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# PHENOMENON OF FORCE IMPULSE RESTITUTION IN COLLISION MODELLING

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The rightness of the Newtonian hypothesis concerning a constat value of the coefficient of restitution $R$ has been confirmed in the paper with reference to collisions in which the loss of energy occurs in conseqence of material damping. For collisions of different nature, when $R$ depends on the density of the energy flux $\Phi$, the paper points to the possibility of extending this notion to the case of eccentric collision. The possibility of describing $R$ as a random function of $\Phi$ has been shown as well.

*Key words:* collision, coefficient of restitution

## 1. Introduction

Despite its universality in nature and technology, the phenomenon of collision of bodies has not been fully and satisfactorily described. The starting point in the analysis of collision is usually Newton's hypothesis, according to which the correlation between an impulse $\Pi_1$ of a contact force $F$ at the stage of increase in deformation of the bodies and an impulse $\Pi_2$ of the same force during the relieving process is given by the following equation

$$R = \frac{\Pi_2}{\Pi_1} \tag{1.1}$$

where

$$\Pi_1 = \int\limits_{t_1}^{t_0} F\; dt \qquad\qquad \Pi_2 = \int\limits_{t_0}^{t_2} F\; dt \tag{1.2}$$

and $R \in [0,1]$ is a constant, called the coefficient of restitution.

The fact that impact forces are generally stronger than any forces acting on the system as well as short duration of the collision make it possible to

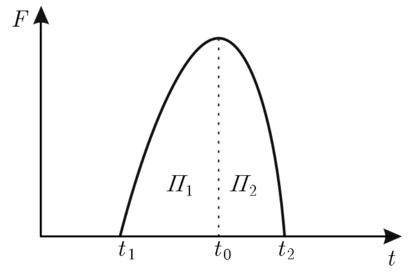


Fig. 1.

omit non-impact forces and relocations of bodies, and regard the system as a free body system (it does not refer to any geometrical constraints which are imposed on motion of the bodies).

For such a system, from equations of conservation of momentum and angular momentum, it is possible to determine kinematic parameters for both bodies at the end of increase of deformation, when the normal components of velocities at the collision point are equalised.

On the base of Newton's hypothesis, which describes the degree of restitution of impulses of collision forces in the second stage of the collision, it is possible to obtain final values of linear and angular velocities for both bodies after collision.

Newton's "time-free" model of the collision enables us to utilize a lot of experimental data concerning impulse restitution and give an "integral look" on the collision, but makes it impossible to determine the process of force growth and local deformations. It is also inconvenient in modelling of dynamical phenomena for computer simulation.

From the theoretical point of view, the basic meaning also has an answer to the question about physical sense of Newton's idea of a constant value of the ratio of impulses in both stages of the collision, i.e. to indicate a physical process which leads to this model or to prove that this hypothesis has no sense.

## 2. $R = \mathrm{const}$ hypothesis and physical phenomena responsible for incomplete restitution

An incomplete restitution of instantaneous forces during the collision is mainly associated with:

1. loss of energy which is associated with a higher or lower degree of stable deformations (e.g. plastic or brittle)

2. flow of energy from the area of collision to further parts of both bodies that takes place by excitation of waves and vibrational phenomena
3. loss of energy which is associated with pressing of contamination, e.g. oxides, dust, etc.
4. loss of energy which results from friction (slip friction – generally associated with different velocities of local deformation of both bodies in the plane of collision) and material damping – in the volumes undergoing deformation of both bodies.

As for factor No. 1, it is possible to asses theoretically the loss of energy in this case, but it does not lead to a constant value of $R$. On the contrary – $R$ forms a decreasing function of relative velocity (Gryboś, 1969, 1971). The actual value of $R$ is burdened with a margin of error which results from its susceptibility to factors which are difficult to be described in a deterministic way due to incomplete repeatibility of the place and angle of contact of both bodies, deviations in microstructure of surface, etc.

The second factor is only essential with regard to bodies of a flabby structure. As for bodies of a compact structure, i.e. when the period of natural vibrations is shorter than the period of contact during collision, this phenomenon does not have any significant contrbution to the balance of loss during the collision (Rayleigh, 1906). In the case of flabby structures, there is an effect of $R$ on velocity (Gryboś, 1969; Harris and Piersol, 2002), so $R$ is not a constant value.

As for the third factor, it can only be described within statistical classes, although there is a strong dependency of statistical characteristics of $R$ on deterministic factors, e.g. area of collision contact (Michalczyk, 1984).

As for factor No. 4, it should be pointed out that the loss of energy, which results from slip friction, shows strong dependence on the intensity of collision, random properties of microstructure and cleanness of the surface, so it can not be considered as a constant value.

Then when $R$ is a constant value? Let us evaluate the loss of energy related to material damping during collision considered as an "in-time" process – development of local deformations in both bodies. Let us also assume that their contact stiffness (dependence of the force $F$ of mutual interactions on total deformations in the bodies $x_w$) could be described by the Hertz-Stajerman model

$$F = kx_w^p \tag{2.1}$$

This model describes a wide range of bodies whose surfaces in the vicinity of the collision point could be regarded as regularly rotational. The exponent $p$

in equation (2.1) depends on the so-called contact tightness of both bodies, and for a surface of the second order it is $p = 3/2$.

When the loss of energy $\Delta L$ occurs according to a model of material damping in which during the vibration period it is proportional to the maximum potential energy $E$ of the system

$$\Delta L = \Psi E \tag{2.2}$$

where $\Psi/2\pi = \eta$ is the loss factor, we can represent the dependence of the force of mutual interactions $F$ between both bodies as shown in Fig. 2.

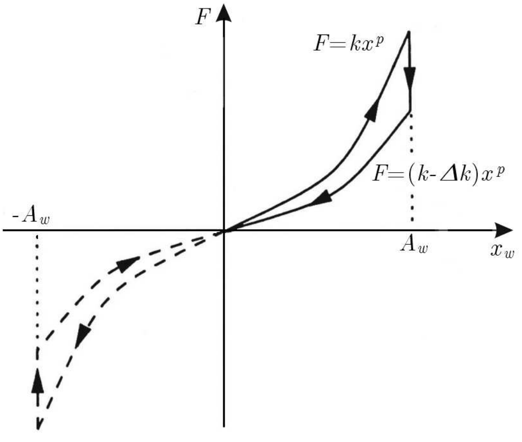


Fig. 2.

To prove the conformity of the accepted form of the force $F$ with the model of material damping, let us calculate the loss of energy for the range $x_w > 0$ (this range refers to change in $x_w$ during the collision) using the model of material damping, and compare it with the loss of energy calculated by integration of the surface area of half of the hysteresis loop for the force determined as above.

In the first case, the loss of energy during the collision (continuous line in Fig. 2) is given by

$$\Delta L = \frac{1}{2}\Psi E = \frac{1}{2}\Psi \int\limits_0^{A_w} F(x_w)\; dx_w = \frac{1}{2}\Psi \int\limits_0^{A_w} kx_w^p\; dx_w = \frac{1}{2}k\Psi \frac{A_w^{p+1}}{p+1} \tag{2.3}$$

where $A_w$ is the maximum value of deformation $x_w$.

In the second case, the loss of energy described by half of the area of the hysteresis loop is given by

$$\Delta L = \int\limits_0^{A_w} [kx_w^p - (k-\Delta k)x_w^p]\, dx_w = \int\limits_0^{A_w} \Delta k x_w^p \, dx_w = \frac{\Delta k}{p+1} A_w^{p+1} \tag{2.4}$$

Comparing (2.3) and (2.4), we obtain

$$\Delta k = \frac{\Psi}{2} k \tag{2.5}$$

Therefore, the force $F$ can be described by the following equation

$$F = kx_w^p \Big\{1 - \frac{\Psi}{4}[1 - \operatorname{sgn}(x_w)\operatorname{sgn}(\dot{x}_w)]\Big\} \qquad \text{(for the collision } \operatorname{sgn}(x_w) = 1) \tag{2.6}$$

We will show that it is possible to obtain an equivalent description of the collision as a "time-free" process according to Newton's formulation characterised by the parameter $R$, to the description of the collision as a dynamic process which takes place in a limited time and is characterised by rheological properties $k$, $\Delta k$ of bodies in the vicinity of the contact area.

Let us consider a simple central collision of bodies of masses $m_i$ and $m_j$, as schematically shown in Fig. 3, where prior to the collision $v_i > v_j$.

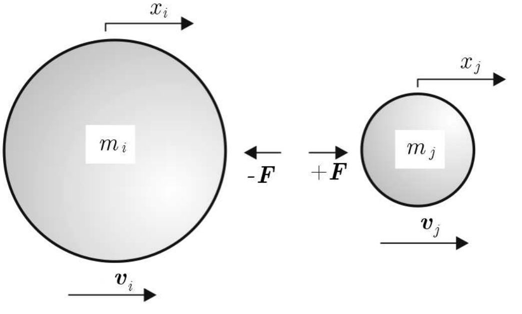


Fig. 3.

During the collision

$$\ddot{x}_i = -\frac{F}{m_i} \qquad\qquad \ddot{x}_j = \frac{F}{m_j} \tag{2.7}$$

By introducing of a relative coordinate $x_w = x_i - x_j$, we obtain

$$\ddot{x}_w = \ddot{x}_i - \ddot{x}_j = -\frac{m_i + m_j}{m_i m_j} F \tag{2.8}$$

If we put

$$\frac{m_i m_j}{m_i + m_j} = m_w \tag{2.9}$$

and taking into consideration Eq. (2.1) we obtain (Gryboś, 1969)

$$m_w \ddot{x}_w + k x_w^p = 0 \tag{2.10}$$

This common equation we can interpret in another way – as an *equation of a non-linear oscillator* of mass $m_w$, non-linear elasticity (2.1) and initial conditions

$$x_w(0) = 0 \qquad \dot{x}_w(0) = v_w \tag{2.11}$$

Then, comparing the maximum value of kinetic energy (beginning of motion) with the maximum potential energy (total deflection $A_w$), it is possible to relate the parameters $v_w$ and $A_w$

$$\frac{1}{2} m_w v_w^2 = \int\limits_0^{A_w} k x_w^p \, dx \tag{2.12}$$

From this

$$m_w v_w^2 = \frac{2k}{p+1} A_w^{p+1} \tag{2.13}$$

The loss of energy during the collision is given by the Carnot theorem

$$\varDelta E = \frac{1 - R^2}{2} m_w v_w^2 \tag{2.14}$$

Taking into consideration (2.13)

$$\varDelta E = \frac{1 - R^2}{p+1} k A_w^{p+1} \tag{2.15}$$

On the other hand, the lost energy can be calculated from Eq. (2.3) as the work of damping forces $\varDelta L$. Comparing it with $\varDelta E$ given by Eq. (2.15), we obtain

$$\varPsi = 2(1 - R^2) \qquad \text{or} \qquad R = \sqrt{1 - \frac{\varPsi}{2}} \tag{2.16}$$

In the limiting cases: $R = 1$ for $\varPsi = 0$ and $R = 0$ for $\varPsi = 2$ (this value results from the fact that during collision the total loss of energy occurs for half of the hysteresis loop).

Equations (2.16) show that *Newton's hypothesis* $R =$ const *is true when the loss of energy arises from material damping mostly.*

*The above relationship allow one to determine the coefficient of restitution $R$ on the base of rheological properties of bodies and enable finding an equivalence between Newton's "time-free" approach with the description of collision as a force interaction with a limited value.*

Substituting $(2.16)_1$ in (2.6), we finally obtain

$$F = kx_w^p\Big\{1 - \frac{1-R^2}{2}[1 - \operatorname{sgn}(\dot{x}_w)]\Big\} \tag{2.17}$$

This approach is advantageous in computer simulation of collision.

If the loss of energy arises from other phenomena than material damping, relationship (2.17) is still true if we put an adequate value of $R$.

## 3. Coefficient of restitution as a function of generalised density of energy flux during collision

The occurence of plastic deformations, microslides, crushing of contaminations, etc. cause that the coefficient of restitution, in general, exhibits strong dependece on mass, velocity and shape of bodies in the region of the collision point.

For overall consideration of the above mentioned factors, it is convenient to use the idea of *density of energy flux $\varPhi$* during collision, which was introduced by Bagrejev (1964) for central collinear collision

$$\varPhi = \frac{m_w v_w^2}{2r_w^3} \tag{3.1}$$

where

| | | |
|---|---|---|
| $m_w$ | – | reduced mass determined from equation (2.9) |
| $v_w$ | – | relative velocity prior to collision, $v_w = v_i - v_j$ |
| $r_w$ | – | reduced radius of curvature in the contact point |

$$r_w = \frac{r_i r_j}{r_i + r_j} \tag{3.2}$$

We will prove that *it is possible to generalize the idea of density of energy flux* formulated for the process of central collision *about the eccentric collision.*

Let us first calculate the acceleration of points $O_i$, $i = 1, 2$ of contact of both bodies in the case of an eccentric collision – Fig. 4.

This acceleration can be found from Euler's equation

$$\mathbf{I}_i \frac{d\boldsymbol{\omega}_i}{dt} + \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i = \boldsymbol{M}_{ci} + \boldsymbol{r}_i \times \boldsymbol{F} \tag{3.3}$$

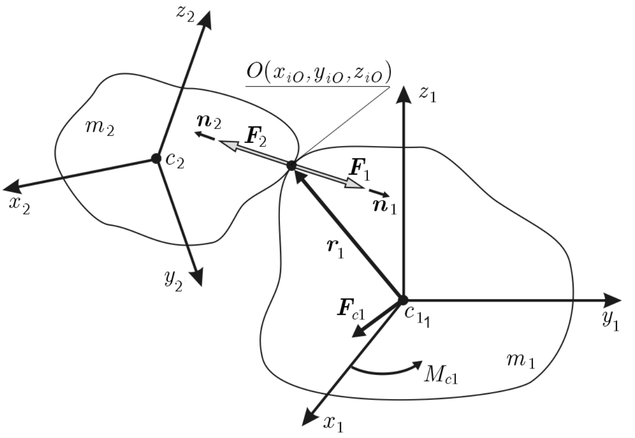


Fig. 4.

where $\mathbf{I}_i$ is the tensor of inertia in the central reference frame $C_ix_iy_iz_i$, $i = 1,2$

$$\mathbf{I}_i = \begin{bmatrix} I_{x_i} & -I_{x_iy_i} & -I_{x_iz_i} \\ -I_{y_ix_i} & I_{y_i} & -I_{y_iz_i} \\ -I_{z_ix_i} & -I_{z_iy_i} & I_{z_i} \end{bmatrix} \tag{3.4}$$

Hence

$$\frac{d\boldsymbol{\omega}_i}{dt} = \mathbf{I}_i^{-1}(\boldsymbol{M}_{ci} + \boldsymbol{r}_i \times \boldsymbol{F} - \boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i) \tag{3.5}$$

and the acceleration of point $O_i$

$$\boldsymbol{a}_{oi} = \frac{\boldsymbol{F}_i + \boldsymbol{F}_{ci}}{m_i} + [\mathbf{I}_i^{-1}(\boldsymbol{M}_{ci} + \boldsymbol{r}_i \times \boldsymbol{F}_i - \boldsymbol{\omega}_i \times \mathbf{I}_i\boldsymbol{\omega}_i)] \times \boldsymbol{r}_i + \boldsymbol{\omega}_i \times (\boldsymbol{\omega}_i \times \boldsymbol{r}_i) \tag{3.6}$$

where $\boldsymbol{F}_{ci}$, $\boldsymbol{M}_{ci}$ are external forces and moments applied to $m_i$. Neglecting limited value terms in (3.6) we get

$$\boldsymbol{a}_{0i} = \frac{\boldsymbol{F}_i}{m_i} + [\mathbf{I}_i^{-1}(\boldsymbol{r}_i \times \boldsymbol{F}_i)] \times \boldsymbol{r}_i \tag{3.7}$$

The projection of $\boldsymbol{a}_{0i}$ against the direction of $\boldsymbol{F}_i$ is then

$$a_{ni} = \boldsymbol{a}_{0i}\boldsymbol{n}_i = \Big\{\frac{\boldsymbol{F}_i}{m_i} + [\mathbf{I}_i^{-1}(\boldsymbol{r}_i \times \boldsymbol{F}_i)] \times \boldsymbol{r}_i\Big\}\boldsymbol{n}_i = F_i\Big\{\frac{\boldsymbol{n}_i}{m_i} + [\mathbf{I}_i^{-1}(\boldsymbol{r}_i \times \boldsymbol{n}_i)] \times \boldsymbol{r}_i\Big\}\boldsymbol{n}_i \tag{3.8}$$

Hence, we can consider the term

$$m_{wi} = \frac{F_i}{a_{ni}} = \frac{1}{\Big\{\frac{\boldsymbol{n}_i}{m_i} + [\mathbf{I}_i^{-1}(\boldsymbol{r}_i \times \boldsymbol{n}_i)] \times \boldsymbol{r}_i\Big\}\boldsymbol{n}_i} \qquad i = 1,2 \tag{3.9}$$

as the reduced mass of $m_i$ in the collision point.

Therefore, the reduced mass $m_w$ in Bagrejev formula (3.1) can be expressed for a general case of colliding solid bodies without friction as

$$\begin{aligned} m_w &= \frac{m_{w1}m_{w2}}{m_{w1}+m_{w2}} = \Big(\frac{1}{m_{w1}} + \frac{1}{m_{w2}}\Big)^{-1} = \\ &= \Big(\frac{1}{m_1} + \frac{1}{m_2} + \{[\mathbf{I}_1^{-1}(\boldsymbol{r}_1 \times \boldsymbol{n}_1)] \times \boldsymbol{r}_1\}\boldsymbol{n}_1 + \{[\mathbf{I}_2^{-1}(\boldsymbol{r}_2 \times \boldsymbol{n}_2)] \times \boldsymbol{r}_2\}\boldsymbol{n}_2\Big)^{-1} \end{aligned} \tag{3.10}$$

If $x_i$, $y_i$, $z_i$ are central principal axes of inertia of $m_1$, $m_2$, then

$$\mathbf{I}_i^{-1} = \begin{bmatrix} 1/I_{xi} & 0 & 0 \\ 0 & 1/I_{yi} & 0 \\ 0 & 0 & 1/I_{zi} \end{bmatrix} \qquad i = 1,2 \tag{3.11}$$

and

$$\begin{aligned} m_w = \Big[&\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{I_{x1}}(z_{1O}\boldsymbol{n}_1\boldsymbol{j}_1 - y_{1O}\boldsymbol{n}_1\boldsymbol{k}_1)^2 + \frac{1}{I_{y1}}(z_{1O}\boldsymbol{n}_1\boldsymbol{i}_1 - x_{1O}\boldsymbol{n}_1\boldsymbol{k}_1)^2 + \\ &+ \frac{1}{I_{z1}}(y_{1O}\boldsymbol{n}_1\boldsymbol{i}_1 - x_{1O}\boldsymbol{n}_1\boldsymbol{j}_1)^2 + \frac{1}{I_{x2}}(z_{2O}\boldsymbol{n}_2\boldsymbol{j}_2 - y_{2O}\boldsymbol{n}_2\boldsymbol{k}_2)^2 + \\ &+ \frac{1}{I_{y2}}(z_{2O}\boldsymbol{n}_2\boldsymbol{i}_2 - x_{2O}\boldsymbol{n}_2\boldsymbol{k}_2)^2 + \frac{1}{I_{z2}}(y_{2O}\boldsymbol{n}_2\boldsymbol{i}_2 - x_{2O}\boldsymbol{n}_2\boldsymbol{j}_2)^2\Big]^{-1} \end{aligned} \tag{3.12}$$

where $\boldsymbol{i}_i$, $\boldsymbol{j}_i$, $\boldsymbol{k}_i$ are the versors of reference frames, and $x_{iO}$, $y_{iO}$, $z_{iO}$ are coordinates of the point $O$, $i = 1,2$.

Another method of calculation of $\Phi$, more convinient if the impulse $\Pi_1$ of the collision force at the first stage of collision is known, may be derived by transformation of the Bagrejev formula

$$\Phi = \frac{m_w v_w^2}{2r_w^3} = \frac{v_w m_w v_w}{2r_w^3} = \frac{v_w \Pi_1}{2r_w^3} \tag{3.13}$$

where $v_w$ is now the normal component of relative velocity of collision points $O_1$ and $O_2$ of $m_1$ and $m_2$, respectively (Michalczyk, 1984).

## 4. Random model of the coefficient of restitution

The strong dependency of $R$ on many difficult to determine and variable factors like microstructure of the surface, its local hardenings and contaminations, changes in the place and angle of collision, etc. results in that the deterministic model of the coefficient of restitution as a function of density of energy flux $R(\Phi)$ does not give satisfactoty description of collision. It is especially important, e.g., in the investigation of stability problems of vibratory machines.

On the other hand, the description of $R$ as a random variable, e.g. Egłajs (1971), leads to loss of functional dependence $R$ on collision conditions.

We suggest to desrcibe $R$ as a random function $\overline{R}(\Phi)$, which enables one to take into consideration both deterministic and random properties of $R$.

Laboratory works (Michalczyk, 1984) on properties of this function led to the following conclusions:

- most often, the probability distribution of the random variable $R$, which corresponds to $\Phi$, can be considered as a normal cut outside the range $[0, 1]$
- due to rapid decay of the normalized autocorrelation function, which corresponds to distant values of $\Phi$, there is a practical possibility, to approximate the random function $\overline{R}(\Phi)$ as a process with independent values.

Therefore, we may describe the random function $\overline{R}(\Phi)$ as a random variable $R(\Phi)$ with the probability density

$$f(R; R_0(\Phi), \sigma(\Phi)) = \begin{cases} 0 & \text{for} \quad R < 0 \\ \dfrac{\exp\Big(-\frac{[R-R_0(\Phi)]^2}{2\sigma^2(\Phi)}\Big)}{\int\limits_0^1 \exp\Big(-\frac{[R-R_0(\Phi)]^2}{2\sigma^2(\Phi)}\Big)\, dR} & \text{for} \quad R \in [0, 1] \\ 0 & \text{for} \quad R > 1 \end{cases} \tag{4.1}$$

where $R_0(\Phi)$ and $\sigma(\Phi)$ are the expected value and standard deviation of the non-cut distribution of the random variable $R(\Phi)$, respectively.

## 5. Conclusions

- The carried out analysis showed the rightness of Newton's $R =$ const hypothesis about the restitution of impulses of instantaneous forces during collision when the loss of energy results from material damping.

  For such a case, relationships (2.16) between $R$ and $\Psi$ for a broad class of non-linear characteristics of contact forces were determined.

- When energy dissipation is influenced by other factors, like stable deformations, changes of the coeficient of restitution, $R$ can be desribed as a function of Bagrejev density of energy flux $\varPhi$.

  It is shown in the paper that it is possible to generalize the idea of $\varPhi$ for eccentric collision, see Eqs. (3.10), (3.13) which allows one to use experimental data obtained for central collision analysis.

- The rheological model based on Hertzian contact forces, which makes it possible to obtain the desired value of $R$ in simulation studies, is proposed, see (2.17).

- It is possible to take into account the strong dependency of $R$ on both measure of collision intensity $\varPhi$ and random factors, if the coefficent of restitution is described as as a random function $\overline{R}(\varPhi)$. In practical considerations, it is possible to present this function as a process of idependent values and one-dimensional Gaussian probability distribution cut outside the range $[0,1]$.

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## Zjawisko restytucji impulsów sił chwilowych w modelowaniu zderzeń


Streszczenie

W pracy wykazano, że Newtonowska idea współczynnika restytucji $R =$ const jest słuszna w odniesieniu do zderzeń, w których strata energii powstaje głównie na skutek tłumienia materiałowego.

Dla zderzeń o innym charakterze, dla których $R$ zależy od gęstości strumienia energii $\varPhi$, wskazano na możliwość rozszerzenia tego pojęcia na przypadek zderzenia mimośrodowego.

W pracy wskazano również na celowość opisania $R$ jako funkcji losowej parametru $\varPhi$ i podano formułę na siłę kontaktową zapewniającą uzyskanie założonej wartości $R$ podczas symulacji cyfrowej procesu zderzenia.


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