous types of smart structures. When subjected to mechanical and magnetical loads in service, piezomagnetic materials may fail prematurely due to their brittleness or due to the presence of defects or flaws produced during their manufacturing process. Therefore, it is important to study the fracture behaviour of piezomagnetic materials.

Among theoretical studies on piezomagnetic bodies, magnetic permeable and impermeable conditions on crack faces are most commonly adopted. For permeable cracks, there is a nonzero magnetic field in the free space inside voids, while for impermeable cracks the magnetic field inside the voids is always zero.

In recent years, the study on magneto-electro-elastic materials with defects or crack has received considerable interest. The magneto-elastic problem of straight cracks lying along the interface of two dissimilar soft ferromagnetic materials subjected to a remote uniform magnetic induction was considered by Lin and Lin (2002). The magneto-elastic coupling effect in an infinite soft ferromagnetic material with a crack was also studied by Liang *et al.* (2002), where the nonlinear effect of magnetic field upon stress and the effect of deformed crack configuration were taken into consideration. Those papers considered the coupling between magnetic and elastic fields. The electro-elastic field inside a piezoelectric material, where the limited electrical permeability inside the crack was taken into account, were considered, and closed form solutions were derived by Rogowski (2007). Rogowski (2008) discussed the limited electric boundary conditions on the crack faces in electro-elastic materials under transient thermal loading and also mechanic and electric (two cases) loadings, and closed form solutions were obtained.

In this paper, a limited permeable crack model is considered. The effects of magnetic boundary conditions (limited permeability) at the crack surfaces on the fracture mechanics of piezomagnetic materials are analysed and some features of the solutions are discussed. In two limiting cases (infinitely large or zero magnetic permeablities of the medium inside the crack) we can obtain the limiting solutions from the general results presented here. For piezoelectric materials, there are two kinds of ideal electric boundary conditions for the crack faces, that is, electrically impermeable crack and electrically permeable crack (Zhang *et al.*, 2002).

Although this paper is a generalisation to magnetoelasticity described by Zhang *et al.* (2002) who dealt with a piezoelectric medium in addition to different results regarding expressions for elastic and magnetic fields, there exists one distinct difference, namely the limited permeable crack boundary conditions are considered.

The physical laws for piezomagnetic materials were explored by Nowacki (1983). Many theoretical problems can be found in the book by Purcell (1965) and Parkus (1972).

2. Fundamental equations for piezomagnetic medium

We consider an axi-symmetric problem. Assume that field variables are functions of r and z in the cylindrical coordinate system (r,θ,z). Constitutive equations for a piezomagnetic material polarised along the z direction subjected to mechanical and magnetic fields can be written as

$$(\sigma_{rr},\sigma_{\theta\theta},\sigma_{zz},\sigma_{rz},B_r,B_z)^\top = \mathbf{C}(\varepsilon_r,\varepsilon_\theta,\varepsilon_z,\varepsilon_{rz},H_r,H_z)^\top \tag{2.1}$$

where $(\cdot)^\top$ denotes the transpose of a matrix and

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & -q_{31} \\ c_{12} & c_{11} & c_{13} & 0 & 0 & -q_{31} \\ c_{13} & c_{13} & c_{33} & 0 & 0 & -q_{33} \\ 0 & 0 & 0 & c_{44} & -q_{15} & 0 \\ 0 & 0 & 0 & q_{15} & \mu_{11} & 0 \\ q_{31} & q_{31} & q_{33} & 0 & 0 & \mu_{33} \end{bmatrix} \tag{2.2}$$

Here σ_{ij}, B_i and H_i are stresses, components of the magnetic induction vector and components of the magnetic field vector; c_{ij}, q_{ij} and μ_{ij} are elastic constants, piezomagnetic constants and magnetic permeabilities, respectively. The strain is related to the mechanical displacements u_r, u_z as follows

$$\begin{aligned} &\varepsilon_r = \frac{\partial u_r}{\partial r} \qquad \varepsilon_\theta = \frac{u_r}{r} \qquad \varepsilon_z = \frac{\partial u_z}{\partial z} \\ &\varepsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \end{aligned} \tag{2.3}$$

The equilibrium equations for stresses and magnetic flux are

$$\begin{aligned} &\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0 \qquad \frac{\partial \sigma_{zr}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r} = 0 \\ &\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{\partial B_z}{\partial z} = 0 \end{aligned} \tag{2.4}$$

Here we neglect the body forces and magnetic sources in piezomagnetic ceramics.

The Maxwell equations in the quasi-static approximation are

$$H_r = -\frac{\partial \phi}{\partial r} \qquad H_z = -\frac{\partial \phi}{\partial z} \tag{2.5}$$

where $\phi(r,z)$ is the magnetic potential.

Substituting equations (2.1), (2.2) and (2.3) into equations (2.4) and using relations (2.5), we obtain the following equilibrium equations

$$
\begin{aligned}
&c_{11}B_1u_r + c_{44}D^2u_r + (c_{13}+c_{44})D\frac{\partial u_z}{\partial r} + (q_{31}+q_{15})D\frac{\partial \phi}{\partial r} = 0\\
&c_{44}B_0u_z + c_{33}D^2u_z + (c_{13}+c_{44})D\frac{\partial (ru_r)}{r\partial r} + q_{15}B_0\phi + q_{33}D^2\phi = 0\\
&(q_{31}+q_{15})D\frac{\partial (ru_r)}{r\partial r} + q_{15}B_0u_z + q_{33}D^2u_z - \mu_{11}B_0\phi - \mu_{33}D^2\phi = 0
\end{aligned}
\tag{2.6}
$$

where the following differential operators have been introduced

$$
B_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k}{r^2} \qquad k = 0,1 \qquad D = \frac{\partial}{\partial z} \tag{2.7}
$$

The quasi-harmonic functions $\varphi_i(r,z)$, such that

$$
\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{\lambda_i^2}\frac{\partial^2}{\partial z^2}\Big)\varphi_i(r,z) = 0 \qquad i = 1,2,3 \tag{2.8}
$$

determine all field variables as follows

$$
\begin{aligned}
u_r(r,z) &= \sum_{i=1}^3 a_{i1}\lambda_i\frac{\partial \varphi_i}{\partial r} & u_z(r,z) &= \sum_{i=1}^3 \frac{1}{\lambda_i}\frac{\partial \varphi_i}{\partial z}\\
\phi(r,z) &= -\sum_{i=1}^3 \frac{a_{i3}}{\lambda_i}\frac{\partial \varphi_i}{\partial z} & \sigma_{rr} &= -\sum_{i=1}^3 \frac{a_{i4}}{\lambda_i}\frac{\partial^2 \varphi_i}{\partial z^2} - (c_{11}-c_{12})\frac{u_r}{r}\\
\sigma_{zz} &= \sum_{i=1}^3 \frac{a_{i4}}{\lambda_i^3}\frac{\partial^2 \varphi_i}{\partial z^2} & \sigma_{\theta\theta} &= -\sum_{i=1}^3 \frac{a_{i4}}{\lambda_i}\frac{\partial^2 \varphi_i}{\partial z^2} - (c_{11}-c_{12})\frac{\partial u_r}{\partial r}\\
\sigma_{zr} &= \sum_{i=1}^3 \frac{a_{i4}}{\lambda_i}\frac{\partial^2 \varphi_i}{\partial r\partial z} & H_r &= -\frac{\partial \phi}{\partial r} = \sum_{i=1}^3 \frac{a_{i3}}{\lambda_i}\frac{\partial^2 \varphi_i}{\partial r\partial z}\\
H_z &= -\frac{\partial \phi}{\partial z} = \sum_{i=1}^3 \frac{a_{i3}}{\lambda_i}\frac{\partial^2 \varphi_i}{\partial z^2} & B_r &= \sum_{i=1}^3 a_{i5}\lambda_i\frac{\partial^2 \varphi_i}{\partial r\partial z}\\
B_z &= \sum_{i=1}^3 \frac{a_{i5}}{\lambda_i}\frac{\partial^2 \varphi_i}{\partial z^2}
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
a_{i1} &= \frac{a_1\lambda_i^2 + b_1}{a_2\lambda_i^4 + b_2\lambda_i^2 + c_2} \qquad\qquad a_{i3} = \frac{c_{13}+c_{44}}{q_{31}+q_{15}} - \frac{c_{11}-c_{44}\lambda_i^2}{q_{31}+q_{15}}a_{i1} \\
a_{i4} &= \frac{q_{31}c_{44}\lambda_i^2 + q_{15}c_{11}}{q_{31}+q_{15}}a_{i1} + \frac{c_{44}q_{31}-c_{13}q_{15}}{q_{31}+q_{15}} \\
a_{i5} &= \frac{q_{33}\mu_{11}-q_{15}\mu_{33}}{\mu_{11}-\mu_{33}\lambda_i^2} - \frac{q_{31}\mu_{11}-q_{15}\mu_{33}\lambda_i^2}{\mu_{11}-\mu_{33}\lambda_i^2}a_{i1} \\
a_1 &= c_{33}(q_{31}+q_{15}) - (c_{13}+c_{44})q_{33} \qquad\qquad a_2 = c_{44}q_{33} \\
b_1 &= c_{13}q_{15} - c_{44}q_{31} \\
b_2 &= (c_{13}+c_{44})q_{31} + c_{13}q_{15} - c_{11}q_{33} \qquad\qquad c_2 = c_{11}q_{15}
\end{aligned}
\tag{2.10}
$$

and λ_i^2 $(i = 1,2,3)$ are the roots of the following cubic algebraic equation in λ_i^2

$$
a_0\lambda^6 + b_0\lambda^4 + c_0\lambda^2 + d_0 = 0 \tag{2.11}
$$

with the coefficients defined by

$$
\begin{aligned}
a_0 &= c_{44}(c_{33}\mu_{33} + q_{33}^2) \\
b_0 &= (q_{31}+q_{15})[2c_{13}q_{33} - c_{33}(q_{31}+q_{15})] + 2c_{44}q_{33}q_{31} - c_{11}q_{33}^2 + \\
&\quad -\mu_{11}c_{33}c_{44} - \mu_{33}c^2 \\
c_0 &= 2q_{15}[c_{11}q_{33} - c_{13}(q_{31}+q_{15})] + c_{44}q_{31}^2 + \mu_{33}c_{11}c_{44} + \mu_{11}c^2 \\
d_0 &= -c_{11}(c_{44}\mu_{11} + q_{15}^2) \\
c^2 &= c_{11}c_{33} - c_{13}(c_{13}+2c_{44})
\end{aligned}
\tag{2.12}
$$

The roots of the above equation for a real material can be expressed for two cases:

(a) $+R_1,\ -R_1,\ +R_2,\ -R_2,\ +R_3,\ -R_3$

(b) $+R_1,\ -R_1,\ R_2+\mathrm{i}R_3,\ R_2-\mathrm{i}R_3,\ -R_2+\mathrm{i}R_3,\ -R_2-\mathrm{i}R_3$

where R_1, R_2, R_3 are positive real numbers and $\mathrm{i} = \sqrt{-1}$.

3. Formulation of the problem

Consider a crack with a finite dimension in a transversely isotropic piezomagnetic solid under combined mechanical (σ_∞) and pure magnetical loads $(B_\infty$ or $H_\infty)$ applied at infinity (Fig. 1).

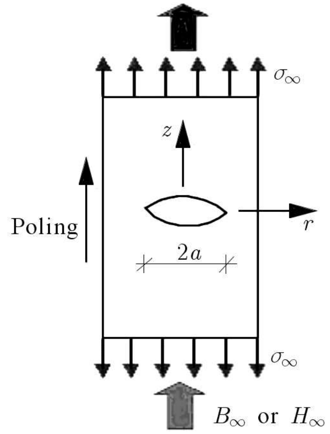
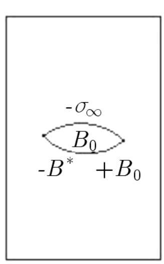


Fig. 1. A crack in a magnetoelastic medium and loading conditions

To solve the crack problem in linear elastic solids, the superposition technique is usually used. Thus, we first solve the stress and magnetic field problem without the cracks in the medium under magnetical and/or mechanical loads. Then, we use equal and opposite stresses and magnetic inductions as the crack surface tractions and solve the crack problem (the so called perturbation problem, Fig. 2)



$$\sigma_\infty = \begin{cases} (1+q_0)\sigma_0 - \widetilde{c}_1 B_\infty & \text{case I} \\ \sigma_0 - q_3 H_\infty & \text{case II} \end{cases}$$

$$B^* = \begin{cases} B_\infty & \text{case I} \\ \mu_0\sigma_0 + \mu_3 H_\infty & \text{case II} \end{cases}$$

$$H^* = \begin{cases} -q_2\sigma_0 + \widetilde{c}_3 B_\infty & \text{case I} \\ H_\infty & \text{case II} \end{cases}$$

$$\widetilde{c}_3 = \frac{c_{11}+c_{12}}{q_1} \qquad q_2 = \widetilde{c}_3\mu_0$$

Fig. 2. Crack loading in the perturbation problem

The material parameters in the above solution are

$$\begin{aligned} & q_1 = \mu_{33}(c_{11}+c_{12}) + 2q_{31}^2 && q_0 = \frac{\widetilde{c}_2^2}{\widetilde{c}_0^2 q_1} = \frac{\widetilde{c}_2\mu_0}{q_1} && \mu_0 = \frac{\widetilde{c}_2}{\widetilde{c}_0^2} \\ & \widetilde{c}_2 = q_{33}(c_{11}+c_{12}) - 2q_{31}c_{13} && \widetilde{c}_1 = \frac{\widetilde{c}_2}{q_1} \\ & \mu_3 = \mu_{33} + \frac{2q_{31}^2}{c_{11}+c_{12}} && q_3 = q_{33} - \frac{2c_{13}}{c_{11}+c_{12}}q_{31} \end{aligned} \tag{3.1}$$

Note that σ_0 is the uniform normal stress at zero magnetical loads.

Employing the superposition principle, one arrives at an equivalent problem with the loading $\sigma_z = -\sigma_\infty$, $B_z = B_0 - B^*$ being applied on both surfaces of the crack.

Inside the crack there is often air or vacuum, and the magnetic induction is usually considered constant under a uniform remote applied load. This unknown component is denoted by B_0 and the following assumption is stated to determine B_0

$$B_0 = \mu_a H_z^c \tag{3.2}$$

where μ_a is the magnetic permeability of the medium inside the crack and H_z^c is the component of the magnetic field vector in the z-direction inside the crack.

The quasi-harmonic function needed for the solution is

$$\varphi_i(r,z) = \int\limits_0^\infty A_i(\xi)\exp(-\lambda_i \xi z) J_0(\xi r)\, d\xi \tag{3.3}$$

4. Solution for a limited magnetically permeable crack problem

The boundary conditions along the crack plane $z = 0$ are stated as follows

$$\begin{array}{lll} u_z(r,0) = 0 \qquad \phi(r,0) = 0 & & r \geqslant a \\ \sigma_{zr}(r,0) = 0 & & r \geqslant 0 \\ B_z(\rho,0) = B_0 - B^* & & 0 \leqslant r < a \\ \sigma_{zz}(r,0) = -\sigma_\infty & & 0 \leqslant r < a \end{array} \tag{4.1}$$

The mechanical crack boundary conditions give

$$\begin{array}{ll} \displaystyle\sum_{i=1}^3 \frac{a_{i4}}{\lambda_i} \int\limits_0^\infty \xi^2 A_i(\xi) J_0(r\xi)\, d\xi = -\sigma_\infty & 0 \leqslant r < a \\ \displaystyle -\sum_{i=1}^3 \int\limits_0^\infty \xi A_i(\xi) J_0(r\xi)\, d\xi = 0 & r \geqslant a \\ \displaystyle\sum_{i=1}^3 a_{i4} \int\limits_0^\infty \xi^2 A_i(\xi) J_1(r\xi)\, d\xi = 0 & r \geqslant 0 \end{array} \tag{4.2}$$

The magnetical crack boundary conditions are

$$
\begin{aligned}
&\sum_{i=1}^{3} a_{i3} \int\limits_0^\infty \xi A_i(\xi) J_0(r\xi)\, d\xi = 0 && r \geqslant a \\
&\sum_{i=1}^{3} a_{i5}\lambda_i \int\limits_0^\infty \xi^2 A_i(\xi) J_0(r\xi)\, d\xi = B_0 - B^* && 0 \leqslant r < a
\end{aligned}
\tag{4.3}
$$

Substituting

$$
\begin{aligned}
&A_1(\xi) + A_2(\xi) + A_3(\xi) = D_1(\xi) \\
&a_{13}A_1(\xi) + a_{23}A_2(\xi) + a_{33}A_3(\xi) = D_2(\xi) \\
&a_{14}A_1(\xi) + a_{24}A_2(\xi) + a_{34}A_3(\xi) = 0
\end{aligned}
\tag{4.4}
$$

and solving this system of algebraic equations, we obtain

$$
m_2 A_i(\xi) = d_i D_1(\xi) + l_i D_2(\xi) \tag{4.5}
$$

where

$$
\begin{aligned}
&m_2 = \sum_{i=1}^{3} d_i && d_1 = a_{24}a_{33} - a_{34}a_{23} \\
&d_2 = a_{13}a_{34} - a_{14}a_{33} && d_3 = a_{14}a_{23} - a_{13}a_{24} \\
&l_1 = a_{34} - a_{24} && l_2 = a_{14} - a_{34} \qquad\qquad l_3 = a_{24} - a_{14}
\end{aligned}
\tag{4.6}
$$

The boundary conditions lead to two pairs of simultaneous dual integral equations

$$
\begin{aligned}
&m \int\limits_0^\infty \xi^2 D_1(\xi) J_0(r\xi)\, d\xi + m_6 \int\limits_0^\infty \xi^2 D_2(\xi) J_0(r\xi)\, d\xi = -\sigma_\infty m_2 && 0 \leqslant r < a \\
&\int\limits_0^\infty \xi D_1(\xi) J_0(r\xi)\, d\xi = 0 && r \geqslant a \\
&m_5 \int\limits_0^\infty \xi^2 D_1(\xi) J_0(r\xi)\, d\xi + m_7 \int\limits_0^\infty \xi^2 D_2(\xi) J_0(r\xi)\, d\xi = (B_0 - B^*) m_2 && 0 \leqslant r < a \\
&\int\limits_0^\infty \xi D_2(\xi) J_0(r\xi)\, d\xi = 0 && r \geqslant a
\end{aligned}
\tag{4.7}
$$

The solution to those equations is

$$
\begin{aligned}
D_1(\xi) &= \frac{2}{\pi}\frac{m_2}{\widetilde{m}}[m_7\sigma_\infty + m_6(B_0 - B^*)]\frac{1}{\xi}\frac{d}{d\xi}\Big(\frac{\sin\xi a}{\xi}\Big)\\
D_2(\xi) &= -\frac{2}{\pi}\frac{m_2}{\widetilde{m}}[m_5\sigma_\infty + m(B_0 - B^*)]\frac{1}{\xi}\frac{d}{d\xi}\Big(\frac{\sin\xi a}{\xi}\Big)
\end{aligned}
\tag{4.8}
$$

where

$$
\begin{aligned}
&\widetilde{m} = mm_7 - m_5m_6 \qquad && m_5 = \sum_{i=1}^{3} a_{i5}\lambda_i d_i \qquad && m_6 = \sum_{i=1}^{3}\frac{a_{i4}l_i}{\lambda_i}\\
&m_7 = \sum_{i=1}^{3} a_{i5}\lambda_i l_i && m = \sum_{i=1}^{3}\frac{a_{i4}d_i}{\lambda_i}
\end{aligned}
\tag{4.9}
$$

The physical quantities are obtained as follows

$$
\begin{aligned}
u_r(r,z) &= \frac{r}{\pi\widetilde{m}}\sum_{i=1}^{3} a_{i1}\lambda_i\widetilde{d}_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i - \frac{\zeta_i}{1+\zeta_i^2}\Big) + r\varepsilon_{r\infty}\\
u_z(r,z) &= \frac{2a}{\pi\widetilde{m}}\sum_{i=1}^{3}\widetilde{d}_i\eta_i\Big[1 - \zeta_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i\Big)\Big] + z\varepsilon_{z\infty}\\
\phi(r,z) &= -\frac{2a}{\pi\widetilde{m}}\sum_{i=1}^{3} a_{i3}\widetilde{d}_i\eta_i\Big[1 - \zeta_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i\Big)\Big] - H^*z\\
\sigma_{zr}(r,z) &= -\frac{2r}{\pi\widetilde{m}a}\sum_{i=1}^{3} a_{i4}\widetilde{d}_i\frac{\eta_i}{(1+\zeta_i^2)(\zeta_i^2+\eta_i^2)}\\
\sigma_{zz}(r,z) &= -\frac{2}{\pi\widetilde{m}}\sum_{i=1}^{3}\frac{a_{i4}}{\lambda_i}\widetilde{d}_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i - \frac{\zeta_i}{\zeta_i^2+\eta_i^2}\Big) + \sigma_\infty\\
\sigma_{rr}(r,z) &= \frac{2}{\pi\widetilde{m}}\sum_{i=1}^{3} a_{i4}\lambda_i\widetilde{d}_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i - \frac{\zeta_i}{\zeta_i^2+\eta_i^2}\Big) - (c_{11}-c_{12})\Big(\frac{u_r}{r} - \varepsilon_{r\infty}\Big)\\
\sigma_{\theta\theta}(r,z) &= \frac{2}{\pi\widetilde{m}}\sum_{i=1}^{3} a_{i4}\lambda_i\widetilde{d}_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i - \frac{\zeta_i}{\zeta_i^2+\eta_i^2}\Big) - (c_{11}-c_{12})\Big(\frac{\partial u_r}{\partial r} - \varepsilon_{r\infty}\Big)\\
H_r(r,z) &= -\frac{2}{\pi\widetilde{m}}\frac{r}{a}\sum_{i=1}^{3} a_{i3}\widetilde{d}_i\frac{\eta_i}{(1+\zeta_i^2)(\zeta_i^2+\eta_i^2)}\\
H_z(r,z) &= -\frac{2}{\pi\widetilde{m}}\sum_{i=1}^{3} a_{i3}\lambda_i\widetilde{d}_i\Big(\frac{\pi}{2} - \tan^{-1}\zeta_i - \frac{\zeta_i}{\zeta_i^2+\eta_i^2}\Big) + H^*
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
B_r(r,z) &= -\frac{2}{\pi\widetilde{m}}\frac{r}{a}\sum_{i=1}^{3} a_{i5}\lambda_i^2\widetilde{d}_i\frac{\eta_i}{(1+\zeta_i^2)(\zeta_i^2+\eta_i^2)}\\
B_z(r,z) &= -\frac{2}{\pi\widetilde{m}}\sum_{i=1}^{3} a_{i5}\lambda_i\widetilde{d}_i\Big(\frac{\pi}{2}-\tan^{-1}\zeta_i-\frac{\zeta_i}{\zeta_i^2+\eta_i^2}\Big)+B^*
\end{aligned}
$$

where

$$
\begin{aligned}
&\widetilde{d}_i = (m_7 d_i - m_5 l_i)\sigma_\infty - (B^* - B_0)(m_6 d_i - m l_i)\\
&\varepsilon_{r\infty} = \begin{cases} -\Big(\dfrac{c_{13}}{\widetilde{c}_0^2}+\dfrac{q_{31}}{q_1}\mu_0\Big)\sigma_0+\dfrac{q_{31}}{q_1}B_\infty & \text{case I}\\ -\dfrac{c_{13}}{\widetilde{c}_0^2}\sigma_0+\dfrac{q_{31}}{c_{11}+c_{12}}H_\infty & \text{case II}\end{cases}\\
&\varepsilon_{z\infty} = \frac{c_{11}+c_{12}}{\widetilde{c}_0^2}\sigma_0 \qquad \text{for case I and case II}\\
&\widetilde{c}_0^2 = c_{33}(c_{11}+c_{12}) - 2c_{13}^2
\end{aligned}
\tag{4.11}
$$

Closed form solutions (4.10) for elastic and magnetic fields are obtained according to the improper integrals presented analytically by oblate spheroidal co-ordinates (Rogowski, 2007)

$$
r^2 = a^2(1+\zeta_i^2)(1-\eta_i^2) \qquad \lambda_i z = a\zeta_i\eta_i \qquad i=1,2,3 \tag{4.12}
$$

5. Solutions for different assumptins on magnetic boundary conditions

Two different assumptions on the magnetic boundary condition on crack surfaces are analysed as described below.

5.1. The notch solution

We assume that the potential at the crack-notch interface is continuous and that along the z direction the magnetic field H_z^c and the magnetic induction B_0 on the upper notch surface can be written as

$$
H_z^c(r) = -\frac{\varphi^+ - \varphi^-}{\delta(r)} \qquad B_0 = -\mu_a\frac{\varphi^+ - \varphi^-}{\delta(r)} \tag{5.1}
$$

where $\delta(r)$ describes the shape of the notch.

Thus

$$B_0 = -\frac{4}{\pi}\frac{m_2}{\widetilde{m}}\mu_a \frac{\sqrt{a^2 - r^2}}{\delta(r)}[m_5\sigma_\infty - m(B^* - B_0)] \tag{5.2}$$

From this equation, we may determine the unknown B_0.

If we assume an elliptic notch, such that

$$\delta(r) = \frac{\delta_0}{a}\sqrt{a^2 - r^2} \tag{5.3}$$

where δ_0 is the thickness of the notch at $r = 0$, then we obtain

$$B_0 = -\frac{m_5\sigma_\infty - mB^*}{m + \frac{\pi}{4}\frac{\widetilde{m}}{m_2}\frac{\delta_0}{a\mu_a}} \tag{5.4}$$

The magnetic induction intensity factor is obtained as follows

$$K_B = \frac{2}{\pi}\sqrt{a}(B^* - B_0) = \frac{2}{\pi}\sqrt{a}\frac{m_5\sigma_\infty + \frac{\pi}{4}\frac{\widetilde{m}}{m_2}\frac{\delta_0}{a\mu_a}B^*}{m + \frac{\pi}{4}\frac{\widetilde{m}}{m_2}\frac{\delta_0}{a\mu_a}} \tag{5.5}$$

If the notch interior is filled with a conductive medium such that μ_a tends to infinity, then K_B and B_0 are

$$K_B^{perm} = \frac{2}{\pi}\frac{m_5}{m}\sigma_\infty\sqrt{a} \qquad\qquad B_0^{perm} = -\frac{m_5}{m}\sigma_\infty + B^* \tag{5.6}$$

which is the permeable crack solution. In this case, $H_r = 0$ and $\phi(r, 0) = 0$ on the whole crack plane. Therefore, the solution for the permeable crack is dependent on the magnetic induction B_∞ (case I) and magnetic field H_∞ (case II) and on the stress in both cases of loading, since

$$\sigma_\infty = \begin{cases} (1 + q_0)\sigma_0 - \widetilde{c}_1 B_\infty & \text{case I} \\ \sigma_0 - q_3 H_\infty & \text{case II} \end{cases} \tag{5.7}$$

The values of m_5/m are $17.84 \cdot 10^{-10}$ m/A and $11.58 \cdot 10^{-10}$ m/A for $CoFe_2O_4$ and composite, respectively. This implies that a tensile stress will produce a negative induction inside the notch (for $B^* = 0$) in a single material and composite. The permeable magnetic induction intensity factor assumes positive values.

If the permeability of the notch is very small, such that we may take $\mu_a = 0$, then

$$K_B^{imp} = \frac{2}{\pi}B^*\sqrt{a} \qquad\qquad B_0^{imp} = 0 \tag{5.8}$$

wich is the impermeable crack solution.

Therefore, the solutions for the impermeable crack depend on the magnetic induction B_∞ (case I) and the magnetic field H_∞ and stress σ_0 (case II), since

$$B^* = \begin{cases} B_\infty & \text{case I} \\ \mu_0\sigma_0 + \mu_3 H_\infty & \text{case II} \end{cases} \tag{5.9}$$

In general

$$K_B = K_B^{imp}\Big[1 - f\Big(\frac{\delta_0}{a\mu_a}\Big)\Big] + K_B^{perm} f\Big(\frac{\delta_0}{a\mu_a}\Big) \tag{5.10}$$

where

$$\begin{aligned} f\Big(\frac{\delta_0}{a\mu_a}\Big) &= \frac{1}{1 + \frac{\pi}{4}\frac{\widetilde{m}}{m_2 m}\frac{\delta_0}{a\mu_a}} \\ K_B^{imp} &= \frac{2}{\pi}B^*\sqrt{a} \qquad\qquad K_B^{perm} = \frac{2}{\pi}\frac{m_5}{m}\sigma_\infty\sqrt{a} \end{aligned} \tag{5.11}$$

we have

$$\begin{aligned} \frac{K_B}{K_I^*} &= \begin{cases} \dfrac{m_5}{m}(1+q_0)\Big[1 - \dfrac{\widetilde{c}_1}{1+q_0}\dfrac{B_\infty}{\sigma_0}\Big] f\Big(\dfrac{\delta_0}{a\mu_a}\Big) + \dfrac{B_\infty}{\sigma_0}\Big[1 - f\Big(\dfrac{\delta_0}{a\mu_a}\Big)\Big] & \text{case I} \\ \dfrac{m_5}{m}\Big[1 - q_3\dfrac{H_\infty}{\sigma_0}\Big] f\Big(\dfrac{\delta_0}{a\mu_a}\Big) + \mu_0\Big(1 + \dfrac{\mu_3}{\mu_0}\dfrac{H_\infty}{\sigma_0}\Big)\Big[1 - f\Big(\dfrac{\delta_0}{a\mu_a}\Big)\Big] & \text{case II} \end{cases} \\ K_I^* &= \frac{2}{\pi}\sigma_0\sqrt{a} \end{aligned} \tag{5.12}$$

Figure 3 shows the dependence of $f(\delta_0/(a\mu_a))$ on $\delta_0/(a\mu_a)$ for composite $BaTiO_3$-$CoFe_2O_4$ and piezomagnetic $CoFe_2O_4$. Material properties for $BaTiO_3$ and $CoFe_2O_4$ are taken from Huang *et al.* (1998). The properties of $BaTiO_3$-$CoFe_2O_4$ composite are obtained by averaging the properties of single-phase $BaTiO_3$ and $CoFe_2O_4$ materials using the rule of mixtures. This implies that the $BaTiO_3$-to-$CoFe_2O_4$ ratio in the composite is roughly 50:50.

We observe that $f(\delta_0/(a\mu_a))$ approaches zero as μ_a tends to zero, and is unity as μ_a tends to infinity. The solution perfectly matches the exact solution in both limiting cases, namely impermeable and permeable magnetic crack boundary conditions. Figure 3 shows the dependence of K_B on $\delta_0/(a\mu_a)$, since K_B is weighted by function $f(\delta_0/(a\mu_a))$ as shown Eqs. (5.12).

Note that

$$K_I = \begin{cases} \Big(1 + q_0 - \widetilde{c}_1\dfrac{B_\infty}{\sigma_0}\Big)K_I^* > 0 & \text{case I} \\ \Big(1 - q_3\dfrac{H_\infty}{\sigma_0}\Big)K_I^* > 0 & \text{case II} \end{cases} \tag{5.13}$$

for the crack tip opening displacement to exist.

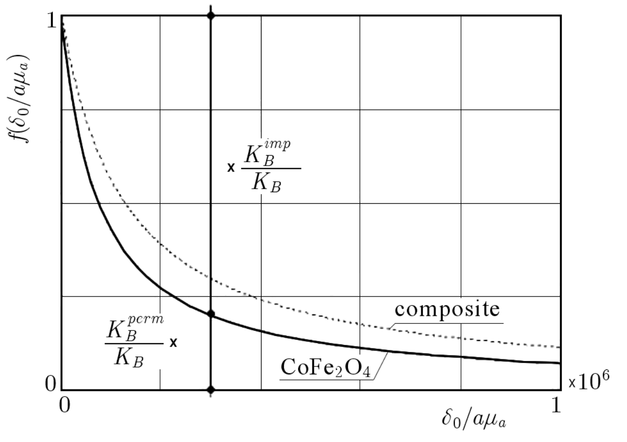


Fig. 3. K_B versus $\delta_0/(a\mu_a)$ for $BaTiO_3$-$CoFe_2O_4$ composite and piezomagnetic $CoFe_2O_4$

The stress intensity factor vanishes if

$$\begin{aligned} \frac{B_\infty}{\sigma_0} &= \frac{1+q_0}{\widetilde{c}_1} && \text{case I} \\ \frac{H_\infty}{\sigma_0} &= \frac{1}{q_3} && \text{case II} \end{aligned} \tag{5.14}$$

The right-hand sides of those equations are $5,303 \cdot 10^{-6}\,\mathrm{m/A}$ and $3.049 \cdot 10^{-6}\,\mathrm{m/A}$ (case I) and $0.033\,\mathrm{Am/N}$ and $0.036\,\mathrm{Am/N}$ (case II) for single material $CoFe_2O_4$ and composite, respectively.

5.2. The effect of crack opening displacement

We assume that the magnetic field inside the crack can be found by

$$H_z^a = -\frac{\varphi^+ - \varphi^-}{u_z^+ - u_z^-} \tag{5.15}$$

Taking into account that

$$B_0 = \mu_a H_z^a \tag{5.16}$$

one arrives at the magnetic condition

$$B_0 u_z(r) = -\mu_a \varphi(r) \tag{5.17}$$

along the crack region.

Using the above obtained results for $u_z(r)$ and $\varphi(r)$ on the crack surface, we obtain

$$B_0 = -\mu_a \frac{m_5\sigma_\infty + m(B_0 - B^*)}{m_7\sigma_\infty + m_6(B_0 - B^*)} \tag{5.18}$$

This gives a quadratic equation with respect to B_0

$$\eta_1 B_0^2 + \eta_2 B_0 + \eta_3 = 0 \tag{5.19}$$

where

$$\begin{aligned} &\eta_1 = m_6 \qquad\qquad \eta_2 = m_7\sigma_\infty - m_6 B^* + m\mu_a \\ &\eta_3 = (m_5\sigma_\infty - mB^*)\mu_a \end{aligned} \tag{5.20}$$

For the two limiting cases, we obtain

(a) $\mu_a = 0$, $B_0^{imp} = 0$, since $m_7\sigma_\infty - m_6 B^* \neq 0$

(b) $\mu_a \to \infty$, $B_0^{perm} = -(m_5/m)\sigma_\infty + B^*$

which are the solutions for impermeable and permeable magnetic crack boundary conditions, respectively. For this model of magnetic boundary conditions, the magnetic induction intensity factors in the extreme cases are the same as given by equations (5.6) and (5.8), respectively.

For $\mu_a \to \infty$ we obtain

$$u_z(r,0) = \frac{2}{\pi}\sigma_\infty \frac{m_2}{\widetilde{m}}\sqrt{a^2 - r^2} \qquad \text{(permeable)} \tag{5.21}$$

For $\mu_a \to 0$, we have

$$u_z(r,0) = \frac{2}{\pi}\frac{m_2}{\widetilde{m}}(m_7\sigma_\infty - m_6 B^*)\sqrt{a^2 - r^2} \qquad \text{(impermeable)} \tag{5.22}$$

Figure 4 shows one half of the crack opening displacement for two analysed materials and permeable or impermeable boundary conditions for $\sigma_\infty = 10\,\mathrm{MPa}$ and $B^* = 0.01\,\mathrm{N/(Am)}$ (case I).

Note that for both materials the crack opening displacement is large for the permeable boundary conditions in comparison to the impermeable case, but the difference is not visible.

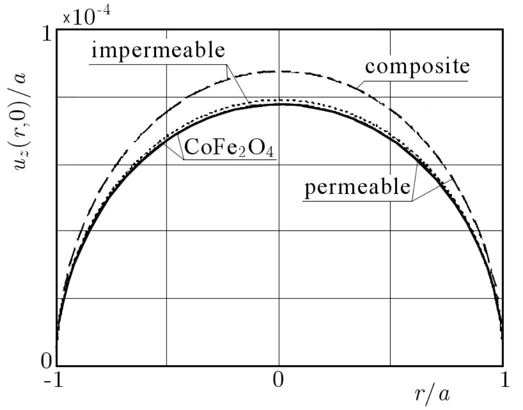


Fig. 4. The crack opening displacement for some magnetic materials

6. Concluding remarks

The following conclusions can be made based on the results obtained in the paper:

(a) The stress intensity factor does not depend on the assuptions applied to the crack-face magnetic boundary condition assumptions.

(b) The stress intensity factor depends on the applied mechanical and magnetic loads and on the material constants (details are given in Section 5 and in Eq. (5.13)).

(c) The stress intensity factor decreases with the magnetic field if the field is applied in the poling direction; in the opposite case K_I increases.

(d) The magnetic induction intensity factor depends on the properties of the material and on the applied magnetic and mechanical loads, as shown by equations (5.12).

(e) The magnetic permeability of air or vacuum inside the crack cannot be ignored while calculating the magnetic induction intensity factor. The effect of finite thickness of a very flat notch or of a crack opening displacement in a realistic structure must be assessed. It can be seen that the function of permittivity and material parameters $f(\cdot)$ describes the ratio of the normal magnetic induction which is stored inside the crack to the total normal magnetic induction B_0^{perm} which may be stored inside the crack. Hence, it can be said that the calculated magnetic induction intensity factor is an average magnetic induction intensity factor weighted by the function $f(\cdot)$. This situation exists for the notch solution model given by solution (5.11) and shown graphically in Fig. 3. However, knowledge of the notch thickness to length ratio is essential for obtaining the correct B_0 in this model. Thus, the crack opening displacement model is more useful. The crack opening displacement is obtained explicitly in a closed-form, and this model may be applied to analysis of fracture of piezomagnetic materials in engineering applications.

(f) Summing up, it must be emphasised that the basic discrepancy exists in the field singularity for a crack and notch. For an elliptic hole, unlike a crack, the field has no singularity. Therefore, it should be noted that the field intensity factors presented in Section 5.1 are only valid for very flat notches (when the notch thickness-to-length ratio δ_0/a is very small).

One cannot find in the open literature a substantial experimental data for piezomagnetic materials about the applicability of permeable or impermeable boundary conditions and about the magnetic permeability inside a crack of a piezomagnetic material. But the comparison with experimental results is critical for the assessment of the most appropriate boundary conditions. The works in scientific laboratories must be underway and test results must be published.

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Analiza kołowej szczeliny w ciele magnetosprężystym

Streszczenie

Rozpatrzono zagadnienie szczeliny w materiale piezomagnetycznym przy obciążeniu mechanicznym i magnetycznym. Dokładne rozwiązanie, otrzymane w tej pracy, zawiera nieznaną *a priori* normalną składową magnetycznej indukcji wewnątrz szczeliny. Fizyczne założenia, odnoszące się do ograniczonej magnetycznej przenikalności ośrodka wypełniającego szczelinę oraz magnetycznych warunków na brzegu szczeliny, prowadzą do wyznaczenia tej magnetycznej indukcji. Otrzymano analityczne wzory określające naprężeniowe i magnetyczne współczynniki intensywności typu I. Zbadano wpływ magnetycznych warunków brzegowych na brzegu szczeliny na parametry mechaniki pękania i przedyskutowano pewne własności rozwiązań. Nieprzepuszczalny i przepuszczalny model szczeliny otrzymuje się jako przypadki graniczne. W pierwszym modelu uproszczonym indukcja magnetyczna w szczelinie jest zawsze równa zeru. W drugim modelu otrzymuje się różne wartości magnetycznej indukcji wewnątrz szczeliny, a tym samym współczynnika intensywności magnetycznej indukcji. Zależy to od warunków, jakie przyjmuje się na powierzchni szczeliny dla określenia w ośrodku szczeliny składowej wektora natężenia pola magnetycznego prostopadłej do jej brzegów.

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