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DYNAMICS OF THE CATENARY MODELLED BY A PERIODICAL STRUCTURE

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In the paper, a dynamic analysis of a two-level model of the catenary which takes into account periodic distribution of hangers and supports is provided. An analytical method is proposed for calculating the response of the catenary to a uniformly moving pantograph. The model of the catenary is composed of two strings (the contact and carrying cables) connected by lumped mass-spring-dashpot elements equidistantly positioned along the strings. These elements are assumed to be visco-elastic. The pantograph is modelled by a concentrated force which moves along the contact cable. The force exerted by the pantograph varies harmonically. This model is capable of describing coupled wave dynamics of the catenary. The proposed method of calculations is based on the Fourier transformation and, therefore, is applicable only to linear models of the catenary. In the analysis, the periodicity condition is used. The spectral analysis is carried out. General results are illustrated by a numerical example in which the effect of wave propagation is visible.

Key words: speed railways, pantograph-catenary system, wave motion

1. Introduction

The pantograph-catenary interaction at high speeds is the critical factor for reliability and safety of high speed railways. In the past years, many researchers studied how to improve current collection quality in order to reduce wear and maintenance costs of both overhead line and pantograph. To improve the pantograph-catenary interface it is essential to better understand the complex behaviour of this couple. Transportation efficiency is for the Polish Railways a constant objective to improve the service to users and competitiveness in Europe. Current collection is, among other railway subsystems, a major functionality. Any failure can have important financial consequences. Therefore, researchers have initiated numerous projects to improve the pantograph-catenary interface.

Recently, it has been proposed to replace conventional droppers by more sophisticated rubber or friction damping hangers. Introduction of the hangers, which are much stiffer in comparison with the conventional droppers, leads to much more intense interaction between the contact and carrying cables. To account for this interaction, coupled vibration of the catenary cables should be examined. Numerous studies on rail vehicles carried out over the years have proved that the processes describing their dynamic state have a complex and non-periodic character.

The aim of the paper is to present a dynamic analysis of a two-level model of the catenary taking into account periodic distribution of hangers and supports. The theoretical problem, which is discussed in the paper, belongs to technical problems connected with dynamics of the periodic systems under moving loads. The emphasis of these studies is placed on the description of wave propagation (Snamina, 2003). Both contact and carrying wire are one-dimensional system, in which mechanical waves excited by moving and fixed sources can propagate.

The presented model belongs to the class of periodically non-homogeneous, continuous systems excited by a uniformly moving load. Such systems were studied in the past by a number of researchers employing different methods. Bogacz *et al.* (1993) based their approach on the Flouquet theorem. Mead (1971), Jezequel (1981) and Szolc (2003) applied the Fourier series technique. The dynamic interaction between a discrete oscillator of two degrees of freedom and a continuous string was studied by Kumaniecka and Snamina (2005). They applied the theory of one-dimensional elastic wave to the description of transient processes caused by an impulsive load. Using the Laplace transform, they solved the equation of motion of the contact wire. Metrikine and Bosch (2006) applied the periodicity condition and analyzed deflection of the contact cable and the contact force between the cable and rubber or friction hangers to estimate the fatigue life of hangers. More reference on that subject can be found in the book by Frýba (1999).

In the paper, an analytical method for calculating the response of a catenary to a uniformly moving pantograph is presented and, because of easier physical interpretation, the periodicity condition method is chosen. To the author's knowledge, the response of such a system to a moving load has not been investigated so far.

The aim of the paper is to obtain a better understanding of the pantograph-catenary system dynamics. A relatively simple analytical model, presented in the paper, is appropriate to gain the physical insight into the pantograph-catenary system. The paper is organised in five sections. Following the introduction, the two-level model of the overhead contact system is presented in Section 2. In Section 3, equations of system vibration are derived and their solution using the method of Fourier's transform is obtained. In Section 4, some results of numerical calculations are presented. Concluding remarks are formulated in Section 5.

2. Model of the system

In the literature, many models of the catenary-pantograph system have been proposed. The contact and carrying cables have been modelled by infinite or non-infinite homogenous strings or Bernoulli-Euler beams. The pantograph has been modelled by a concentrated force or by an oscillator of two or four degrees of freedom. A review paper describing the pantograph-catenary systems was presented by Poetsch *et al.* (1997) and by Kumaniecka (2004).

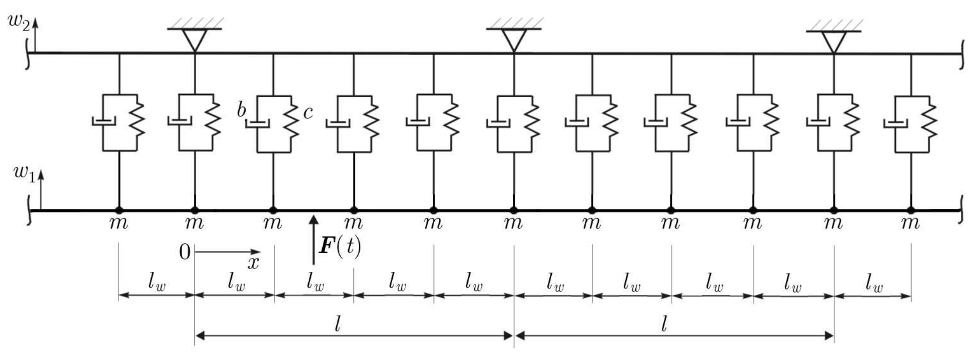

Fig. 1. Physical model of the system

The simplified model of the catenary, introduced in the paper shown in Fig. 1, is composed of two parallel, infinitely long homogenous strings (the contact and carrying cables) connected by lumped mass-spring-dashpot elements (the suspension rods), which are positioned equidistantly along the strings. The upper string (the carrying cable) is fixed at periodically spaced support points (supports) $x = nl$, $n = 0, \pm 1, \pm 2, \ldots$. The lower string (the contact wire) is suspended from the upper string by means of visco-elastic elements. These elements are used as a model of suspension rods. They are periodically placed at points $x = kl_w$, $k = 0, \pm 1, \pm 2, \ldots$ along the strings. It is assumed that the distance l is a multiple of l_w. It can be written then $l = rl_w$, $r \in N$.

Thus the system in question has two spatial periods. The larger period is introduced by the supports of the carrying cable, whereas the smaller period is associated with the droppers.

The system in question is subjected to a concentrated force (model of the pantograph), which is applied to the lower string. This load moves along the lower string at a constant velocity v and oscillates in time harmonically. The force appearing between the contact wire and the pantograph is the moving source of waves, and forces in the suspension elements are fixed sources of the waves.

Between the contact wire and the pantograph there also appears the friction force. In the present study, the friction force is neglected.

A description of motion of the whole system can be obtained by means of equations which govern small vertical motion of each spring about its equilibrium and continuity conditions at the suspension and support points.

3. Equations of the system vibration

The analysis of processes associated with the pantograph motion is provided in the fixed co-ordinate (x, y) system. The equations which govern small transverse vibrations of the strings about their equilibrium, induced by the transversal force moving along the lower string, have the following form

$$
\begin{aligned}
&\rho_1 A_1 \frac{\partial^2 w_1}{\partial t^2} - T_1 \frac{\partial^2 w_1}{\partial x^2} = F(t)\delta(x - vt) \\
&\rho_2 A_2 \frac{\partial^2 w_2}{\partial t^2} - T_2 \frac{\partial^2 w_2}{\partial x^2} = 0
\end{aligned}
\tag{3.1}
$$

In the above equations x denotes the spatial horizontal co-ordinate, t – time, subscripts 1 and 2 are related to the lower and upper wire, respectively, w_i denotes the vertical displacement of the strings, ρ_i – material density, A_i – cross-sectional area, $\rho_i A_i$ – mass per unit length of the strings, T_i – tension of the strings, $F(t)$ is the moving force and $\delta(\cdot)$ is the Dirac delta function.

At the suspension points, where the visco-elastic elements are fixed, $x = kl_w \neq nl$, the following boundary conditions have to be satisfied

$$
\begin{aligned}
& w_1\big|_{x=kl_w^+} - w_1\big|_{x=kl_w^-} = 0 \qquad\qquad w_2\big|_{x=kl_w^+} - w_2\big|_{x=kl_w^-} = 0 \\
& T_1\Big(\frac{\partial w_1}{\partial x}\Big|_{x=kl_w^+} - \frac{\partial w_1}{\partial x}\Big|_{x=kl_w^-}\Big) = \Big[c(w_1 - w_2) + b\frac{\partial}{\partial t}(w_1 - w_2) + m\frac{\partial^2 w_1}{\partial t^2}\Big]\Big|_{x=kl_w} \\
& T_2\Big(\frac{\partial w_2}{\partial x}\Big|_{x=kl_w^+} - \frac{\partial w_2}{\partial x}\Big|_{x=kl_w^-}\Big) = \Big[c(w_2 - w_1) + b\frac{\partial}{\partial t}(w_2 - w_1)\Big]\Big|_{x=kl_w}
\end{aligned}
\tag{3.2}
$$

The first two equations are the continuity conditions at the suspension points. The second two follow from the dynamic equilibrium conditions of vertical forces at the suspension points. These conditions can be obtained as a result of balancing the inner forces acting on the left and right-hand sides of the suspension points.

The fixation of the upper wire at $x = nl$ can be described as follows

$$
w_2\big|_{x=nl} = 0 \tag{3.3}
$$

Additionally, for $x = nl$ (support points of the upper string), the boundary conditions may be expressed in the form

$$
\begin{aligned}
& w_1\big|_{x=nl^+} - w_1\big|_{x=nl^-} = 0 \\
& T_1\Big(\frac{\partial w_1}{\partial x}\Big|_{x=nl^+} - \frac{\partial w_1}{\partial x}\Big|_{x=nl^-}\Big) = \Big(cw_1 + b\frac{\partial w_1}{\partial t} + m\frac{\partial^2 w_1}{\partial t^2}\Big)\Big|_{x=nl}
\end{aligned}
\tag{3.4}
$$

In Eqs. (3.1)-(3.4) the following notation is used: c, b – stiffness and damping coefficients of the suspension rods, m – equivalent mass of the suspension rods, l and l_w – spatial periods of the support and suspension points, respectively, $k, n \in N$.

The model shown in Fig. 1 is periodic, i.e. its parameters vary periodically with the co-ordinate x. In steady-state motion, under a harmonically oscillating load

$$
F(t) = F_0 e^{i\Omega t} \tag{3.5}
$$

that moves along the strings at a constant speed, the following periodicity condition must be satisfied (Metrikine and Bosch, 2006)

$$
w_i(x,t) = w_i\Big(x + nl, t + \frac{nl}{v}\Big)e^{\frac{-i\Omega nl}{v}} \qquad\qquad i = 1,2 \tag{3.6}
$$

Physically, the periodicity condition implies that the displacement pattern of the strings repeats itself in time with the period l/v. Displacements at a given point x at the moment t are strictly connected with displacements at the point $x+nl$ at the moment $t+nl/v$. The time nl/v and distance nl are associated with the load that moves along the lower string at a constant speed v. The factor $\exp(-\mathrm{i}\Omega nl/v)$ introduces a phase shift between the displacements.

The steady-state solution was obtained analytically by transforming the problem into a frequency domain by means of Fourier transform. The frequency analysis of system motion can be then carried out.

Denote the Fourier transform of the displacement $w_i(x,t)$ by $\widetilde{w}_i(x,\omega)$, $i=1,2$

$$\widetilde{w}_i(x,\omega) = \int\limits_{-\infty}^{\infty} w_i(x,t)\mathrm{e}^{-\mathrm{i}\omega t}\;dt \tag{3.7}$$

Application of this transform to equations (3.1) gives

$$\frac{\partial^2 \widetilde{w}_1}{\partial x^2} + \frac{\omega^2}{c_1^2}\widetilde{w}_1 = -\frac{F_0}{T_1 v}\mathrm{e}^{\frac{\mathrm{i}(\Omega-\omega)}{v}x} \qquad\qquad \frac{\partial^2 \widetilde{w}_2}{\partial x^2} + \frac{\omega^2}{c_2^2}\widetilde{w}_2 = 0 \tag{3.8}$$

where $c_i = \sqrt{T_i/(\rho_i A_i)}$ are the wave speeds in the strings $i=1,2$.

These equations are linear with respect to $\widetilde{w}_i(x,\omega)$, $i=1,2$.

Applying Fourier's transformation, boundary conditions (3.2) can be written as

$$\begin{aligned}
&\widetilde{w}_1\big|_{x=kl_w^+} - \widetilde{w}_1\big|_{x=kl_w^-} = 0 \qquad\qquad \widetilde{w}_2\big|_{x=kl_w^+} - \widetilde{w}_2\big|_{x=kl_w^-} = 0\\
&T_1\Big(\frac{\partial \widetilde{w}_1}{\partial x}\Big|_{x=kl_w^+} - \frac{\partial \widetilde{w}_1}{\partial x}\Big|_{x=kl_w^-}\Big) =\\
&\qquad = [c(\widetilde{w}_1-\widetilde{w}_2) + \mathrm{i}\omega b(\widetilde{w}_1-\widetilde{w}_2) - m\omega^2\widetilde{w}_1]\big|_{x=kl_w}\\
&T_2\Big(\frac{\partial \widetilde{w}_2}{\partial x}\Big|_{x=kl_w^+} - \frac{\partial \widetilde{w}_2}{\partial x}\Big|_{x=kl_w^-}\Big) = [c(\widetilde{w}_2-\widetilde{w}_1) + \mathrm{i}\omega b(\widetilde{w}_2-\widetilde{w}_1)]\big|_{x=kl_w}
\end{aligned} \tag{3.9}$$

Application of Fourier's transformation to equations (3.3) and (3.4), gives

$$\begin{aligned}
&\widetilde{w}_2\big|_{x=nl} = 0 \qquad\qquad \widetilde{w}_1\big|_{x=nl^+} - \widetilde{w}_1\big|_{x=nl^-} = 0\\
&T_1\Big(\frac{\partial \widetilde{w}_1}{\partial x}\Big|_{x=nl^+} - \frac{\partial \widetilde{w}_1}{\partial x}\Big|_{x=nl^-}\Big) = (c\widetilde{w}_1 + \mathrm{i}\omega b\widetilde{w}_1 - m\omega^2\widetilde{w}_1)\big|_{x=nl}
\end{aligned} \tag{3.10}$$

The periodicity condition, after application of Fourier's transformation, may be expressed in the form

$$\widetilde{w}_i(x,\omega) = \widetilde{w}_i(x+nl,\omega)\mathrm{e}^{\frac{-\mathrm{i}(\Omega-\omega)nl}{v}} \tag{3.11}$$

Solving equations (3.8), the functions which describe Fourier's transforms of displacements can be written as

$$
\begin{aligned}
&\widetilde{w}_1(x,\omega) = C_1^{(k)}\mathrm{e}^{\frac{\mathrm{i}\omega x}{c_1}} + C_2^{(k)}\mathrm{e}^{\frac{-\mathrm{i}\omega x}{c_1}} + C_s\mathrm{e}^{\frac{\mathrm{i}(\Omega-\omega)x}{v}}\\
&kl_w \leqslant x \leqslant (k+1)l_w \qquad\qquad k = 0, \pm 1, \pm 2, \dots \\
&\widetilde{w}_2(x,\omega) = D_1^{(k)}\mathrm{e}^{\frac{\mathrm{i}\omega x}{c_2}} + D_2^{(k)}\mathrm{e}^{\frac{-\mathrm{i}\omega x}{c_2}}
\end{aligned}
\tag{3.12}
$$

with the constant C_s

$$
C_s = \frac{F_0 v}{T_1}\frac{1}{(\Omega-\omega)^2 - \frac{\omega^2 v^2}{c_1^2}}
\tag{3.13}
$$

Equations (3.12) in each neighbouring intervals, constrained by the support points of the upper string, can be coupled using periodicity condition (3.11). It is sufficient to express $\widetilde{w}_i(x,\omega)$, $i = 1,2$ in the interval $x \in [0,l]$. The solution can be extended to the interval $x \in [l,2l]$ by employing periodicity condition (3.11).

As it follows from equations (3.12), there are $4r$ unknown coefficients which should be determined. Taking into account boundary conditions (3.9) and (3.10) the number of which in the integral $x \in [0,l]$ is $4r+2$, and substituting (3.12) and (3.13) into the boundary conditions, a set of $4r+2$ linear algebraic equations is obtained with respect to the unknown coefficients. Since the boundary conditions for $x = 0$ and $x = l$ have constants associated with the solutions for neighbouring intervals $x \in [-l,0]$ and $x \in [l,2l]$, there occurs $4r+4$ unknown coefficients in the set of algebraic equations. Apart from the equations following from the boundary conditions, some equations resulting from the periodicity condition should be taken into consideration. The linear set of equations with respect to the unknown constants is non-homogeneous and can be readily solved numerically for given parameters of the system and arbitrarily chosen values of ω. Substituting the determined constants to equation (3.12), the Fourier transform $\widetilde{w}_i(x,\omega)$, $i = 1,2$ can be derived. The displacement spectra for any points of the system can be found. These amplitude spectra of the displacements are the base for analysis of the system in question.

4. Numerical analysis

The equations presented in the previous section can be used to describe motion of the considered system. In this section, a steady-state dynamical response of the catenary is studied. The numerical calculations are carried out for the following parameters of the system

$l = 64\,\mathrm{m}$	$r = 4$	$l_w = 16\,\mathrm{m}$
$v = 50\,\mathrm{m/s}$	$T_1 = 19600\,\mathrm{N}$	$T_2 = 9800\,\mathrm{N}$
$c = 1500\,\mathrm{kg/m}$	$\Omega = 2\,\mathrm{Hz}$	$\rho_1 A_1 = 2.175\,\mathrm{kg/m}$
$\rho_2 A_2 = 1.5\,\mathrm{kg/m}$	$b = 50\,\mathrm{Ns/m}$	$c_1 = 94.93\,\mathrm{m/s}$
$c_2 = 80.83\,\mathrm{m/s}$	$m = 0.4\,\mathrm{kg}$	

These data correspond to the parameters of real overhead power lines for high-speed trains.

Making use of equations (3.12), amplitude spectra of displacements of two points were derived. The x-coordinate of each point is equal to 8 m. The points are placed on the contact wire and the carrying cable. The results of calculations are presented in Fig. 2.

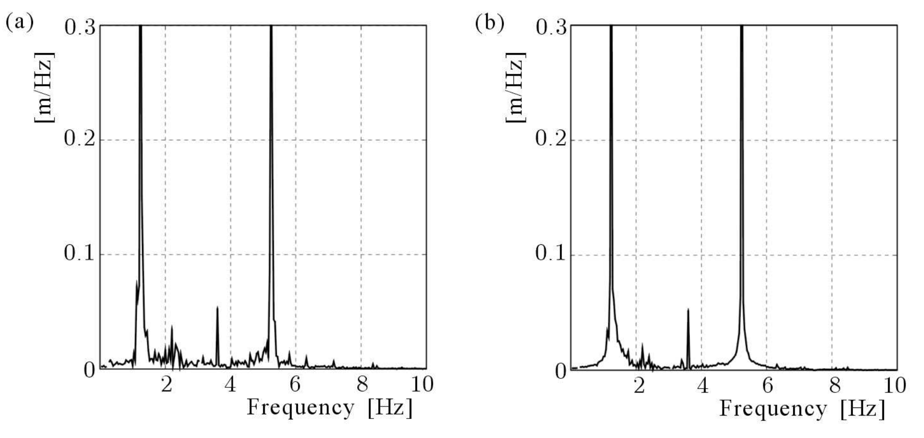

Fig. 2. Amplitude spectrum of displacement at $x = 8\,\mathrm{m}$: (a) contact wire, (b) carrying cable

The presented spectra have a similar pattern. Motion of the contact and carrying cable is dominated by two harmonic components. Their frequencies are equal to 1.2 Hz and 5.2 Hz. Spectra of other points of the cable system are similar. The harmonically varying concentrated force moves at the constant velocity $v = 50\,\mathrm{m/s}$, which is about a half of the wave speed in the contact cable. The cyclic frequency Ω of the load is equal to $2\,\mathrm{Hz}$.

5. Conclusions

In paper the application of one-dimensional elastic wave propagation theory to the analysis of pantograph-catenary interaction is discussed. A simplified model of the pantograph and catenary has been proposed. The equations of motion are based on the string model with a concentrated, harmonically varying force moving along the contact wire at a constant velocity.

Calculations have been done for the case when the load velocities are lower than the speed of transverse waves in the cable. Concluding, we can state that any time motion of the cable has a wavy character. The calculations confirmed that the travelling force is a source of two waves propagating leftwards and rightwards with different frequencies (Kumaniecka and Snamina, 2005). The wave propagating in front of the load has a frequency greater than Ω, whereas the wave propagating behind the load has a frequency lower than Ω. Because of wave reflection and motion of hangers, the amplitude spectra of each point show two dominant harmonic components. The described phenomena should be taken into account in real applications concerning high-speed railway structures.

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Dynamika sieci trakcyjnej modelowanej strukturą periodyczną

Streszczenie

W pracy przedstawiono wyniki analizy dynamicznej modelu sieci trakcyjnej z uwzględnieniem charakterystycznej powtarzalności fragmentów konstrukcji zawieszenia przewodu jezdnego i wieszaków. Podano rozwiązanie problemu współdziałania sieci z poruszającym się ze stałą prędkością odbierakiem prądu. Model sieci składa się z dwóch nieskończenie długich strun oddziaływujących ze sobą poprzez sprężysto-tłumiące elementy rozłożone periodycznie. Oddziaływanie odbieraka prądu na dolną strunę modelowane jest harmonicznie zmienną siłą. Stosując transformację Fouriera, przeprowadzono analizę częstotliwościową ruchu układu. Wykorzystano warunek periodyczności. Rezultaty rozważań analitycznych zobrazowano przykładem numerycznym.

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