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THE PROPERTIES OF COUPLED WAVES PROPAGATING IN LONG SUSPENDED CABLES

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In this paper, the properties of coupled waves travelling along a long cable are analysed. Since the tension and curvature in the equilibrium position of the cable are slowly varying functions of the arc co-ordinate, the problems concerning the travelling waves can be solved using the Wentzel-Kramers-Brillouin (WKB) method. The waves propagating in the plane of the equilibrium curve are coupled. The wave associated with displacements perpendicular to the plane is uncoupled from the remaining waves. Applying the WKB method, the dispersion relation and equations describing the amplitudes of waves are determined. For a longitudinal-dominated pair of waves, there exist two cut-off frequencies depending on the arc co-ordinate. The results of calculations of wavelengths and amplitudes are presented in the form of plots.

Key words: long cables, coupled waves, dispersion relation, cut-off frequencies

1. Introduction

Cables are used in many engineering applications. For instance, they are applied in ship equipment, cable railway, bridge suspensions and lift devices. Above all, cables are used in the overhead transmission lines.

Unfortunately, cables transmit waves induced by relatively small disturbances. Oscillations of the overhead transmission line can cause considerable damage in suspension towers and lead to failures of the conductor and electrical or mechanical subsystems due to material fatigue. There are many reasons that can induce waves in cables – for example: gusts of wind, interruptions associated with installation works and various failures of devices mounted to the cable.

A wide range of cable applications leads to formulation of many research strategies and equations describing motion of the cable. The equations have been derived using various techniques and coordinate systems.

Many papers are devoted to the analysis of vibrations of cables. Motion of the cables is usually analysed as a superposition of modes (Irvine and Caughey, 1974; Perkins and Mote, 1986; Burgess and Triantafyllou, 1988). This approach is very efficient for describing the response of short cables. For very long cables and high frequencies, the wavelength is small relative to the cable length, and the time in which the waves pass along the cable is relatively long. In this case, motion of the cable (especially the transient response) is better and more naturally described using the technique of superposition of travelling waves.

In several papers, the overhead cable is modelled as a taut string. In these papers, the effect of equilibrium curvature and non-constant tension along the cable are ignored. However, the majority of papers takes into account the equilibrium curvature in cable dynamics.

Many interesting problems in cable motion are associated with waves travelling along the cables. Perkins in works (Perkins and Mote, 1987; Perkins and Behbahani-Nejad, 1995, 1996) considered the wave propagation in elastic cables with a small curvature. The equations of motion derived in (Perkins and Mote, 1987) were simplified to the form of linear equations with a constant coefficient, neglecting variability of the tension and curvature along the cable. The coupled longitudinal and transverse waves were determined.

Large-amplitude free vibrations of a suspended cable were investigated e.g. in Luongo *et al.* (1984), Rega *et al.* (1984), Srinil *et al.* (2004). Wave interactions in a non-linear elastic string were considered in Young (2002).

The aim of the present work is an investigation of the properties of travelling waves in cables taking into account effects associated with variability of cable tension and curvature along the cable. Linear equations of motion with variable coefficients are considered. The cable is treated as a non-uniform wave medium. The dispersion relations will be analysed. The lengths and amplitudes of waves as functions of the arc co-ordinate will be derived.

2. General linear equations of motion

Observers usually describe motion of the cable in relation to its equilibrium position using reference axes associated with the line of the cable equilibrium. In this co-ordinate system, the points of the cable are identified by the arc-coordinate s, and the base unit vectors $\boldsymbol{e}_\tau$, $\boldsymbol{e}_n$, $\boldsymbol{e}_b$ have the tangent, normal and binormal direction, as shown in Fig. 1.

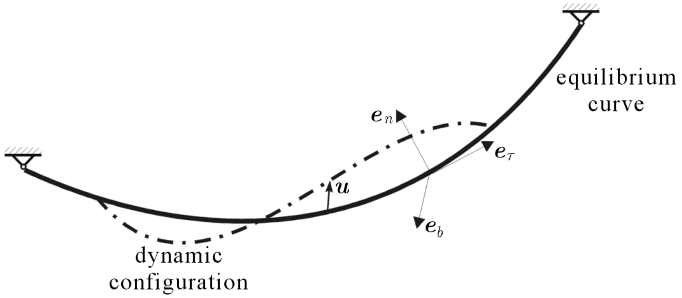

Fig. 1. Equilibrium curve and the dynamic configuration of the cable

The displacement vector $\boldsymbol{u}$ has three components $u_1(s,t)$, $u_2(s,t)$, $u_3(s,t)$ that describe displacements of cable points in the tangent, normal and binormal directions. In order to simplify the equations, one has assumed that the arc-coordinate s is equal to zero in the lowest point of the cable.

The general equations of cable motion are non-linear. They have been derived in Perkins and Mote (1987). In this paper, it is assumed that the components of displacement $\boldsymbol{u}$ are small. Thus the non-linear terms in general equations can be neglected, and finally the linear equations can be written in the form

$$
\begin{aligned}
\mu\frac{\partial^2 u_1}{\partial t^2} &= EA_0\frac{\partial^2 u_1}{\partial s^2} - (N + EA_0)\kappa\frac{\partial u_2}{\partial s} - N\kappa^2 u_1 - EA_0\frac{d\kappa}{ds}u_2 \\
\mu\frac{\partial^2 u_2}{\partial t^2} &= N\frac{\partial^2 u_2}{\partial s^2} + (N + EA_0)\kappa\frac{\partial u_1}{\partial s} + \frac{dN}{ds}\frac{\partial u_2}{\partial s} + \frac{d(N\kappa)}{ds}u_1 - EA_0\kappa^2 u_2 \\
\mu\frac{\partial^2 u_3}{\partial t^2} &= N\frac{\partial^2 u_3}{\partial s^2} + \frac{dN}{ds}\frac{\partial u_3}{\partial s}
\end{aligned}
\tag{2.1}
$$

where: μ is the mass per unit length of the cable, N is the cable tension, κ is the curvature of the equilibrium curve, A_0 is the area of the cable cross-section and E is the Young modulus.

Using the equilibrium equations, one can determine the cable tension N and the curvature κ of the equilibrium curve as functions of the arc-coordinate s. The equations describing the cable tension and the curvature are presented e.g. in Perkins and Behbahani-Nejad (1996). They can be written in the following forms

$$
N(s) = \sqrt{N_0^2 + (\mu g s)^2} \qquad\qquad \kappa(s) = \frac{N_0\mu g}{N_0^2 + (\mu g s)^2}
\tag{2.2}
$$

where: g is the gravitational acceleration and N_0 is the horizontal component of cable tension. Functions $N(s)$, $\kappa(s)$ are even. This property is associated with symmetry of the cable. The waves propagating in both directions from the lowest point are symmetric. For this reason, only waves propagating in the positive direction are investigated in the next sections.

3. Wave propagation in a long cable

The typical wavelength λ appearing in long cables is considerably smaller than the cable length L. For convenience, we introduce a small coefficient ε defined as the ratio of λ to L. The changes of parameters $N(s)$, $\kappa(s)$ are small if the change of s is comparable with λ.

The calculation can be generalised by introducing an infinite cable in which the horizontal component of the equilibrium tension is constant and it has the same value N_0 as in the considered cable. In this case, the cable can be treated as an infinite non-uniform medium with slowly varying parameters $N(s)$, $\kappa(s)$. The distance L can be interpreted as such a distance that causes significant changes of parameters $N(s)$, $\kappa(s)$.

The problems concerning the travelling waves in the cable can be solved using the well-known Wentzel-Kramers-Brillouin (WKB) approximation method. In this method, presented e.g. in Yang (1990), the slowly varying co-ordinate η and slowly varying time τ are introduced. The relation between the arc co-ordinate s and the slowly varying co-ordinate η is as follows

$$\eta = \varepsilon s \tag{3.1}$$

By analogy, the relationship between time t and the slowly varying time τ is

$$\tau = \varepsilon t \tag{3.2}$$

In the above expressions, the small parameter ε is used.

We assume that a wave is called the normal wave, if it develops only normal displacements of cable points. By analogy, the tangent wave results in tangent displacements only and the binormal wave develops only binormal displacements, as shown in Fig. 1.

From Eqs. (2.1) it is apparent that the tangent and normal waves are coupled (the first and second equation) whereas the binormal wave is not coupled to the remaining waves (the third equation). These properties of waves propagating in long cables have been described in the literature. Thus, the binormal wave can appear by itself, without tangent and binormal waves. In accordance with the WKB method, the binormal wave can be expressed in the form

$$u_3 = U_3 \exp\Big[\mathrm{i}\theta(\eta,\tau)\frac{1}{\varepsilon}\Big] \tag{3.3}$$

where U_3 is the wave amplitude and $\theta(\eta,\tau)$ is a slowly varying phase of the wave.

In the proposed description of waves, it is convenient to define, as in Whitham (1999), a local wave number k and a local frequency ω

$$k = \frac{\partial\theta}{\partial\eta} \qquad\qquad \omega = -\frac{\partial\theta}{\partial\tau} \tag{3.4}$$

From equations (3.4), the following relationship, known as the eikonal equation, can be derived

$$\frac{\partial k}{\partial\tau} + \frac{\partial\omega}{\partial\eta} = 0 \tag{3.5}$$

The described non-uniform medium depends on the spatial co-ordinate s and it does not depend on time t because tension N associated with the equilibrium configuration depends on the arc co-ordinate s alone. Therefore, the dispersion relation can be written in the form

$$\omega = \omega(k,N) \tag{3.6}$$

Substituting relationship (3.6) into Eq. (3.5), one can obtain a partial differential equation describing the wave number k

$$\frac{\partial k}{\partial\tau} + v_g\frac{\partial k}{\partial\eta} = -\frac{\partial\omega}{\partial N}\frac{dN}{d\eta} \tag{3.7}$$

where v_g stands for the group velocity defined as

$$v_g = \frac{\partial\omega}{\partial k} \tag{3.8}$$

Differentiating relationship (3.6) with respect to the slowly varying time τ and using eikonal equation (3.5), one can obtain an equation describing the frequency ω

$$\frac{\partial\omega}{\partial\tau} + v_g\frac{\partial\omega}{\partial\eta} = 0 \tag{3.9}$$

The left-hand sides of Eqs. (3.7) and (3.9) are analogous. Hence, both equations have the same characteristic curves which can be described by the following relationship

$$\eta - \int v_g\,d\tau = \mathrm{const} \tag{3.10}$$

Using Eqs. (3.8), (3.9) and (3.10), one can determine time derivatives of the wave number and frequency along the characteristic curve. They have the forms

$$\Big(\frac{dk}{d\tau}\Big)_{ch} = -\frac{\partial\omega}{\partial N}\frac{dN}{d\eta} \qquad \Big(\frac{d\omega}{d\tau}\Big)_{ch} = 0 \tag{3.11}$$

The above equations show that the wave number k varies and the frequency ω is conserved along the characteristic curve. If additionally the frequency wave source is invariable, the wave frequency is constant at each point of the cable. Thus the binormal wave can be expressed in a more convenient form

$$u_3 = U_3(\eta)\exp\Big[\mathrm{i}\Big(\omega t - \frac{\psi_3(\eta)}{\varepsilon}\Big)\Big] \tag{3.12}$$

where $U_3(\eta)$ is the wave amplitude and $\psi_3(\eta)$ is a slowly varying component of the wave phase depending on the spatial coordinate.

It is apparent that the amplitude $U_3(\eta)$ and the local wave number k_3 are slowly varying functions of the spatial co-ordinate η, but the displacement u_3 varies due to fast oscillations as well. In order to incorporate this requirement, the phase of the wave is written in the form $\omega t - \varepsilon^{-1}\psi_3(\eta)$. The first term provides the fast oscillation depending on time and the second one describes the changes depending on the spatial co-ordinate. Using Eq. (3.4) and taking into account the form of the phase function, the local wave number can be calculated as follows

$$k_3 = \frac{d\psi_3}{d\eta} \tag{3.13}$$

The phase dependence on the spatial co-ordinate can also be expressed using the function $\varphi_3(s)$. The following relation holds

$$\psi_3(\eta) = \varepsilon\varphi_3(s) \tag{3.14}$$

It is easy to prove that the first derivative of $\varphi_3(s)$ with respect to s is equal to the local wave number k_3.

A great advantage of the WKB method in practical calculations is an easy way of arranging terms in the expressions according to their rate of change. Equation (3.12) describing the binormal wave can be substituted into the third of Eqs. (2.1), in which the derivative of slowly varying cable tension N with respect to s should be transformed in the following way

$$\frac{dN}{ds} = \frac{dN}{d\eta}\frac{d\eta}{ds} = N'\varepsilon \tag{3.15}$$

In the above expression and in the following description, the sign $(\cdot)'$ represents the first derivative of the slowly varying function with respect to η.

By analogy, the first and second derivative of the function $u_3(s,t)$ with respect to s should be written in terms of derivatives with respect to η. After rearranging, they take the form

$$\begin{aligned}\frac{\partial u_3}{\partial s} &= (U_3'\varepsilon - \mathrm{i}U_3k_3)\exp\Big[\mathrm{i}\Big(\omega t - \frac{\psi_3(\eta)}{\varepsilon}\Big)\Big]\\ \frac{\partial^2 u_3}{\partial s^2} &= [U_3''\varepsilon^2 - U_3k_3^2 - \mathrm{i}(2U_3'k_3 + U_3k_3')\varepsilon]\exp\Big[\mathrm{i}\Big(\omega t - \frac{\psi_3(\eta)}{\varepsilon}\Big)\Big]\end{aligned} \tag{3.16}$$

After substituting Eqs. (3.12), (3.15) and (3.16) into the third of Eqs. (2.1), the expression obtained can be written as a polynomial of ε. Equating the free term and the coefficient at the first power of ε to zero, one obtains

$$\mu\omega^2 - Nk_3^2 = 0 \qquad\qquad 2Nk_3U_3' + U_3(Nk_3)' = 0 \tag{3.17}$$

The above equations can be expressed in non-dimensional forms

$$\widetilde{\omega}^2 - \widetilde{N}\widetilde{k}_3^2 = 0 \qquad\qquad 2\widetilde{N}\widetilde{k}_3\widetilde{U}_3' + \widetilde{U}_3(\widetilde{N}\widetilde{k}_3)' = 0 \tag{3.18}$$

where the following dimensionless quantities

$$\begin{array}{lll}\widetilde{\omega} = \omega\sqrt{\dfrac{L}{g}} & \qquad \widetilde{k}_3 = k_3L & \qquad \widetilde{s} = \dfrac{s}{L}\\ \widetilde{N} = \dfrac{N}{\mu gL} & \qquad \widetilde{\eta} = \dfrac{\eta}{L} & \qquad \widetilde{U}_3 = \dfrac{U_3}{L}\end{array} \tag{3.19}$$

are used.

Equation $(3.18)_1$ is the dispersion relation for the binormal wave. Equation $(3.18)_2$ is the differential equation containing the unknown wave amplitude as a function of slowly varying co-ordinate $\widetilde{\eta}$. Using dispersion relation $(3.18)_1$, one can determine the dimensionless phase velocity $\widetilde{v}_{f3}$ (the frequency $\widetilde{\omega}$ divided by the wave number $\widetilde{k}_3$) and the dimensionless group velocity $\widetilde{v}_{g3}$ (the first derivative of $\widetilde{\omega}$ with respect to $\widetilde{k}_3$)

$$\widetilde{v}_{f3} = \widetilde{v}_{g3} = \sqrt{\widetilde{N}} \tag{3.20}$$

These velocities are functions of the arc co-ordinate. Their plots are presented in Fig. 2 for two values of the horizontal component of the equilibrium cable tensions: $\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$.

Differential equation $(3.18)_2$ is equivalent to the condition that the energy of the binormal wave is conserved during propagation along the cable. Assuming the boundary condition $\widetilde{U}_3(0) = \widetilde{U}_{30}$ (the amplitude in the lowest point

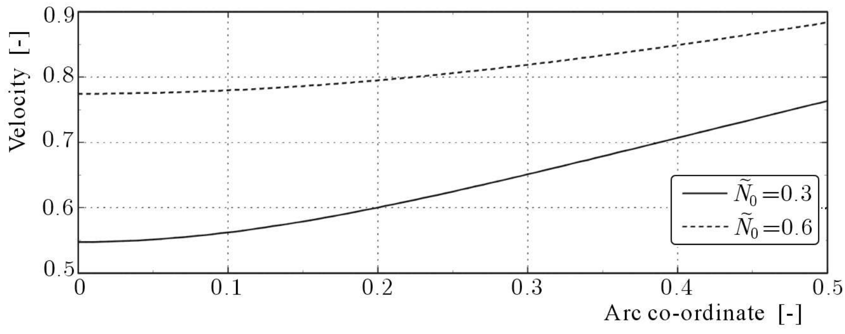

Fig. 2. Phase velocity and group velocity versus non-dimensional arc co-ordinate $\widetilde{s}$ for: $\widetilde{N}_0 = 0.3$ and $\widetilde{N}_0 = 0.6$

of the cable is equal to $\widetilde{U}_{30}$), Eq. $(3.18)_2$ can be solved with respect to the amplitude of the wave as a function of the dimensionless arc co-ordinate $\widetilde{s}$

$$\widetilde{U}_3(\widetilde{s}) = \widetilde{U}_{30} \sqrt[4]{\frac{\widetilde{N}_0}{\widetilde{N}(\widetilde{s})}} \tag{3.21}$$

The wave number $\widetilde{k}_3(\widetilde{s})$ and the amplitude $\widetilde{U}_3(\widetilde{s})$ are plotted in Fig. 3 (for the amplitude in the lowest point $\widetilde{U}_{30} = 10^{-4}$).

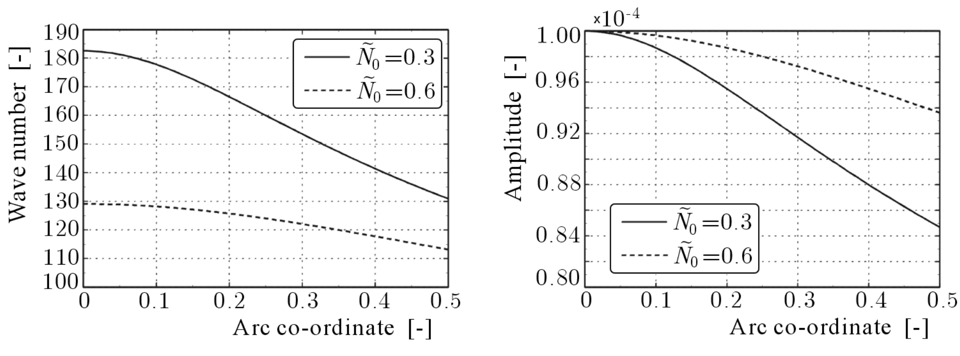

Fig. 3. Wave number and amplitude versus non-dimensional arc co-ordinate $\widetilde{s}$ for: $\widetilde{\omega} = 100$, $\widetilde{N}_0 = 0.3$ and $\widetilde{N}_0 = 0.6$

The wave number and the amplitude become smaller when the wave moves away from the lowest point of the cable. These phenomena are more explicit for lower values of the horizontal component of the cable tension, see the plots for $\widetilde{N}_0 = 0.3$. The change of the wave number is much greater than the change of the wave amplitude. Thus, the varying cable tension N has a bigger influence on the wave number than on the amplitude of the binormal wave.

Using Eqs. (3.1), (3.12), (3.13) and (3.14), the binormal wave can be written in the form

$$\widetilde{u}_3(\widetilde{s},\widetilde{t}) = \widetilde{U}_3(\widetilde{s}) \exp\Big[\mathrm{i}\Big(\widetilde{\omega}\widetilde{t} - \int\limits_0^{\widetilde{s}} \widetilde{k}_3(\widetilde{s}_1)\, d\widetilde{s}_1\Big)\Big] \tag{3.22}$$

Figure 4 shows the shape of the wave for $\widetilde{N}_0 = 0.3$ and $\widetilde{\omega} = 100$. It is easy to notice the increase of the wavelength and the decrease of the amplitude. Taking into account that the frequency of the wave is constant, the increase of the wavelength results in an increase of the phase velocity, as shown in Fig. 2.

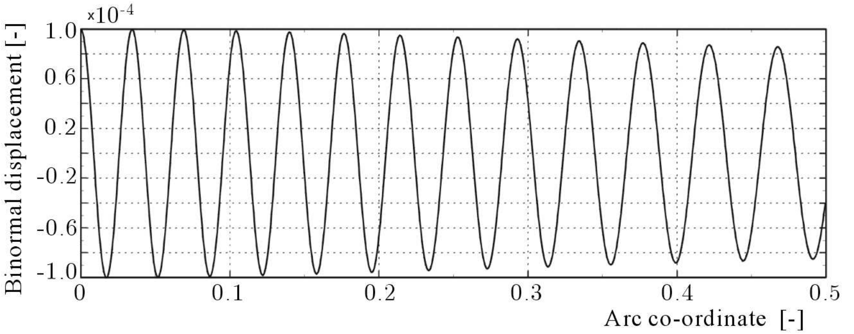

Fig. 4. Plot of the travelling wave for: $\widetilde{N}_0 = 0.3$ and $\widetilde{\omega} = 100$

In accordance with previous considerations, the tangent- and the normal waves can be assumed in the following forms

$$u_1 = U_1(\eta) \exp\Big[\mathrm{i}\Big(\omega t - \frac{\psi(\eta)}{\varepsilon}\Big)\Big] \qquad\qquad u_2 = U_2(\eta) \exp\Big[\mathrm{i}\Big(\omega t - \frac{\psi(\eta)}{\varepsilon}\Big)\Big] \tag{3.23}$$

The waves have different amplitudes. Since the waves are coupled, the phases are described by the same expressions, but taking into account that the functions $U_1(\eta)$, $U_2(\eta)$ describing the amplitudes are complex functions the physical phases of the waves cannot be the same. Their difference can be constant or a slowly varying function.

Relations (3.23) have to satisfy the first and the second of Eqs. (2.1). Derivatives of displacements u_1 and u_2 with respect to time and with respect to the arc co-ordinate should be determined similarly to derivatives of the displacement u_3 presented in Eqs. (3.16). The derivatives of slowly varying functions in Eqs. (2.1) should be replaced by the following derivatives with respect to the co-ordinate η

$$\frac{dN}{ds} = N'\varepsilon \qquad\qquad \frac{d\kappa}{ds} = \kappa'\varepsilon \qquad\qquad \frac{d(N\kappa)}{ds} = (N\kappa)'\varepsilon \tag{3.24}$$

Substituting Eqs. (3.23) and (3.24) into the first and second equation of Eqs. (2.1) and comparing the free terms on both sides and the expressions at ε^1, one obtains two systems of equations. The first system consists of the following algebraic equations

$$
\begin{aligned}
&(EA_0k^2 + N\kappa^2 - \mu\omega^2)U_1 - \mathrm{i}(N + EA_0)\kappa k U_2 = 0 \\
&\mathrm{i}(N + EA_0)\kappa k U_1 + (EA_0\kappa^2 + Nk^2 - \mu\omega^2)U_2 = 0
\end{aligned}
\tag{3.25}
$$

The second system includes two differential equations

$$
\begin{aligned}
&(N + EA_0)\kappa U_1' + (N\kappa)'U_1 - \mathrm{i}[2NkU_2' + (Nk)'U_2] = 0 \\
&(N + EA_0)\kappa U_2' + EA_0\kappa' U_2 + \mathrm{i}(2EA_0kU_1' + EA_0k'U_1) = 0
\end{aligned}
\tag{3.26}
$$

Introducing the following non-dimensional quantities

$$
\widetilde{k} = kL \qquad \widetilde{\kappa} = \kappa L \qquad \varepsilon_g = \frac{\mu g L}{EA_0}
\tag{3.27}
$$

and using appropriate dimensionless quantities taken from (3.19), system (3.25) can be written in the form

$$
\begin{aligned}
&(\widetilde{k}^2 + \widetilde{N}\varepsilon_g\widetilde{\kappa}^2 - \varepsilon_g\widetilde{\omega}^2)\widetilde{U}_1 - \mathrm{i}(\varepsilon_g\widetilde{N} + 1)\widetilde{\kappa}\widetilde{k}\widetilde{U}_2 = 0 \\
&\mathrm{i}(\varepsilon_g\widetilde{N} + 1)\widetilde{\kappa}\widetilde{k}\widetilde{U}_1 + (\widetilde{\kappa}^2 + \widetilde{N}\varepsilon_g\widetilde{k}^2 - \varepsilon_g\widetilde{\omega}^2)\widetilde{U}_2 = 0
\end{aligned}
\tag{3.28}
$$

The calculations were done for $\varepsilon_g = 0.00018$.

The non-trivial solution to equations (3.28) exists if and only if the determinant of the system is equal to zero. Two wave numbers $\widetilde{k}_1$, $\widetilde{k}_2$ can be calculated from this condition. They have the following forms

$$
\begin{aligned}
\widetilde{k}_1 &= \sqrt{\widetilde{\kappa}^2 + \frac{1}{2}(\varepsilon_g\widetilde{N} + 1)\frac{\widetilde{\omega}^2}{\widetilde{N}} - \frac{\widetilde{\omega}}{\sqrt{\widetilde{N}}}\sqrt{2(\varepsilon_g\widetilde{N} + 1)\widetilde{\kappa}^2 + \frac{1}{4}(1 - \varepsilon_g\widetilde{N})^2\frac{\widetilde{\omega}^2}{\widetilde{N}}}} \\
\widetilde{k}_2 &= \sqrt{\widetilde{\kappa}^2 + \frac{1}{2}(\varepsilon_g\widetilde{N} + 1)\frac{\widetilde{\omega}^2}{\widetilde{N}} + \frac{\widetilde{\omega}}{\sqrt{\widetilde{N}}}\sqrt{2(\varepsilon_g\widetilde{N} + 1)\widetilde{\kappa}^2 + \frac{1}{4}(1 - \varepsilon_g\widetilde{N})^2\frac{\widetilde{\omega}^2}{\widetilde{N}}}}
\end{aligned}
\tag{3.29}
$$

The above expressions describe the dispersion properties of two coupled waves travelling along the cable in the positive direction of the arc co-ordinate (which defines the positive direction of wave propagation). It is apparent that the second wave number $\widetilde{k}_2$ is a positive real number for each frequency of the wave, whereas the first wave number $\widetilde{k}_1$ can be a positive real number or an imaginary number depending on the frequency $\widetilde{\omega}$ and arc co-ordinate $\widetilde{s}$. The results of the wave number calculation are presented in Figs. 5, 6, 7 and 8. Figures 5, 6 show the wave numbers as functions of the frequency (for $\widetilde{s} = 0$). Figures 7, 8 illustrate the wave numbers as functions of the arc co-ordinate $\widetilde{s}$.

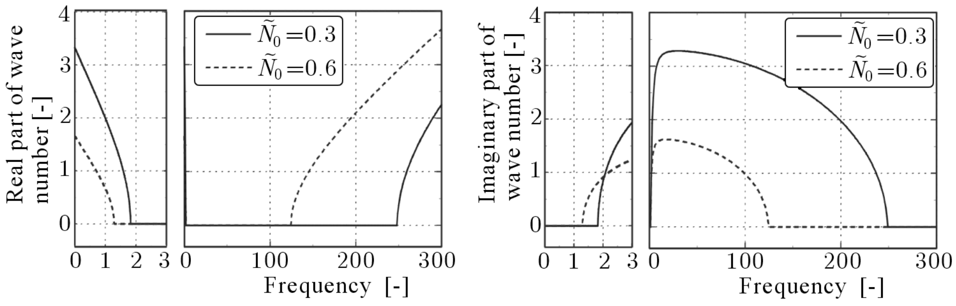

Fig. 5. The wave number $\widetilde{k}_1$ versus frequency ($\widetilde{s} = 0$ and $\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$)

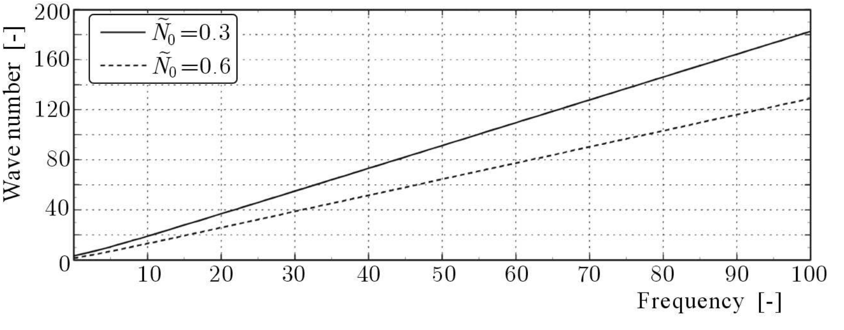

Fig. 6. The wave number $\widetilde{k}_2$ versus frequency ($\widetilde{s} = 0$ and $\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$)

When the curvature of the cable approaches zero, the coupling between waves vanishes and the wave number $\widetilde{k}_1$ approaches the wave number of longitudinal waves in a slender rod whereas the wave number $\widetilde{k}_2$ approaches the wave number of the transverse wave in a taut string. Hence, it can be concluded that the wave number $\widetilde{k}_1$ is associated with the longitudinal-dominant pair of waves and $\widetilde{k}_2$ is associated with the transverse-dominant pair of waves.

The results of calculations show that the waves belonging to the transverse-dominant pair are dispersive, whereas the waves belonging to the longitudinal-dominant pair can be dispersive in the pass band frequency ranges

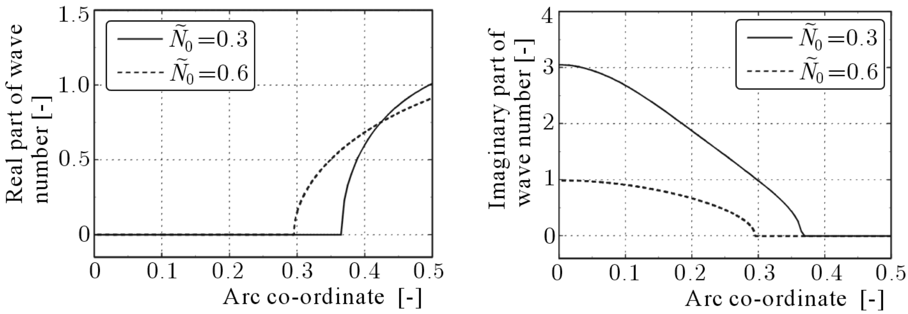

Fig. 7. The wave number $\widetilde{k}_1$ versus non-dimensional arc co-ordinate ($\widetilde{\omega} = 100$ and $\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$)

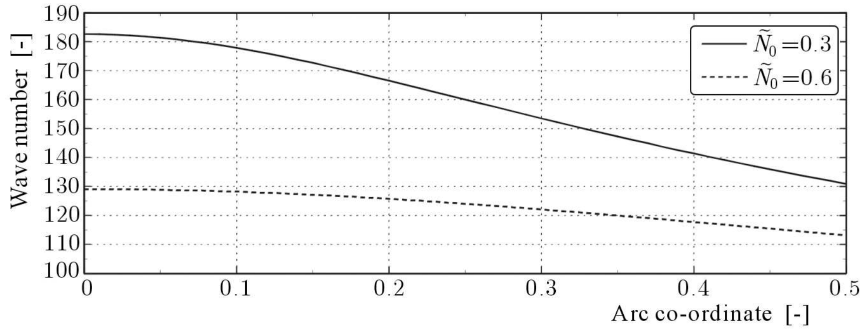

Fig. 8. The wave number $\widetilde{k}_2$ versus non-dimensional arc co-ordinate ($\widetilde{\omega} = 100$ and $\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$)

$\widetilde{\omega} \in (0; \widetilde{\omega}_{g1}) \cup (\widetilde{\omega}_{g2}; \infty)$ or non-propagating (exponentially decaying with distance) in the stop band frequency range $\widetilde{\omega} \in (\widetilde{\omega}_{g1}; \widetilde{\omega}_{g2})$. The cut-off frequencies $\widetilde{\omega}_{g1}$, $\widetilde{\omega}_{g2}$ can be calculated using Eq. $(3.29)_1$. Their non-dimensional form is given by the following formulas

$$\widetilde{\omega}_{g1} = \widetilde{\kappa}\sqrt{\widetilde{N}} \qquad\qquad \widetilde{\omega}_{g2} = \widetilde{\kappa}\frac{1}{\sqrt{\varepsilon_g}} \tag{3.30}$$

Figure 9 shows the graphs of the cut-off frequencies as functions of the arc co-ordinate.

Since the cut-off frequencies depend on the arc co-ordinate, there exists a neighbourhood of the lowest point of the cable where the longitudinal-dominant pair consists of two waves decaying exponentially. At points laying outside this neighbourhood, the same longitudinal-dominant pair consists of two dispersive waves.

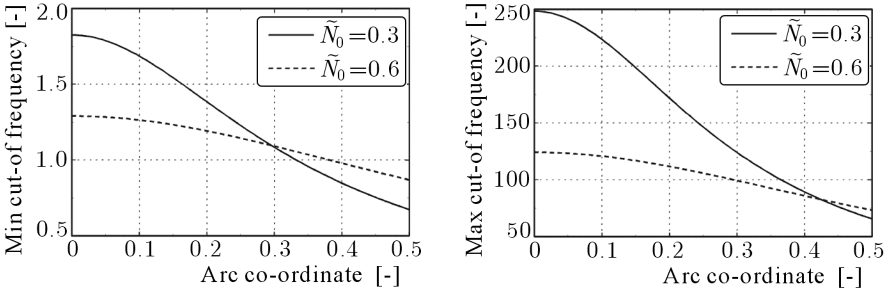

Fig. 9. Cut-of frequencies versus non-dimensional arc co-ordinate ($\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$)

Using Eqs. (3.29), one can calculate the phase and group velocity for each pair of waves. The results of such calculations performed for $\widetilde{N}_0 = 0.3$ are presented in Figs. 10 and 11. For the longitudinal-dominant pair, the velocities have been calculated assuming the wave frequency greater than the cut-off frequency $\widetilde{\omega}_{g2}$.

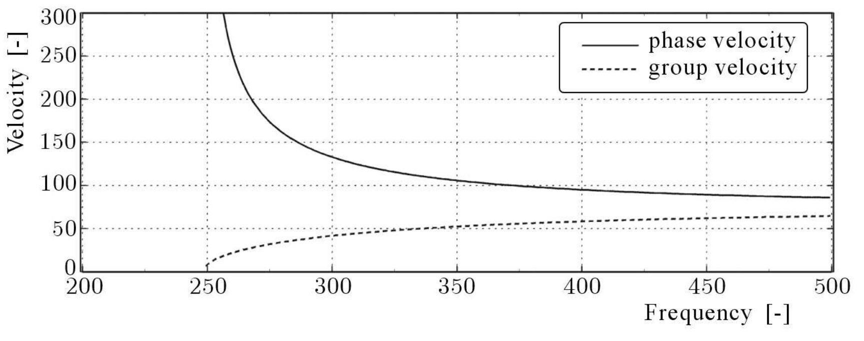

Fig. 10. The phase and group velocity of longitudinal-dominant waves versus frequency ($\widetilde{s} = 0$, $\widetilde{N}_0 = 0.3$)

Analysing the graphs, it is apparent that the group velocity is lower than the phase velocity for the longitudinal-dominant pair (normal dispersion) and higher than the phase velocity for the transversal-dominant pair (anomalous dispersion).

In the case of short waves (high frequencies), the group velocity approaches the phase velocity for both transversal-dominant and longitudinal-dominant pairs, see Figs. 10 and 11. Therefore, the dispersion phenomenon can be better observed for lower than for higher frequencies. The difference between the group- and phase frequencies diminishes when the waves move away from the lowest point of the cable, as shown in Figs. 12 and 13. The velocities of the

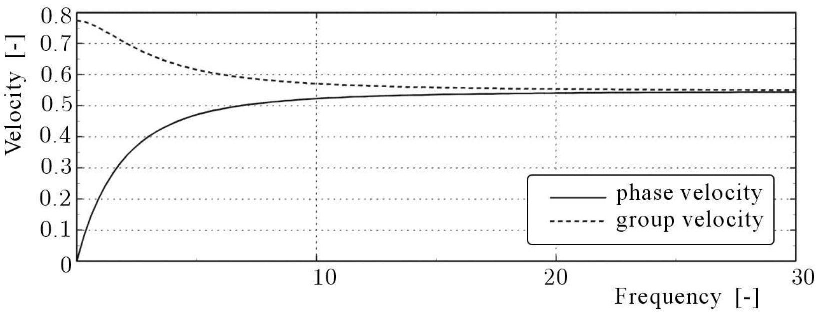

Fig. 11. The phase and group velocity of transverse-dominant waves versus frequency $(\widetilde{s} = 0,\ \widetilde{N}_0 = 0.3)$

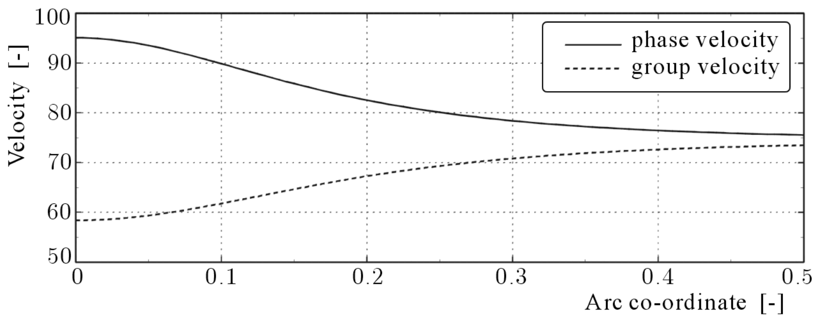

Fig. 12. The phase and group velocity of longitudinal-dominant waves versus non-dimensional arc co-ordinate $(\widetilde{N}_0 = 0.3,\ \widetilde{\omega} = 400)$

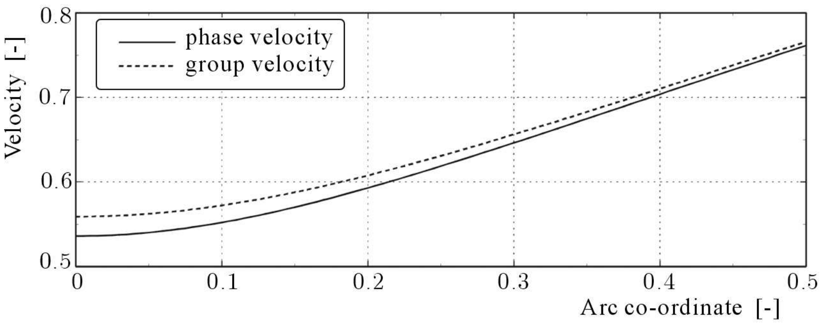

Fig. 13. The phase and group velocity of transversal-dominant waves versus non-dimensional arc co-ordinate $(\widetilde{N}_0 = 0.3,\ \widetilde{\omega} = 15)$

longitudinal-dominant pair are significantly greater than the corresponding velocities of the transversal-dominant pair.

Taking into account Eqs. (3.23) and the results of the wave number calculation (Eqs. (3.29)), the formulas describing the displacements of cable points in the tangent and normal directions can be written in the form

$$
\begin{aligned}
u_1 &= U_{11}\exp\Big[\mathrm{i}\Big(\omega t - \int_0^s k_1(s_1)\,ds_1\Big)\Big] + U_{22}\delta_{12}\exp\Big[\mathrm{i}\Big(\omega t - \int_0^s k_2(s_1)\,ds_1\Big)\Big] \\
u_2 &= U_{11}\delta_{21}\exp\Big[\mathrm{i}\Big(\omega t - \int_0^s k_1(s_1)\,ds_1\Big)\Big] + U_{22}\exp\Big[\mathrm{i}\Big(\omega t - \int_0^s k_2(s_1)\,ds_1\Big)\Big]
\end{aligned}
\tag{3.31}
$$

The displacements are linear combinations of two waves travelling along the cable in the positive direction. The waves have the same frequencies and a different length.

In Eqs. (3.31), we have taken into account the previous conclusion that each tangent wave (with an amplitude equal to U_{11}) is coupled to the corresponding normal wave (with an amplitude equal to $\delta_{21}U_{11}$). Both waves have the same frequency and wave number. By analogy, each normal wave (with an amplitude equal to U_{22}) is coupled to the corresponding tangent wave (with an amplitude equal to $\delta_{12}U_{22}$).

The coefficient δ_{21} is the amplitude ratio in the longitudinal-dominant pair, and δ_{12} in the transversal-dominant pair. They can be determined from any of Eqs. (3.28). Their final expressions are as follows

$$
\delta_{12} = \mathrm{i}\frac{\widetilde{\kappa}^2 + \varepsilon_g\widetilde{N}\widetilde{k}_2^2 - \varepsilon_g\widetilde{\omega}^2}{(\varepsilon_g\widetilde{N}+1)\widetilde{k}_2\widetilde{\kappa}} \qquad \delta_{21} = -\mathrm{i}\frac{\widetilde{k}_1^2 + \varepsilon_g\widetilde{N}\widetilde{\kappa}^2 - \varepsilon_g\widetilde{\omega}^2}{(\varepsilon_g\widetilde{N}+1)\widetilde{k}_1\widetilde{\kappa}} \tag{3.32}
$$

Taking into account that both the above coefficients are imaginary, the differences between wave phases in each pair are equal to $\pi/2$.

Graphs of modules of the ratios δ_{21} and δ_{12} are shown in Figs. 14 and 15. They are good measures of the wave coupling in each pair. Using these measures, it can be asserted that waves are not coupled, if the ratios $|\delta_{21}|$, $|\delta_{12}|$ are less than the assumed limit value δ_m. The discussion about δ_m is associated with the aim and accuracy of calculations. Taking into account Eqs (3.32) and (3.29), the limit values for the horizontal component of the cable tension N_0 or curvature κ can be determined. The calculations were done for $\widetilde{N}_0 = 0.3$ and $\widetilde{N}_0 = 0.6$. These values of the horizontal component of the cable tension correspond to the cable curvature (at $\widetilde{s} = 0$) which is equal to $\widetilde{\kappa}(0) \cong 3.3$ and $\widetilde{\kappa}(0) \cong 1.7$.

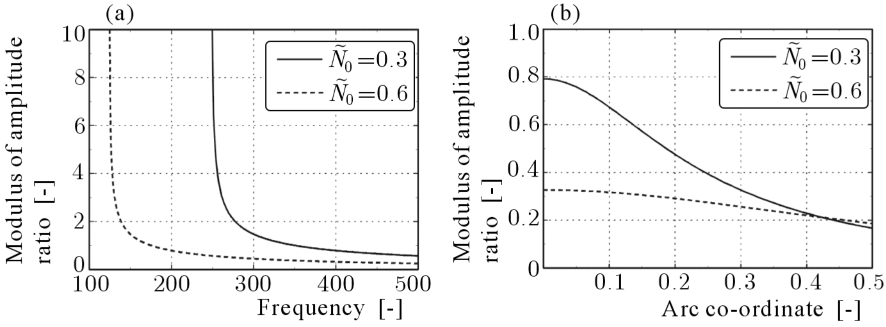

Fig. 14. The modulus of amplitude ratio in longitudinal-dominant pair of waves ($\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$), (a) for $\widetilde{s} = 0$, (b) for $\widetilde{\omega} = 400$

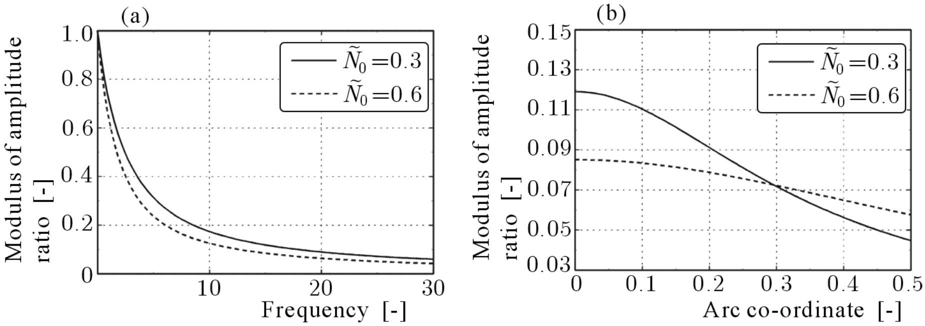

Fig. 15. The modulus of amplitude ratio in transversal-dominant pair of waves ($\widetilde{N}_0 = 0.3$, $\widetilde{N}_0 = 0.6$), (a) for $\widetilde{s} = 0$, (b) for $\widetilde{\omega} = 15$

The system of differential equations (3.26) has an integral that is associated with the energy of a pair of waves. In order to determine this integral, the following transformations are proposed. The first equation is pre-multiplied by $(\mathrm{i}U_2)$, the second equation is pre-multiplied by $(\mathrm{i}U_1)$ and the resultant equations are added. The expression on the left-hand side of the obtained equation can be rearranged to give the derivative of the double energy of coupled waves with respect to the arc co-ordinate. Since the energy is conserved, the derivative is equal to zero. Taking into account the non-dimensional quantities defined so far (relationships (3.19), (3.27)) and denoting $(\mathrm{i}\widetilde{U}_1)$ as $\widetilde{U}_1^*$, the integral of system (3.26) can be expressed in the following form

$$\widetilde{k}(\widetilde{U}_1^*)^2 + \widetilde{N}\varepsilon_g\widetilde{k}\widetilde{U}_2^2 + \widetilde{\kappa}(1 + \widetilde{N}\varepsilon_g)\widetilde{U}_1^*\widetilde{U}_2 = \text{const} \tag{3.33}$$

Equation (3.33) is the basic equation used in calculations of wave amplitudes. Additionally, in order to determine the amplitudes, Eqs. (3.29) and relationships (3.32) should be taken into account.

The results of such calculations are shown in Figs. 16 and 17. The amplitudes of waves in the longitudinal-dominated pair are presented in Fig. 16, whereas the amplitudes of waves in the transversal-dominated pair are shown in Fig. 17.

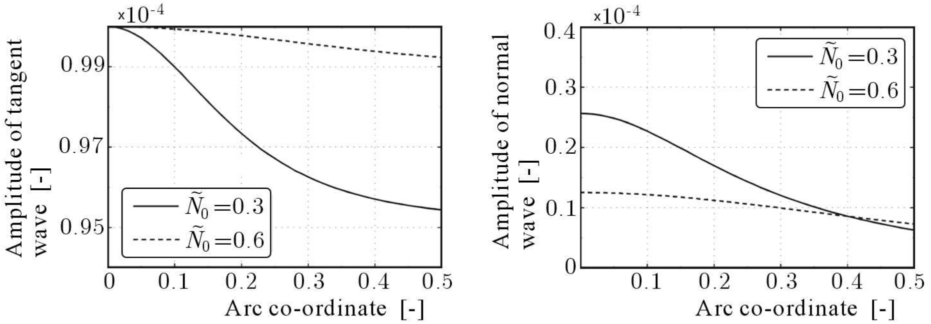

Fig. 16. Amplitudes of waves in the longitudinal-dominant pair versus non-dimensional arc co-ordinate, for $\widetilde{\omega} = 1000$, $\widetilde{N}_0 = 0.3$ and $\widetilde{N}_0 = 0.6$

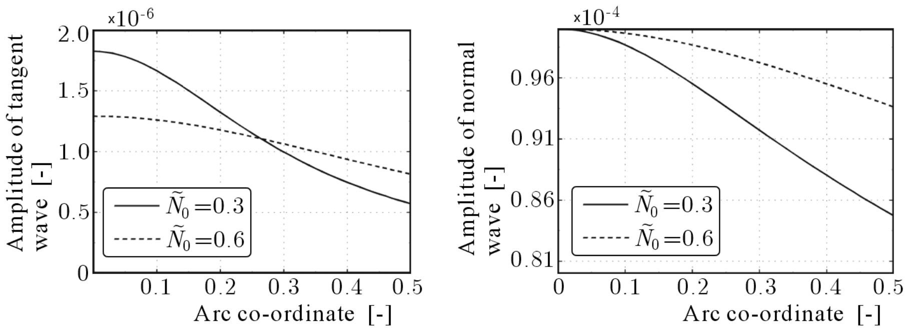

Fig. 17. Amplitudes of waves in the transversal-dominant pair versus non-dimensional arc co-ordinate, for $\widetilde{\omega} = 100$, $\widetilde{N}_0 = 0.3$ and $\widetilde{N}_0 = 0.6$

The calculations for the longitudinal-dominated pair have been done for $\widetilde{\omega} = 1000$, i.e. the frequency higher than the cut-off frequency $\widetilde{\omega}_{g2}$.

It follows from the calculations that the amplitudes of both waves in each pair diminish when the waves move away from the lowest point of the cable. The amplitude of the dominant wave in each pair diminishes slower than the amplitude of the wave coupled to it. The differences between the velocity of the normal wave in the transverse-dominant pair and the velocity of the binormal wave are small as shown in Fig. 2 and Fig. 13.

4. Conclusions

A suspended elastic cable is a non-uniform wave medium. The features of this medium can be described accounting for the curvature and tension of the sagged cable. They are slowly varying functions of the arc co-ordinate. In this case, the WKB method is an effective tool to derive the dispersion relations and amplitude equations. The wavelengths and the amplitude vary along the cable.

The curvature causes that the waves travelling in the plane of the equilibrium line are coupled. There are two distinct pairs of coupled waves propagating along the cable – the longitudinal-dominant and transverse-dominant pair. When the curvature tends to zero, the coupling gets smaller and the model simplifies to well known models of lateral waves in a taut string and longitudinal waves in an elastic rod. The coupling of waves is very weak when the frequency tends to infinity or when the waves move away from the lowest point of the cable.

The transverse-dominant pair of waves is dispersive, whereas the longitudinal-dominant pair can be dispersive in two pass band frequency ranges or exponentially decaying in the stop band frequency range. Two cut-off frequencies depend on the curvature and the arc co-ordinate. Since the cut-off frequencies depend on the arc co-ordinate, the longitudinal-dominant pair of waves can exponentially decay in the neighbourhood of the lowest point of the cable and, for the same frequency, can be dispersive at points with a longer distance from the lowest point.

The curvature does not influence the waves associated with the displacement perpendicular to the plane of the equilibrium line. These waves are not coupled with the in-plane waves.

The amplitudes of waves in each pair diminish when the waves move away from the lowest point of the cable. The amplitude of the dominant wave in each pair diminishes slower than the amplitude of the wave coupled with it.

The method applied in the present calculations can also be used in the analysis of wave motion appearing in other continuous systems.

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Własności sprzężonych fal rozprzestrzeniających się w linach o znacznej długości

Streszczenie

W pracy przedstawiono analizę własności fal mechanicznych rozprzestrzeniających się w linach o znacznej długości. W liniowej teorii fale rozchodzące się w płaszczyźnie zwisu są ze sobą sprzężone, a fala wywołująca przemieszczenia punktów liny w kierunku prostopadłym do płaszczyzny zwisu rozprzestrzenia się niezależnie od pozostałych fal. W rozważaniach uwzględniono wolno-zmienną zależność siły osiowej i krzywizny od współrzędnej łukowej, w położeniu równowagi statycznej liny. Zagadnienia dotyczące przemieszczających się wzdłuż przewodu fal rozwiązano metodą

WKB (Wentzel-Kramers-Brillouin). Wyznaczono związki dyspersyjne charakteryzujące ruch falowy w linach, wyprowadzono prędkości fazowe i grupowe z uwzględnieniem sprzężenia fal rozchodzących się w płaszczyźnie zwisu. Analizie poddano zależność liczb falowych oraz amplitud od współrzędnej łukowej. W rozważaniach wykorzystano symetrię ruchu fal. Wyniki obliczeń zostały zobrazowane na szeregu wykresach.

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