JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
46, 1, pp. 3-20, Warsaw 2008

COUPLED AND UNCOUPLED CONSTITUTIVE EQUATIONS OF LINEAR ELASTICITY AND VISCOELASTICITY OF ORTHOTROPIC MATERIALS

Marian Klasztorny

Faculty of Mechanical Engineering, Military University of Technology, Warsaw, Poland
e-mail: m.klasztorny@wme.wat.edu.pl

The study concerns the linear elastic and viscoelastic constitutive modelling of homogeneous orthotropic solid bodies. The considerations are based on well-known coupled standard/inverse constitutive equations of elasticity. The author has derived new uncoupled standard/inverse constitutive equations of elasticity, new uncoupled standard/inverse constitutive equations of viscoelasticity, and new coupled standard/inverse constitutive equations of viscoelasticity of orthotropic materials. A homogeneous orthotropic material is described by 9 elastic and 18 viscoelastic constants, clearly interpreted physically. Simpler materials, i.e. monotropic and isotropic solid bodies, are also considered. In addition, the separation of shear and bulk strains in the uncoupled constitutive equations of elasticity has been examined numerically for exemplary materials.

Key words: solid body, orthotropy, constitutive modelling, elastic properties, viscoelastic properties

1. Introduction

Modern high-performance materials, e.g. xFRP cross-ply laminates, are usually homogenised and modelled as orthotropic solid bodies (Jones, 1975; Tsai, 1987; Daniel and Ishai, 1994). However, only the theory of linear elasticity (Timoshenko and Goodier, 1951) and viscoelasticity (Ferry, 1970; Garbarski, 1990; Klasztorny, 2004a,b) of isotropic materials is advanced in literature. Garbarski (1990) presented state-of-the-art in viscoelastic modelling of isotropic plastics exhibiting substantial viscoelastic/viscous deformations. Klasztorny (2004a,b) developed coupled/uncoupled constitutive equations of elasticity/viscoelasticity of isotropic thermohardening plastics as well as formulated numerical algorithms for transforming standard constitutive equations into inverse constitutive equations.

In literature, viscoelastic modelling of orthotropic materials was formulated in reference to fibre-reinforced plastics (xFRP), in the form of unidirectional fibre composites or cross-ply laminates. Such materials exhibit sensible viscoelastic properties.

Sobotka (1980) presented a two-dimensional rheological model for an orthotropic viscoelastic thin plate. The basic model consists of one planar Hookean elastic and one Newtonian viscous region in the unit representative area, and is described by 4 relaxation and 2 retardation times. The writer generalizes this model via incorporating several elastic and viscous regions.

Holzapfel and Gasser (2001) developed a general viscoelastic model for the three-dimensional stress state of orthotropic materials on the assumption of finite strains. The authors have developed the expressions for the fourth-order elasticity tensor. Papers (Zaoutsos *et al.*, 1998; Papanicolaou *et al.*, 1999, 2004) concern unidirectional fibre-polymer matrix composites. The considerations are restricted to uniaxially tensioned samples. Zaoutsos *et al.* (1998) analysed a nonlinear viscoelastic response of a unidirectional CFRP, employing a one-dimensional viscoelastic model and modified Schapery's nonlinear constitutive relationship. Creep-recovery tests in tension were executed for stress levels of 30-70% of the ultimate tensile stress. This approach has been advanced in the next papers, in which a methodology for predicting the nonlinear viscoelastic behaviour of xFRP composites was developed (Papanicolaou *et al.*, 1999), and uniaxial tension of samples for different fibre orientations was tested (Papanicolaou *et al.*, 2004).

To the author's knowledge based on the literature review, a gap in the theory of linear elasticity and viscoelasticity of orthotropic materials is observed. So far, only coupled standard/inverse constitutive equations of linear elasticity of an orthotropic material have been formulated (Jones, 1975; Tsai, 1987; Daniel and Ishai, 1994). Coupled standard constitutive equations of linear viscoelasticity derived from generalisation of respective equations of elasticity can be formulated relatively easy, as shown in this study. However, analytic inversion of these equations, using well-known classic procedures, is impossible.

This paper presents a new approach to the problem of viscoelastic modelling of homogeneous orthotropic materials. Uncoupled standard constitutive equations of linear elasticity of an orthotropic solid body are derived. These equations enable one to formulate uncoupled standard constitutive equations of linear viscoelasticity. Both groups of uncoupled equations are reversed analytically in order to obtain uncoupled inverse constitutive equations of linear elasticity and viscoelasticity. Finally, respective uncoupled equations are transformed into coupled standard/inverse constitutive equations of linear viscoelasticity.

2. Coupled standard constitutive equations of linear viscoelasticity of orthotropic solid bodies formulated by generalization of respective elastic equations

A homogeneous orthotropic solid body in isothermal conditions in the $x_1x_2x_3$ Cartesian co-ordinate system with the x_1, x_2, x_3 axes coinciding the orthotropy directions is examined. The considerations are restricted to stress levels inducing linear behaviour of the material.

Coupled standard constitutive equations of linear viscoelasticity of an orthotropic material can be formulated by generalization of Eqs. (A.2), (A3), i.e.

$$\boldsymbol{\varepsilon}(t) = \mathbf{S}(t) \otimes \boldsymbol{\sigma}(t) \tag{2.1}$$

where

$$\mathbf{S}(t) = \begin{bmatrix} S_{11}(t) & S_{12}(t) & S_{13}(t) & 0 & 0 & 0 \\ & S_{22}(t) & S_{23}(t) & 0 & 0 & 0 \\ & & S_{33}(t) & 0 & 0 & 0 \\ & & & S_{s4}(t) & 0 & 0 \\ & & & & S_{s5}(t) & 0 \\ \text{symm.} & & & & & S_{s6}(t) \end{bmatrix} \tag{2.2}$$

is termed as a viscoelastic compliance matrix whose elements are defined by the formulae

$$\begin{aligned} S_{ii}(t) &= \frac{1}{E_{ii}}\Big[1 + \omega_{ii}\int\limits_0^t F_{ii}(t-\vartheta)\ d\vartheta\Big] \qquad && ii = 11, 22, 33 \\ S_{ij}(t) &= -\frac{\nu_{ij}}{E_{jj}}\Big[1 + \omega_{ij}\int\limits_0^t F_{ij}(t-\vartheta)\ d\vartheta\Big] \qquad && ij = 23, 13, 12 \\ S_{s4}(t) &= \frac{1}{2G_{23}}\Big[1 + \omega_{s4}\int\limits_0^t F_{s4}(t-\vartheta)\ d\vartheta\Big] \\ S_{s5}(t) &= \frac{1}{2G_{13}}\Big[1 + \omega_{s5}\int\limits_0^t F_{s5}(t-\vartheta)\ d\vartheta\Big] \\ S_{s6}(t) &= \frac{1}{2G_{12}}\Big[1 + \omega_{s6}\int\limits_0^t F_{s6}(t-\vartheta)\ d\vartheta\Big] \end{aligned} \tag{2.3}$$

where $F_{11}(t)$, $F_{22}(t)$, $F_{33}(t)$, $F_{23}(t)$, $F_{13}(t)$, $F_{12}(t)$, $F_{s4}(t)$, $F_{s5}(t)$, $F_{s6}(t)$ are termed as generic functions or stress-history functions or memory functions of a viscoelastic material (Ferry, 1970). The generic functions are described by respective retardation times. Moreover, t is a time variable, $\otimes$ – convolution operator.

Summing up, Eqs. (2.3) contain 9 elastic constants (set up in Appendix) and 18 viscoelastic constants, i.e. 9 long-term creep coefficients ω_{11}, ω_{22}, ω_{33}, ω_{23}, ω_{13}, ω_{12}, ω_{s4}, ω_{s5}, ω_{s6} and 9 retardation times τ_{11}, τ_{22}, τ_{33}, τ_{23}, τ_{13}, τ_{12}, τ_{s4}, τ_{s5}, τ_{s6}. Equations (2.1)-(2.3) show high complexity of the viscoelastic modelling of orthotropic bodies. Equations (2.1) are coupled, so the analytic reversal of these equations using the available classic methods is impossible.

For a monotropic solid body, the following relationships resulting from Eqs. (A.6) are satisfied

$$S_{33}(t) = S_{22}(t) \qquad S_{13}(t) = S_{12}(t) \qquad S_{55}(t) = S_{66}(t) \tag{2.4}$$

In this case, the number of viscoelastic constants reduces itself to 12.

3. Uncoupled standard/inverse constitutive equations of elasticity of orthotropic solid bodies

Equations $(\mathrm{A.2})_{1,2,3}$ are coupled. Uncoupled standard constitutive equations of linear elasticity of an orthotropic solid body are searched in the form of two matrix equations

$$\boldsymbol{\varepsilon}_s = \{\mathbf{S}_s\}\boldsymbol{\sigma}_s \qquad \boldsymbol{\varepsilon}_b = \{\mathbf{S}_b\}\boldsymbol{\sigma}_b \tag{3.1}$$

where

$$\begin{aligned} \boldsymbol{\varepsilon}_s &= (\mathbf{I}-\mathbf{B})\boldsymbol{\varepsilon} \qquad & \boldsymbol{\sigma}_s &= (\mathbf{I}-\mathbf{A})\boldsymbol{\sigma} \\ \boldsymbol{\varepsilon}_b &= \mathbf{B}\boldsymbol{\varepsilon} \qquad & \boldsymbol{\sigma}_b &= \mathbf{A}\boldsymbol{\sigma} \end{aligned} \tag{3.2}$$

with

$$\mathbf{I} = \mathrm{diag}\,(1,1,1,1,1,1) \qquad \mathbf{A} = \begin{bmatrix} \dfrac{1}{3} & \dfrac{1}{3\lambda_2} & \dfrac{1}{3\lambda_3} & 0 & 0 & 0 \\ \dfrac{\lambda_2}{3} & \dfrac{1}{3} & \dfrac{\lambda_2}{3\lambda_3} & 0 & 0 & 0 \\ \dfrac{\lambda_3}{3} & \dfrac{\lambda_3}{3\lambda_2} & \dfrac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.3}$$

and

$$
\begin{aligned}
\{\mathbf{S}_s\} &= \operatorname{diag}(S_{s1}, S_{s2}, S_{s3}, S_{s4}, S_{s5}, S_{s6}) \\
\{\mathbf{S}_b\} &= \operatorname{diag}(S_{b1}, S_{b2}, S_{b3}, 1, 1, 1)
\end{aligned}
\tag{3.4}
$$

Matrices defined by Eqs. (3.4) may be termed as diagonal matrices of elastic compliances. The x_1 direction has been privileged in order to reflect the monotropy direction for monotropic materials. Equations (3.2) result in the following relationships

$$
\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_s + \boldsymbol{\varepsilon}_b \qquad\qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_s + \boldsymbol{\sigma}_b \tag{3.5}
$$

Coefficients λ_2, λ_3 and elastic compliances S_{s1}, S_{s2}, S_{s3}, S_{b1}, S_{b2}, S_{b3} will be derived from compatibility conditions related to the coupled and uncoupled constitutive equations. Comparing Eqs. (A.2) and (3.1), taking into consideration Eqs. (3.5), one obtains

$$
\mathbf{S}\boldsymbol{\sigma} = \{\mathbf{S}_s\}\boldsymbol{\sigma}_s + \{\mathbf{S}_b\}\boldsymbol{\sigma}_b \tag{3.6}
$$

Substituting Eqs. $(3.2)_2$, $(3.2)_4$ into Eq. (3.6) results in

$$
\mathbf{S} = \{\mathbf{S}_s\}(\mathbf{I} - \mathbf{A}) + \{\mathbf{S}_b\}\mathbf{A} \tag{3.7}
$$

Matrix equation (3.7) constitutes the compatibility conditions. Taking into account Eqs. (A.1), the explicit form of Eq. (3.7) related to sub-blocks $i,j = 1,2,3$, has the following form

$$
\begin{bmatrix}
\dfrac{1}{E_{11}} & -\dfrac{\nu_{12}}{E_{22}} & -\dfrac{\nu_{13}}{E_{33}} \\
-\dfrac{\nu_{12}}{E_{22}} & \dfrac{1}{E_{22}} & -\dfrac{\nu_{23}}{E_{33}} \\
-\dfrac{\nu_{13}}{E_{33}} & -\dfrac{\nu_{23}}{E_{33}} & \dfrac{1}{E_{33}}
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{1}{3}(2S_{s1} + S_{b1}) & \dfrac{1}{3\lambda_2}(S_{b1} - S_{s1}) & \dfrac{1}{3\lambda_3}(S_{b1} - S_{s1}) \\
\dfrac{\lambda_2}{3}(S_{b2} - S_{s2}) & \dfrac{1}{3}(2S_{s2} + S_{b2}) & \dfrac{\lambda_2}{3\lambda_3}(S_{b2} - S_{s2}) \\
\dfrac{\lambda_3}{3}(S_{b3} - S_{s3}) & \dfrac{\lambda_3}{3\lambda_2}(S_{b3} - S_{s3}) & \dfrac{1}{3}(2S_{s3} + S_{b3})
\end{bmatrix}
\tag{3.8}
$$

Comparing respective elements in Eq. (3.8), one obtains 9 algebraic equations with one equation (optional) stating identity. The analytic solution to the remaining 8 equations has the form

$$
\begin{aligned}
\lambda_2 &= \frac{\nu_{13}}{\nu_{23}} & \lambda_3 &= \frac{\nu_{12}}{\nu_{32}} \\
S_{s1} &= \frac{1}{E_{11}}\Big(1 + \frac{\nu_{21}\nu_{13}}{\nu_{23}}\Big) & S_{s2} &= \frac{1}{E_{22}}\Big(1 + \frac{\nu_{12}\nu_{23}}{\nu_{13}}\Big) \\
S_{s3} &= \frac{1}{E_{33}}\Big(1 + \frac{\nu_{13}\nu_{32}}{\nu_{12}}\Big) & S_{b1} &= \frac{1}{E_{11}}\Big(1 - 2\frac{\nu_{21}\nu_{13}}{\nu_{23}}\Big)
\end{aligned}
\tag{3.9}
$$

$$S_{b2} = \frac{1}{E_{22}}\Big(1 - 2\frac{\nu_{12}\nu_{23}}{\nu_{13}}\Big) \qquad S_{b3} = \frac{1}{E_{33}}\Big(1 - 2\frac{\nu_{13}\nu_{32}}{\nu_{12}}\Big)$$

The remaining equations in Eq. $(3.1)_1$, related to shear stresses, are directly uncoupled.

Taking into account Eqs. $(3.2)_4$, (A.9), Eq. $(3.1)_2$ can be transformed to the following form

$$\boldsymbol{\varepsilon}_b = \{\mathbf{S}_b\}\boldsymbol{\sigma}_b = \{\mathbf{S}_b\}\mathbf{A}\boldsymbol{\sigma} = \{\mathbf{S}_b\}\mathbf{AC}\boldsymbol{\varepsilon} = \mathbf{B}\boldsymbol{\varepsilon} \tag{3.10}$$

Therefore,

$$\mathbf{B} = \{\mathbf{S}_b\}\mathbf{AC} \tag{3.11}$$

The matrix $\mathbf{B}$ is of a block structure analogous to the structure of the matrix $\mathbf{A}$, i.e.

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.12}$$

For a monotropic material, the conditions written in Eqs. (A.5), (A.7) give the relationships

$$\lambda_3 = \lambda_2 \qquad S_{s2} = S_{s3} = S_{s4} \qquad S_{s5} = S_{s6} \qquad S_{b3} = S_{b2} \tag{3.13}$$

In this case, Eqs. (3.4) simplify themselves to the form

$$\begin{aligned} \{\mathbf{S}_s\} &= \operatorname{diag}(S_{s1}, S_{s4}, S_{s4}, S_{s4}, S_{s6}, S_{s6}) \\ \{\mathbf{S}_b\} &= \operatorname{diag}(S_{b1}, S_{b2}, S_{b2}, 1, 1, 1) \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} S_{s1} &= \frac{1}{E_{11}}\Big(1 + \frac{\nu_{21}\nu_{12}}{\nu_{32}}\Big) \qquad & S_{s4} &= \frac{1}{2G_{23}} \qquad & S_{s6} &= \frac{1}{2G_{12}} \\ S_{b1} &= \frac{1}{E_{11}}\Big(1 - 2\frac{\nu_{21}\nu_{12}}{\nu_{32}}\Big) \qquad & S_{b2} &= \frac{1}{3B_{22}} \end{aligned} \tag{3.15}$$

with

$$G_{23} = \frac{E_{22}}{2(1+\nu_{32})} \qquad B_{22} = \frac{E_{22}}{3(1-2\nu_{32})} \qquad \nu_{12} = \nu_{21}\frac{E_{22}}{E_{11}} \tag{3.16}$$

Equations (A.8) for an isotropic material yield

$$\lambda_2 = \lambda_3 = 1 \qquad \mathbf{B} = \mathbf{A} \tag{3.17}$$

In this case, Eqs. (3.2) take the matrix form equivalent to the classic results (Timoshenko and Goodier, 1951; Klasztorny, 2004a)

$$\begin{aligned} \boldsymbol{\varepsilon}_s &= (\mathbf{I} - \mathbf{A})\boldsymbol{\varepsilon} \qquad & \boldsymbol{\sigma}_s &= (\mathbf{I} - \mathbf{A})\boldsymbol{\sigma} \\ \boldsymbol{\varepsilon}_b &= \mathbf{A}\boldsymbol{\varepsilon} \qquad & \boldsymbol{\sigma}_b &= \mathbf{A}\boldsymbol{\sigma} \end{aligned} \tag{3.18}$$

where

$$\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ & & \frac{1}{3} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & \text{symm.} & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \tag{3.19}$$

and

$$\begin{aligned} \{\mathbf{S}_s\} &= \operatorname{diag}(S_s, S_s, S_s, S_s, S_s, S_s) \qquad & \{\mathbf{S}_b\} &= \operatorname{diag}(S_b, S_b, S_b, 1, 1, 1) \\ S_s &= \frac{1}{2G} & S_b &= \frac{1}{3B} \\ G &= \frac{E}{2(1+\nu)} & B &= \frac{E}{3(1-2\nu)} \end{aligned} \tag{3.20}$$

According to the classic theory of elasticity (Timoshenko and Goodier, 1951), S_s is termed as an elastic shear compliance, S_b – elastic bulk compliance, G – Kirchhoff's modulus, B – Helmholtz's modulus of an isotropic medium.

The reversal of Eqs. (3.1) gives uncoupled inverse constitutive equations of elasticity of an orthotropic solid body, i.e.

$$\boldsymbol{\sigma}_s = \{\mathbf{C}_s\}\boldsymbol{\varepsilon}_s \qquad \boldsymbol{\sigma}_b = \{\mathbf{C}_b\}\boldsymbol{\varepsilon}_b \tag{3.21}$$

where

$$\begin{aligned} \{\mathbf{C}_s\} &= \{\mathbf{S}_s\}^{-1} \qquad & \{\mathbf{C}_b\} &= \{\mathbf{S}_b\}^{-1} \\ C_{si} &= S_{si}^{-1} & i &= 1,2,3,4,5,6 \\ C_{bj} &= S_{bj}^{-1} & j &= 1,2,3 \end{aligned} \tag{3.22}$$

Matrices defined by Eqs. $(3.22)_{1,2}$ may be termed as diagonal matrices of the elastic stiffness.

For a monotropic material, Eqs. $(3.22)_{1,2}$ take the following form

$$
\begin{aligned}
\{\mathbf{C}_s\} &= \operatorname{diag}(C_{s1}, 2G_{23}, 2G_{23}, 2G_{23}, 2G_{12}, 2G_{12}) \\
\{\mathbf{C}_b\} &= \operatorname{diag}(C_{b1}, 3B_{22}, 3B_{22}, 1, 1, 1)
\end{aligned}
\tag{3.23}
$$

with

$$
C_{s1} = S_{s1}^{-1} \qquad\qquad C_{b1} = S_{b1}^{-1} \tag{3.24}
$$

For an isotropic material, diagonal matrices of the elastic stiffness in Eqs (3.21) reduce themselves to the following form (see Eqs. (3.20) and (3.22))

$$
\begin{aligned}
\{\mathbf{S}_s\} &= \operatorname{diag}(C_s, C_s, C_s, C_s, C_s, C_s) \qquad\qquad C_s = 2G \\
\{\mathbf{S}_b\} &= \operatorname{diag}(C_b, C_b, C_b, 1, 1, 1) \qquad\qquad C_b = 2B
\end{aligned}
\tag{3.25}
$$

According to the classic theory of elasticity (Timoshenko and Goodier, 1951), C_s is termed as the elastic shear stiffness, and C_b – elastic bulk stiffness of an isotropic material.

4. Analysis of separation of shear and bulk strains in uncoupled equations of elasticity of orthotropic solid bodies

Shear strains of an orthotropic material are fully expressed in terms of the vector $\boldsymbol{\varepsilon}_s$. This statement results from Eqs. (3.2), (3.12). The bulk (volumetric) strains are expressed in terms of the vector $\boldsymbol{\varepsilon}_b$, but a minor part of these strains is included into the vector $\boldsymbol{\varepsilon}_s$. The following quantities constitute the measures of the bulk strains corresponding to the vectors $\boldsymbol{\varepsilon}_b$, $\boldsymbol{\varepsilon}_s$, respectively

$$
\begin{aligned}
e_b &= \varepsilon_{b1} + \varepsilon_{b2} + \varepsilon_{b3} = \rho_1\varepsilon_{11} + \rho_2\varepsilon_{22} + \rho_3\varepsilon_{33} \\
e_s &= e - e_b = (1-\rho_1)\varepsilon_{11} + (1-\rho_2)\varepsilon_{22} + (1-\rho_3)\varepsilon_{33}
\end{aligned}
\tag{4.1}
$$

where

$$
e = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{4.2}
$$

is termed as dilatation (the full volumetric strain of a unit element) (Timoshenko and Goodier, 1951). From Eqs. $(3.2)_3$ and (3.12), one obtains

$$
\rho_1 = B_{11} + B_{21} + B_{31} \qquad \rho_2 = B_{12} + B_{22} + B_{32} \qquad \rho_3 = B_{13} + B_{23} + B_{33} \tag{4.3}
$$

Taking into consideration Eqs. $(3.18)_3$ and (3.19) for an isotropic material, one obtains

$$\begin{array}{lll} \rho_1 = 1 & \rho_2 = 1 & \rho_3 = 1 \\ e_b = e & e_s = 0 & \end{array} \tag{4.4}$$

hence, the shear and bulk strains are fully separated, as expected.

In this study, the separation of shear and bulk strains is examined for the following materials:

a) monotropic materials

$$\begin{array}{lll} E_{11} = 10-100 \text{ GPa} & E_{22} = 10 \text{ GPa} & E_{33} = 10 \text{ GPa} \\ \nu_{32} = 0.4 & \nu_{31} = 0.4 & \nu_{21} = 0.4 \end{array}$$

b) orthotropic materials

$$\begin{array}{lll} E_{11} = 10-100 \text{ GPa} & E_{22} = 40 \text{ GPa} & E_{33} = 10 \text{ GPa} \\ \nu_{32} = 0.4 & \nu_{31} = 0.4 & \nu_{21} = 0.1 \end{array}$$

The calculations were performed using author's computer programme.

Diagrams of the coefficients ρ_1, ρ_2, ρ_3 are presented in Fig. 1 and Fig. 2 for monotropic and orthotropic materials, respectively. Full separation of the shear and bulk strains in the isotropic material has been confirmed. Values of the coefficients ρ_1, ρ_2, ρ_3 depend on the orthotropy level. The vector $\boldsymbol{\varepsilon}_b$ incorporates 80–100% of the bulk strains, whereas the remaining part of these strains is included in the vector $\boldsymbol{\varepsilon}_s$. For a monotropic material, one obtains $\rho_2 = \rho_3$.

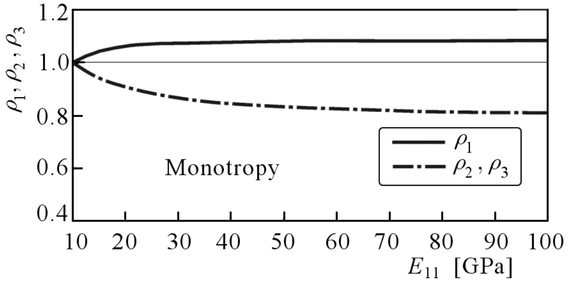

Fig. 1. Values of coefficients ρ_1, ρ_2, ρ_3 vs. longitudinal Young's modulus E_{11} for monotropic materials

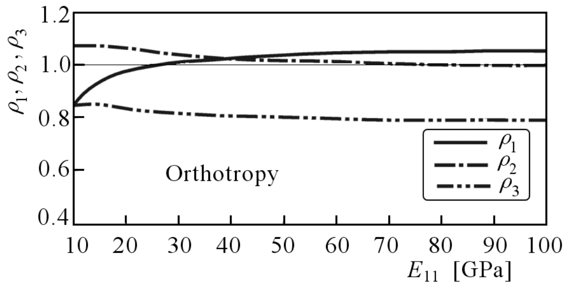

Fig. 2. Values of coefficients ρ_1, ρ_2, ρ_3 vs. longitudinal Young's modulus E_{11} for orthotropic materials

5. Uncoupled/coupled standard constitutive equations of viscoelasticity of orthotropic solid bodies

In this study, uncoupled standard constitutive equations of linear viscoelasticity of an orthotropic material are derived from generalisation of Eqs. (3.1) and, in matrix notation, take the following form

$$\boldsymbol{\varepsilon}_s(t) = \{\mathbf{S}_s(t)\} \otimes \boldsymbol{\sigma}_s(t) \qquad \boldsymbol{\varepsilon}_b(t) = \{\mathbf{S}_b(t)\} \otimes \boldsymbol{\sigma}_b(t) \tag{5.1}$$

where $\boldsymbol{\varepsilon}_s$, $\boldsymbol{\sigma}_s$, $\boldsymbol{\varepsilon}_b$, $\boldsymbol{\varepsilon}_b$ are defined by Eqs. (3.2), and

$$\begin{aligned}
\{\mathbf{S}_s(t)\} &= \operatorname{diag}[S_{s1}(t), S_{s2}(t), S_{s3}(t), S_{s4}(t), S_{s5}(t), S_{s6}(t)] \\
\{\mathbf{S}_b(t)\} &= \operatorname{diag}[S_{b1}(t), S_{b2}(t), S_{b3}(t), 1, 1, 1]
\end{aligned} \tag{5.2}$$

The matrices defined by Eqs. (5.2) may be termed as a diagonal viscoelastic quasi-shear compliance matrix and a diagonal viscoelastic quasi-bulk compliance matrix, respectively. General formulae for these compliances have the form

$$\begin{aligned}
S_{si}(t) &= S_{si}\Big[1 + \omega_{si} \int_0^t F_{si}(t-\vartheta)\, d\vartheta\Big] \qquad i = 1,2,3,4,5,6 \\
S_{bj}(t) &= S_{bj}\Big[1 + \omega_{bj} \int_0^t F_{bj}(t-\vartheta)\, d\vartheta\Big] \qquad j = 1,2,3
\end{aligned} \tag{5.3}$$

where $F_{si}(t)$, $F_{bj}(t)$ are termed as quasi-deviatoric and quasi-axiatoric stress-history functions, respectively. For each function, a retardation time is to be specified.

Summing up, an orthotropic material is described by 9 elastic constants and 18 viscoelastic constants, i.e. 9 long-term creep coefficients ω_{si},

$i = 1,2,3,4,5,6$; ω_{bj}, $j = 1,2,3$ and 9 retardation times τ_{si}, $i = 1,2,3,4,5,6$; τ_{bj}, $j = 1,2,3$.

For a monotropic solid body, the following relationships resulting form Eqs. (3.13) are valid

$$S_{s2}(t) = S_{s3}(t) = S_{s4}(t) \qquad S_{s5}(t) = S_{s6}(t) \qquad S_{b2}(t) = S_{b3}(t) \tag{5.4}$$

In this case, the number of viscoelastic constants drops to 10.

Coupled standard constitutive equations of linear viscoelasticity of an orthotropic solid body are obtained by summing matrix equations $(5.1)_1$, $(5.1)_2$, i.e

$$\boldsymbol{\varepsilon}(t) = \mathbf{S}(t) \otimes \boldsymbol{\sigma}(t) \tag{5.5}$$

where

$$\begin{aligned}\mathbf{S}(t) &= \{\mathbf{S}_s(t)\}(\mathbf{I} - \mathbf{A}) + \{\mathbf{S}_b(t)\}\mathbf{A} = \\ &= \begin{bmatrix} S_{11}(t) & S_{12}(t) & S_{13}(t) & 0 & 0 & 0 \\ S_{21}(t) & S_{22}(t) & S_{23}(t) & 0 & 0 & 0 \\ S_{31}(t) & S_{32}(t) & S_{33}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{s4}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{s5}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{s6}(t) \end{bmatrix}\end{aligned} \tag{5.6}$$

is named as in Section 2, i.e. the viscoelastic compliance matrix.

6. Uncoupled/coupled inverse constitutive equations of viscoelasticity of orthotropic solid bodies

The exact analytic reversal of uncoupled standard constitutive equations of viscoelasticity (Eqs. (5.1)) is possible for a number of generic functions. As a result, one obtains uncoupled inverse constitutive equations of viscoelasticity of an orthotropic material in the form

$$\boldsymbol{\sigma}_s(t) = \{\mathbf{C}_s(t)\} \otimes \boldsymbol{\varepsilon}_s(t) \qquad \boldsymbol{\sigma}_b(t) = \{\mathbf{C}_b(t)\} \otimes \boldsymbol{\varepsilon}_b(t) \tag{6.1}$$

where

$$\begin{aligned}\{\mathbf{C}_s(t)\} &= \operatorname{diag}[C_{s1}(t), C_{s2}(t), C_{s3}(t), C_{s4}(t), C_{s5}(t), C_{s6}(t)] \\ \{\mathbf{C}_b(t)\} &= \operatorname{diag}[C_{b1}(t), C_{b2}(t), C_{b3}(t), 1, 1, 1]\end{aligned}$$

$$C_{si}(t) = C_{si}\Big[1 - \kappa_{si}\int\limits_0^t K_{si}(t-\vartheta)\;d\vartheta\Big] \qquad i = 1,2,3,4,5,6 \tag{6.2}$$

$$C_{bj}(t) = C_{bj}\Big[1 - \kappa_{bj}\int\limits_0^t K_{bj}(t-\vartheta)\;d\vartheta\Big] \qquad j = 1,2,3$$

The matrices defined by Eqs. (6.2) may be termed as a diagonal viscoelastic quasi-shear stiffness matrix and a diagonal viscoelastic quasi-bulk stiffness matrix, respectively. Quantities $K_{si}(t)$, $K_{bj}(t)$ may be termed as quasi-shear and quasi-bulk strain-history functions, respectively. For each function, a relaxation time is to be specified.

From the point of view of inverse equations, an orthotropic material is still described by 9 elastic constants and 18 viscoelastic constants, i.e. 9 long-term relaxation coefficients κ_{si}, $i = 1,2,3,4,5,6$; κ_{bj}, $j = 1,2,3$ and 9 relaxation times θ_{si}, $i = 1,2,3,4,5,6$; θ_{bj}, $j = 1,2,3$.

For some genering functions, formulae transforming quantities related to the standard equations (Eqs. (5.1)) into quantities related to the inverse equations (Eqs. (6.1)) are known. For example, normal exponential generic functions

$$\begin{aligned} F_{si}(t) &= \alpha_{si}\mathrm{e}^{-\alpha_{si}t} & \qquad \alpha_{si} &= \frac{1}{\tau_{si}} & \qquad i &= 1,2,3,4,5,6 \\ F_{bj}(t) &= \alpha_{bj}\mathrm{e}^{-\alpha_{bj}t} & \qquad \alpha_{bj} &= \frac{1}{\tau_{bj}} & \qquad j &= 1,2,3 \end{aligned} \tag{6.3}$$

result in (Timoshenko and Goodier, 1951; Klasztorny, 2004a,b)

$$\begin{aligned} &\left.\begin{aligned} \kappa_{si} &= \frac{\omega_{si}}{1+\omega_{si}} & \qquad K_{si}(t) &= \beta_{si}\mathrm{e}^{-\beta_{si}t} \\ \beta_{si} &= (1+\omega_{si})\alpha_{si} & \qquad \beta_{si} &= \frac{1}{\theta_{si}} \end{aligned}\right\} \quad i = 1,2,3,4,5,6 \\ &\left.\begin{aligned} \kappa_{bj} &= \frac{\omega_{bj}}{1+\omega_{bj}} & \qquad K_{bj}(t) &= \beta_{bj}\mathrm{e}^{-\beta_{bj}t} \\ \beta_{bj} &= (1+\omega_{bj})\alpha_{bj} & \qquad \beta_{bj} &= \frac{1}{\theta_{bj}} \end{aligned}\right\} \quad j = 1,2,3 \end{aligned} \tag{6.4}$$

Summing Eqs. $(6.1)_1$, $(6.2)_2$ and taking into account formulae (3.2), one obtains coupled inverse constitutive equations of linear viscoelasticity of an orthotropic solid body in the form

$$\boldsymbol{\sigma}(t) = \mathbf{C}(t) \otimes \boldsymbol{\varepsilon}(t) \tag{6.5}$$

where

$$\begin{gathered}\mathbf{C}(t) = \{\mathbf{C}_s(t)\}(\mathbf{I} - \mathbf{B}) + \{\mathbf{C}_b(t)\}\mathbf{B} = \\ + \begin{bmatrix} C_{11}(t) & C_{12}(t) & C_{13}(t) & 0 & 0 & 0 \\ C_{21}(t) & C_{22}(t) & C_{23}(t) & 0 & 0 & 0 \\ C_{31}(t) & C_{32}(t) & C_{33}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{s4}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{s5}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{s6}(t) \end{bmatrix}\end{gathered} \tag{6.6}$$

is termed as a viscoelastic stiffness matrix.

7. Conclusions

The study concerns the linear elastic/viscoelastic constitutive modelling of homogeneous orthotropic solid bodies. Coupled standard/inverse constitutive equations of elasticity have consituted the basis for deriving the following equations:

- uncoupled standard constitutive equations of elasticity,
- uncoupled inverse constitutive equations of elasticity,
- uncoupled standard constitutive equations of viscoelasticity,
- uncoupled inverse constitutive equations of viscoelasticity,
- coupled standard constitutive equations of viscoelasticity,
- coupled inverse constitutive equations of viscoelasticity.

The uncoupled/coupled constitutive equations of viscoelasticity of a homogeneous orthotropic material are described by 9 elastic and 18 viscoelastic constants. These constants have been clearly interpreted physically. Two particular cases have been considered, i.e. monotropic and isotropic solid bodies.

In addition, the separation of shear and bulk strains in the uncoupled constitutive equations of elasticity has been examined numerically for selected monotropic and orthotropic materials.

A. Coupled standard/inverse constitutive equations of linear elasticity of orthotropic solid bodies

This appendix is based on Jones (1975), Tsai (1987), Daniel and Ishai (1994) and slightly develops respective equations via unification of Poisson's ratios. A homogeneous orthotropic solid body in isothermal conditions in the $x_1x_2x_3$ Cartesian co-ordinate system with the x_1, x_2, x_3 axes coinciding the orthotropy directions is examined. The considerations are restricted to stress levels inducing linear behaviour of the material. The stress and strain states are described by the following vectors (reflecting stress and strain tensors, respectively)

$$
\begin{aligned}
\boldsymbol{\sigma} &= \operatorname{col}(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}) \\
\boldsymbol{\varepsilon} &= \operatorname{col}(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12})
\end{aligned}
$$

where, for $i, j = 1, 2, 3$, σ_{ii} is the normal stress, σ_{ij} – shear stress $(i \neq j)$, ε_{ii} – relative elongation, ε_{ij} – half of the shear strain angle of the $dx_1dx_2dx_3$ differential element $(i \neq j)$.

A homogeneous orthotropic material is described by 9 independent elastic constants, i.e. E_{11}, E_{22}, E_{33} – Young's longitudinal moduli of elasticity; ν_{32}, ν_{31}, ν_{21} – Poisson's ratios; G_{23}, G_{13}, G_{12} – Kirchhoff's shear moduli. The remaining Poisson's ratios $(\nu_{23}, \nu_{13}, \nu_{12})$ are derived from the symmetry conditions, i.e.

$$
\frac{\nu_{ij}}{E_{jj}} = \frac{\nu_{ji}}{E_{ii}} \qquad i \neq j; \quad i, j = 1, 2, 3 \tag{A.1}
$$

where $(-\nu_{ji})$ denotes shortening in the x_j direction induced by tensioning in the x_i direction.

Coupled standard constitutive equations of linear elasticity of an orthotropic solid body, written in matrix notation, have the following form

$$
\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} \tag{A.2}
$$

where

$$
\mathbf{S} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
 & S_{22} & S_{23} & 0 & 0 & 0 \\
 & & S_{33} & 0 & 0 & 0 \\
 & & & S_{s4} & 0 & 0 \\
 & & & & S_{s5} & 0 \\
\text{symm.} & & & & & S_{s6}
\end{bmatrix} \tag{A.3}
$$

is termed as an elastic compliance matrix, whose elements are defined by the following formulae

$$
\begin{aligned}
&S_{11} = \frac{1}{E_{11}} && S_{12} = -\frac{\nu_{12}}{E_{22}} && S_{13} = -\frac{\nu_{13}}{E_{33}} \\
&S_{21} = -\frac{\nu_{21}}{E_{11}} && S_{22} = \frac{1}{E_{22}} && S_{23} = -\frac{\nu_{23}}{E_{33}} \\
&S_{31} = -\frac{\nu_{31}}{E_{11}} && S_{32} = -\frac{\nu_{32}}{E_{22}} && S_{33} = \frac{1}{E_{33}} \\
&S_{s4} = \frac{1}{2G_{23}} && S_{s5} = \frac{1}{2G_{13}} && S_{s6} = \frac{1}{2G_{12}}
\end{aligned}
\tag{A.4}
$$

A monotropic material, also termed as a transverse isotropic material, satisfies the following relationships

$$
E_{33} = E_{22} \qquad G_{13} = G_{12} \qquad \nu_{13} = \nu_{12} \qquad G_{23} = \frac{E_{22}}{2(1+\nu_{32})}
\tag{A.5}
$$

resulting in

$$
S_{33} = S_{22} \qquad S_{13} = S_{12} \qquad S_{55} = S_{66}
\tag{A.6}
$$

with x_1 being the direction of monotropy, x_2x_3 – plane of isotropy. A monotropic material is described by 5 independent elastic constants, i.e.: E_{11} – Young's longitudinal modulus, E_{22} – Young's transverse modulus ($E_{22} \leqslant E_{11}$), ν_{32} – Poisson's ratio in the x_2x_3 plane, ν_{21} – greater Poisson's ratio in the x_1x_2 plane, G_{12} – Kirchhoff's shear modulus in the x_1x_2 plane. The remaining 7 elastic constants for a monotropic material depend on the constants E_{11}, E_{22}, ν_{32}, ν_{21}, G_{12} according to Eqs. (A.1), (A.5). In this case, one obtains

$$
\nu_{12} = \nu_{21}\frac{E_{22}}{E_{11}} \leqslant \nu_{21} \qquad \nu_{31} = \nu_{21} \qquad \nu_{23} = \nu_{32}
\tag{A.7}
$$

The coefficient ν_{12} is termed as a smaller Poisson's ratio in the x_1x_2 plane.

An isotropic material is described by two classic elastic constants, E, ν, and satisfies the following relationships

$$
\begin{aligned}
&E_{ii} = E && ii = 11, 22, 33 \\
&\nu_{ij} = \nu_{ji} = \nu && ij = 23, 13, 12 \\
&G_{ij} = G = \frac{E}{2(1+\nu)} && j = 23, 13, 12
\end{aligned}
\tag{A.8}
$$

The coupled inverse constitutive equations describing behaviour of an orthotropic solid body, written in matrix notation, have the following form

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \tag{A.9}$$

where

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{s4} & 0 & 0 \\ & & & & C_{s5} & 0 \\ \text{symm.} & & & & & C_{s6} \end{bmatrix} \tag{A.10}$$

is termed as an elastic stiffness matrix, which elements are defined by the following formulae

$$\begin{array}{lll} C_{11} = E_{11}\dfrac{1-\nu_{23}\nu_{32}}{\varDelta} & C_{22} = E_{22}\dfrac{1-\nu_{13}\nu_{31}}{\varDelta} & \\ C_{33} = E_{33}\dfrac{1-\nu_{12}\nu_{21}}{\varDelta} & C_{12} = E_{22}\dfrac{\nu_{21}+\nu_{23}\nu_{31}}{\varDelta} & \\ C_{13} = E_{33}\dfrac{\nu_{31}+\nu_{21}\nu_{32}}{\varDelta} & C_{23} = E_{33}\dfrac{\nu_{32}+\nu_{12}\nu_{31}}{\varDelta} & \\ C_{s4} = 2G_{23} & C_{s5} = 2G_{13} & C_{s6} = 2G_{12} \end{array} \tag{A.11}$$

with

$$\varDelta = 1 - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - \nu_{12}\nu_{21} - \nu_{12}\nu_{23}\nu_{31} - \nu_{21}\nu_{32}\nu_{13} \tag{A.12}$$

and $\mathbf{C} = \mathbf{S}^{-1}$.

For a monotropic material, one obtains

$$C_{33} = C_{22} \qquad C_{13} = C_{12} \qquad C_{55} = C_{66} \tag{A.13}$$

Acknowledgements

This work has been supported by Ministry of Science and Higher Education, Poland, through grant No. 3 T08E 001 27. This support is gratefully acknowledged.

References

1. Aboudi J., 1990, Micromechanical characterization of the non-linear viscoelastic behaviour of resin matrix composites, *Composites Science and Technology*, **38**, 4, 371-386
2. Daniel I.M., Ishai O., 1994, *Engineering Mechanics of Composite Materials*, Oxford University Press, New York-Oxford

3. Ferry J.D., 1970, *Viscoelastic Properties of Polymers*, J. Wiley & Sons, Inc., New York
4. Garbarski J., 1990, *Description of Linear Viscoelasticity Phenomena in Constructional Polymers*, Warsaw University of Technology Publishers, Warsaw [in Polish]
5. Holzapfel G.A., Gasser T.C., 2001, A viscoelastic model for fiber-reinforced composites at finite strains: continuum basis, computational aspects and applications, *Computer Methods in Applied Mechanics and Engineering*, **190**, 4379-4403
6. Jones R.M., 1975, *Mechanics of Composite Materials*, McGraw-Hill Book Co
7. Klasztorny M., 2004a, Constitutive modelling of resins in compliance domain, *J. Mechanics of Composite Materials*, **40**, 4, 349-358
8. Klasztorny M., 2004b, Constitutive modelling of resins in stiffness domain, *J. Mechanics of Composite Materials*, **40**, 5, 443-452
9. Papanicolaou G.C., Zaoutsos S.P., Cardon A.H., 1999, Prediction of the non-linear viscoelastic response of unidirectional fiber composites, *Composites Science and Technology*, **59**, 1311-1319
10. Papanicolaou G.C., Zaoutsos S.P., Kontou E.A., 2004, Fiber orientation dependence of continuous carbon/epoxy composites nonlinear viscoelastic behaviour, *Composites Science and Technology*, **64**, 2535-2545
11. Sobotka Z., 1980, Two-dimensional rheological models for simple and fibre-reinforced viscoelastic materials, *Acta Technica CSAV*, **25**, 5, 608-635
12. Timoshenko S., Goodier J.N., 1951, *Theory of Elasticity*, 2nd Edn., New York-Toronto-London, Mc Graw-Hill Book Company, Inc.
13. Tsai S.W., 1987, *Composites Design*, IV Edn. Think Composites, Dayton
14. Zaoutsos S.P., Papanicolaou G.C., Cardan A.H., 1998, On the non-linear visoelastic behaviour of polymer-matrix composites, *Composites Science and Technology*, **58**, 883-889

Sprzężone i niesprzężone równania konstytutywne liniowej sprężystości i lepkosprężystości materiałów ortotropowych

Streszczenie

Praca dotyczy modelowania konstytutywnego jednorodnych ortotropowych ciał stałych w zakresie linowym, sprężystym i lepkosprężystym. Podstawą rozważań są znane sprzężone standardowe/odwrotne równania konstytutywne liniowej sprężystości tych materiałów. Wyznaczono niesprzężone standardowe/odwrotne równania konstytutywne liniowej sprężystości, sformułowano niesprzężone standardowe/odwrotne równania konstytutywne liniowej lepkosprężystości, a następnie wyznaczono sprzężone standardowe/odwrotne równania konstytutywne liniowej lepkosprężystości ortotropowych ciał stałych. Jednorodny materiał ortotropowy opisano za pomocą 9 stałych sprężystości i 18 stałych lepkosprężystości z podaniem przejrzystej interpretacji fizycznej tych stałych. Rozważono również przypadki szczególne materiału monotropowego i izotropowego. Dodatkowo, przetestowano numerycznie rozdzielenie odkształceń postaciowych i objętościowych w przypadku niesprzężonych równań konstytutywnych sprężystości materiałów ortotropowych i monotropowych.

Manuscript received June 18, 2007; accepted for print October 23, 2007