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ON THE MODELLING AND OPTIMIZATION OF FUNCTIONALLY GRADED LAMINATES

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The object of investigations are Functionally Graded Materials (FGM) which on the microstructural level are made of two kinds of very thin laminae. These FGM will be referred to as the Functionally Graded Laminates (FGL). The aim of this contribution is to formulate discrete-continuum and continuum 3D-models of elastodynamics of FGL. The proposed modelling procedure constitutes a certain generalization of the approach to the modelling of periodic structures leading to a system of finite difference equations and then to their continuum approximation, Rychlewska and Woźniak (2003). The obtained results are applied to the analysis of a certain layered structure with a FGL transition zone. The optimization problem related to the position of the transition zone is discussed.

Key words: functionally graded laminates, dynamics, modelling

1. Introduction

Functionally Graded Materials (FGM) are usually regarded as heterogeneous composites having effective (macroscopic) properties varying smoothly in space. A review of researches on FGM can be found in Suresh and Mortensen (1998). In this paper, the object of considerations are micro-layered linear elastic solids made of two materials and having macroscopic properties continuously varying in the direction normal to the layering. These solids will be referred to as the Functionally Graded Laminates (FGL). A fragment of FGL on the macro and micro-level is shown in Fig. 1. The modelling approach presented in this contribution takes into account some concepts and assumptions of the tolerance averaging technique formulated and applied in Woźniak and Wierzbicki (2000) for periodic structures. This technique was also used in the modelling of elastodynamics of functionally graded laminated shells, Woźniak *et al.* (2005), functionally graded laminated plates, Jędrysiak *et al.* (2005) and functionally graded laminates with interlaminar microdefects, Rychlewska *et al.* (2006). The purpose of this contribution is to formulate discrete-continuum models of FGL which state a basis for continuum models. To this end, a certain generalization of the known approach to the modelling of periodic structures is applied. For the aforementioned periodic structures, the periodic simplicial division technique was used in Rychlewska and Woźniak (2003). In this case, the system of finite difference equations is obtained. These equations constitute foundations of different continuum models represented by equations with constant coefficients. For functionally graded laminates, the proposed continuum model equations have slowly-varying, smooth coefficients.

Notations. The index n run over $1,\dots,N$ unless otherwise stated and is assigned to the n-th layer of FGL. Subscripts $\alpha,\beta,\gamma,\delta$ run over the sequence $1,2$ and subscripts i,j,k,l over $1,2,3$. For an arbitrary sequence $\{f^m\}$, $m=0,\dots,N$, we define the difference operators

$$\varDelta f^m = \frac{f^{m+1}-f^m}{l} \qquad\qquad m=0,\dots,N-1$$

$$\overline{\varDelta} f^m = \frac{f^{m}-f^{m-1}}{l} \qquad\qquad m=1,\dots,N$$

where the superscript m is related to the interface $z=ml$ between the m-th and $(m+1)$-th layer (provided that $m=1,\dots,N-1$) and $m=0$, $m=N$ are related to the boundary planes $z=0$, $z=Nl$, respectively.

In the physical space, we introduce the cartesian orthogonal coordinate system $0x_1x_2x_3$ with the x_3 axis normal to the lamina interfaces. Let $\partial_\alpha f$ and $\partial_k f$ stand for partial differentiation of the function $f(x_1,x_2,x_3)$ with respect to x_α and x_k, respectively. We also use gradient operators $\nabla=(\partial_1,\partial_2,\partial_3)$, $\overline{\nabla}=(\partial_1,\partial_2,0)$, gradient-difference operators $D=(\partial_1,\partial_2,\varDelta)$, $\overline{D}=(\partial_1,\partial_2,\overline{\varDelta})$ and introduce notations $z=x_3$, $\boldsymbol{x}=(x_1,x_2)$. The time coordinate is denoted by t and time differentiation by the overdot. Small bold-face letters represent vectors and points in 3D space, capital bold-face letters stand for second order tensors, and block letters are used for higher order tensors. In the paper, the absolute notations one used.

2. Preliminaries

Let $\varOmega\times(0,L)$, $\varOmega\subset R^2$, stand for a region occupied in the physical space by the laminated medium under consideration in its natural configuration. The subject of analysis is a FGL medium composed of two linear-elastic materials distributed in N layers $\varLambda_1,\dots,\varLambda_N$ of the same thickness l. It is assumed that $N^{-1}\ll 1$. Every layer $\varLambda_n$ is made of two homogeneous laminae $\varLambda'_n$, $\varLambda''_n$ having different thicknesses l'_n, l''_n, respectively, $n = 1, \ldots, N$. By ρ', $\mathbb{C}'$, ρ'', $\mathbb{C}''$ we denote mass densities and tensors of elastic moduli in every pair of adjacent laminae, cf. Fig. 1. The material volume fractions in the laminae Λ'_n, Λ''_n are denoted by $\nu'_n = l'_n/l$ and by $\nu''_n = l''_n/l$, respectively, $\nu'_n + \nu''_n = 1$. Moreover, we introduce the phase distribution sequence $\{\nu_n\}$ setting $\nu_n = \sqrt{\nu'_n \nu''_n}$. By $\nu'(\cdot)$, $\nu''(\cdot)$ we denote smooth functions defined on $[0, L]$ representing distributions of the mean volume fractions of lamina materials, $\nu'(z) + \nu''(z) = 1$, $z \in [0, L]$. It means that $\nu'(z_n) = \nu'_n$, $\nu''(z_n) = \nu''_n$ for some $z_n \in [(n-1)l, nl]$, $n = 1, \ldots, N$. We also define $\nu(z) = \sqrt{\nu'(z)\nu''(z)}$, $z \in [0, L]$.

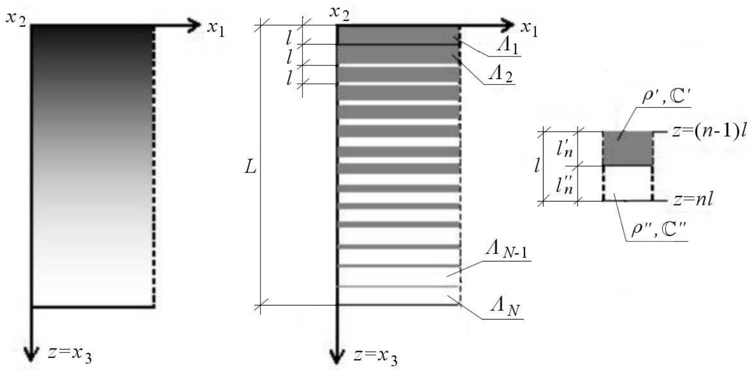

Fig. 1. Fragments of FGL on the macro- and micro-structural level together with a scheme of the n-th layer, $n = 1, \ldots, N$

The sequence $\{f^m\}$ will be referred to as slowly-varying (with a certain tolerance $0 < \varepsilon \ll 1$) if the condition $l|\Delta f^m| \leqslant \varepsilon \max\{|f^0|, \ldots, |f^N|\}$ holds for every $m = 0, 1, \ldots, N-1$. In this case, we shall write $\{f^m\} \in SV_\varepsilon$. For a detailed discussion of the concept of slowly-varying function cf. Woźniak and Wierzbicki (2000). Crucial assumptions related to the material volume sequences are:

1° sequence $\{\nu'_n\}$, $n = 1, \ldots, N$, is strongly monotone;

2° sequence $\{\nu'_n\}$, $n = 1, \ldots, N$, is slowly varying.

Similar requirements are satisfied by the sequence $\{\nu''_n\}$, $n = 1, \ldots, N$. Under the above conditions, the laminated medium represents a certain Functionally Graded Laminate (FGL). Moreover, it is assumed that the lamina materials have elastic symmetry planes parallel to the lamina interfaces and that the laminae are perfectly bonded.

3. Modelling assumptions

Let $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, z, t)$, $\boldsymbol{x} = (x_1, x_2) \in \overline{\Omega}$, $z \in [0, L]$ stand for the displacement field at time t. The restriction of this field to the n-th layer Λ_n will be denoted by $\boldsymbol{w}_n$ i. e., $\boldsymbol{w}_n = \boldsymbol{w}(\boldsymbol{x}, z, t)$, $\boldsymbol{x} \in \overline{\Omega}$, $z \in [(n-1)l, nl]$, $n = 1, \ldots, N$. Let us also denote $\boldsymbol{w}'_n = \boldsymbol{w}(\boldsymbol{x}, z'_n, t)$, $\boldsymbol{x} \in \overline{\Omega}$, $z'_n \in [(n-1)l, l'_n + (n-1)l]$, $n = 1, \ldots, N$ and $\boldsymbol{w}''_n = \boldsymbol{w}(\boldsymbol{x}, z''_n, t)$, $\boldsymbol{x} \in \overline{\Omega}$, $z''_n \in [l'_n + (n-1)l, nl]$, $n = 1, \ldots, N$. Moreover, let us denote by $\widetilde{\boldsymbol{w}}_n(\boldsymbol{x}, z, t)$, $z = \widetilde{c}_n$, $n = 0, 1, \ldots, N$, the restriction of the displacement field to interfaces between the layers Λ_n and by $\overline{\boldsymbol{w}}_n(\boldsymbol{x}, z, t)$, $z = \overline{c}_n$, $n = 1, \ldots, N$, the restriction of this field to interfaces between laminae in Λ_n, $n = 1, \ldots, N$, cf. Fig. 2, where $\widetilde{c}_n = nl$, $\overline{c}_n = nl + l'_n$. We introduce functions $\boldsymbol{u}_n(\boldsymbol{x}, t)$, $\boldsymbol{v}_n(\boldsymbol{x}, t)$ satisfying conditions

$$
\begin{aligned}
\widetilde{\boldsymbol{w}}_n &= \boldsymbol{u}_n && n = 1, \ldots, N \\
\overline{\boldsymbol{w}}_n &= \boldsymbol{u}_n + 2\sqrt{3} l \nu_n \boldsymbol{v}_n + l'_n \Delta \boldsymbol{u}_n && n = 0, 1, \ldots, N-1
\end{aligned}
$$

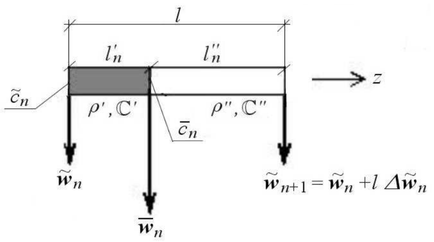

Fig. 2. A scheme of the n-th layer and displacements on interfaces, $\widetilde{c}_n = nl$, $\overline{c}_n = nl + l'_n$, $n = 1, \ldots, N$, $N^{-1} \ll 1$

The first modelling assumption states that displacements in every laminae belonging to the n-th layer, $n = 1, \ldots, N$, are linear functions of z. This statement is satisfied if for every pair of laminae Λ'_n, Λ''_n, pertinent displacements $\boldsymbol{w}'_n$, $\boldsymbol{w}''_n$ are assumed respectively in the form

$$
\begin{aligned}
\boldsymbol{w}'_n &= \Big(2\sqrt{3}\frac{\nu_n}{\nu'_n}\boldsymbol{v}_n + \Delta \boldsymbol{u}_n\Big) z'_n + \boldsymbol{u}_n \\
\boldsymbol{w}''_n &= \Big(\Delta \boldsymbol{u}_n - 2\sqrt{3}\frac{\nu_n}{\nu''_n}\boldsymbol{v}_n\Big) z''_n + \boldsymbol{u}_n + 2l\sqrt{3}\nu_n \boldsymbol{v}_n + l'_n \Delta \boldsymbol{u}_n
\end{aligned}
\tag{3.1}
$$

where $\boldsymbol{u}_n = \boldsymbol{u}_n(\boldsymbol{x}, t)$, $\boldsymbol{v}_n = \boldsymbol{v}_n(\boldsymbol{x}, t)$, $\Delta \boldsymbol{u}_n = \Delta \boldsymbol{u}_n(\boldsymbol{x}, t)$, $z'_n \in [(n-1)l, l'_n + (n-1)l]$, $z''_n \in [l'_n + (n-1)l, nl]$, $n = 1, \ldots, N$. Moreover, conditions $\boldsymbol{v}_0 = \boldsymbol{v}_1$ and $\boldsymbol{v}_{N-1} = \boldsymbol{v}_N$ are implied by the postulated homogeneity of the layers Λ_1 and Λ_N, respectively.

The second modelling assumption states that the continuous mass distribution in the laminate can be approximated by a proportional mass distribution *only on the interfaces between adjacent layers.* This assumption can be applied if macroscopic wavelengths are large when compared to the lamina thicknesses. Let ρ_n, $n = 1,\dots,N$, stand for the mass density on the interface $z = \widetilde{c}_n$ between layers. Then, the above assumption implies that the kinetic energy density will take the form

$$\kappa_n = \frac{1}{2}\rho_n(\dot{\boldsymbol{u}}_n)^2 \tag{3.2}$$

where

$$\rho_n = \frac{1}{2}\rho'[(\nu'_{n-1})^2 + \nu'_n(2-\nu'_n)] + \frac{1}{2}\rho''[\nu''_{n-1}(2-\nu''_{n-1}) + (\nu''_n)^2] \tag{3.3}$$

The mean strain energy density in the n-th layer is given by

$$\sigma_n = \nu'_n\sigma'_n + \nu''_n\sigma''_n \tag{3.4}$$

where

$$\sigma'_n = \frac{1}{2}\mathbf{E}' : \mathbb{C}' : \mathbf{E}' \qquad\qquad \sigma''_n = \frac{1}{2}\mathbf{E}'' : \mathbb{C}'' : \mathbf{E}''$$

and where

$$\mathbf{E}' = \frac{1}{2}\Big(\nabla\boldsymbol{w}'_n + (\nabla\boldsymbol{w}'_n)^\top\Big) \qquad\qquad \mathbf{E}'' = \frac{1}{2}\Big(\nabla\boldsymbol{w}''_n + (\nabla\boldsymbol{w}''_n)^\top\Big)$$

stand for pertinent linearized strain tensors.

4. Model equations

4.1. Discrete-continuum model

The governing equations for the basic unknowns $\boldsymbol{u}^n(\boldsymbol{x},t)$, $n = 0,1,\dots,N$ and $\boldsymbol{v}^n(\boldsymbol{x},t)$, $n = 0,1,\dots,N$, $\boldsymbol{x}\in\Omega$, $t\in R$, will be derived from the principle of stationary action for the action functional

$$\begin{aligned} \mathcal{A} &= \sum_{n=1}^{N}\int\limits_{t_0}^{t_1}\int\limits_{\Omega}\mathcal{L}_n(\boldsymbol{v}^n, \overline{\nabla}\boldsymbol{u}^n, \varDelta\boldsymbol{u}^n, \dot{\boldsymbol{u}}^n)\, d\boldsymbol{x}dt \\ \mathcal{L}_n &= \kappa_n - \sigma_n \end{aligned} \tag{4.1}$$

where κ_n and σ_n are determined by formulae (3.2), (3.4). Let us introduce the following denotations

$$\begin{aligned} &\langle\mathbb{C}^n\rangle = \nu'_n\mathbb{C}' + \nu''_n\mathbb{C}'' && [\mathbb{C}^n] = 2\sqrt{3}\nu_n(\mathbb{C}'-\mathbb{C}'')\cdot\boldsymbol{i}_3 \\ &[\mathbb{C}^n]^\top = 2\sqrt{3}\nu_n\boldsymbol{i}_3\cdot(\mathbb{C}'-\mathbb{C}'') && \{\boldsymbol{C}^n\} = 12\boldsymbol{i}_3\cdot(\nu''_n\mathbb{C}' + \nu'_n\mathbb{C}'')\cdot\boldsymbol{i}_3 \end{aligned} \tag{4.2}$$

where $\boldsymbol{i}_3 = (0,0,1)$ is a unit normal to the layering. It can be shown that the Euler-Lagrange equations related to (4.1) take the form

$$\begin{aligned} &\rho_n \ddot{\boldsymbol{u}}^n - \overline{D} \cdot (\langle \mathbb{C}^n \rangle : D\boldsymbol{u}^n + [\mathbb{C}^n] \cdot \boldsymbol{v}^n) = \boldsymbol{0} \\ &\{\boldsymbol{C}^n\} \cdot \boldsymbol{v}^n + [\mathbb{C}^n]^\top : D\boldsymbol{u}^n = \boldsymbol{0} \end{aligned} \tag{4.3}$$

Equations (4.3) represent *the discrete- continuum model* of the FGL under consideration. The above model equations have to be considered together with relevant boundary and initial conditions. After obtaining a solution to the specific boundary/initial value problem, the distribution of displacements in the laminae Λ'_n, Λ''_n, $n = 1,\ldots,N$, is described by formulae (3.1).

Let us observe that the unknowns $\boldsymbol{v}^n$, $n = 0,1,\ldots,N$, can be eliminated from the governing equations. We obtain

$$\boldsymbol{v}^n = -\{\boldsymbol{C}^n\}^{-1} \cdot [\mathbb{C}^n]^\top : D\boldsymbol{u}^n \tag{4.4}$$

Introducing the following tensors of effective elastic moduli

$$\mathbb{C}_0^n = \langle \mathbb{C}^n \rangle - [\mathbb{C}^n] \cdot \{\boldsymbol{C}^n\}^{-1} \cdot [\mathbb{C}^n]^\top$$

we obtain equations

$$\rho_n \ddot{\boldsymbol{u}}^n - \overline{D} \cdot (\mathbb{C}_0^n : D\boldsymbol{u}^n) = \boldsymbol{0} \tag{4.5}$$

Equations (4.5) and (4.4) represent an alternative form of the general model equations (4.3). It has to be emphasized that the solutions $\boldsymbol{u}^n$, $n = 0,1,\ldots,N$, have a physical sense only if the sequences $\{\nu'_n\}$, $\{\nu''_n\}$ are slowly-varying. Then, the mass density ρ_n reduces to the form

$$\rho_n = \rho'\nu'_n + \rho''\nu''_n$$

Equations (4.4), (4.5) constitute the foundations of subsequent analysis leading to a continuum model of the FGL under consideration.

4.2. Continuum model

We shall assume that for the finite sequence $\{f^n\}$, $n = 1,\ldots,N$, in equations (4.5), (4.4) there exists a continuous function $f(z)$, $z \in [0,L]$ such that f^n are approximated by $f(nl)$ for $n = 1,\ldots,N$. Moreover, for macroscopic deformation wavelengths large when compared to the lamina thicknesses we assume that the function $f(\cdot)$ is differentiable, and we shall approximate Δf^n by $\partial_3 f(nl)$. Under the above conditions, equations (4.4), (4.5) can be interpreted as a certain finite difference approximation of the equations

$$\rho\ddot{\boldsymbol{u}} - \nabla \cdot (\mathbb{C}^h : \nabla\boldsymbol{u}) = \boldsymbol{0} \tag{4.6}$$

and

$$\boldsymbol{v} = -\{\boldsymbol{C}\}^{-1} \cdot [\mathbb{C}]^{\top} : \nabla \boldsymbol{u} \tag{4.7}$$

where $\mathbb{C}^h$ is the tensor of effective elastic moduli

$$\mathbb{C}^h = \langle \mathbb{C} \rangle - [\mathbb{C}] \cdot \{\boldsymbol{C}\}^{-1} \cdot [\mathbb{C}]^{\top}$$

and

$$\rho = \rho' \nu' + \rho'' \nu''$$

Equations (4.6) and (4.7) represent *the continuum model* equations of the FGL under consideration. In the subsequent section, the proposed models will be compared with models obtained by using a similar discretization approach and presented in Rychlewska (2006).

5. Comparison of models

The modelling procedure proposed in Rychlewska (2006) is based on the concepts of the tolerance averaging technique formulated and applied in Woźniak and Wierzbicki (2000) for periodic composites. Moreover, this approach is a certain generalization of the modelling technique leading to a system of finite difference/differential equations. To make this paper self-consistent, we outline below the basic concepts and results presented in Rychlewska (2006).

Instead of (3.1), the displacements $\boldsymbol{w}'_n$, $\boldsymbol{w}''_n$ are assumed respectively in the form

$$\begin{aligned} \boldsymbol{w}'_n &= [(\boldsymbol{u}_n - l\sqrt{3}\nu_n \boldsymbol{v}_n) z'_n + (\boldsymbol{u}_n - l'_n \Delta \boldsymbol{u}_n + l\sqrt{3}\nu_n \boldsymbol{v}_n)(l'_n - z'_n)] \frac{1}{l'_n} \\ \boldsymbol{w}''_n &= [(\boldsymbol{u}_n - l\sqrt{3}\nu_n \boldsymbol{v}_n)(l''_n - z''_n) + \\ &\quad + (\boldsymbol{u}_n + l''_n \Delta \boldsymbol{u}_n + l\sqrt{3}\nu_n \boldsymbol{v}_n + l^2\sqrt{3}\nu_n \Delta \boldsymbol{v}_n - l\Delta(l'_n \Delta \boldsymbol{u}_n)) z''_n] \frac{1}{l''_n} \end{aligned} \tag{5.1}$$

where $\boldsymbol{u}_n = \boldsymbol{u}_n(\boldsymbol{x},t)$, $\boldsymbol{v}_n = \boldsymbol{v}_n(\boldsymbol{x},t)$, $\Delta \boldsymbol{u}_n = \Delta \boldsymbol{u}_n(\boldsymbol{x},t)$, $\Delta \boldsymbol{v}_n = \Delta \boldsymbol{v}_n(\boldsymbol{x},t)$, $z'_n \in [(n-1)l, l'_n + (n-1)l]$, $z''_n \in [l'_n + (n-1)l, nl]$, $n = 1, \dots, N$.

On the assumption that the sequences $\{\nu'_n\}$, $\{\nu''_n\}$ of component volume fractions in the FGL are slowly-varying, it was stated that $\{\boldsymbol{u}_n\}$, $\{\nu_n \boldsymbol{v}_n\}$, $\{l'_n \Delta \boldsymbol{u}_n\}$ are slowly-varying (in a certain tolerance ε). Hence, the displacements on interfaces between the adjacent layers are

$$\widetilde{\boldsymbol{w}}_n \cong \boldsymbol{u}_n + l\sqrt{3}\nu_n \boldsymbol{v}_n \qquad \overline{\boldsymbol{w}}_n \cong \boldsymbol{u}_n - l\sqrt{3}\nu_n \boldsymbol{v}_n \qquad \widetilde{\boldsymbol{w}}_{n+1} \cong \boldsymbol{u}_n + l\sqrt{3}\nu_n \boldsymbol{v}_n \tag{5.2}$$

and strains in the laminae Λ'_n, Λ''_n of the n-th layer are obtained in the form

$$\varepsilon'_n = \boldsymbol{\Delta u}_n - 2l\sqrt{3}\nu_n(\nu'_n)^{-1}\boldsymbol{v}_n \qquad \varepsilon'_n \cong \boldsymbol{\Delta u}_n + 2l\sqrt{3}\nu_n(\nu''_n)^{-1}\boldsymbol{v}_n \tag{5.3}$$

The strain energy density is taken in the form analogous to that given by (3.4), while the kinetic energy density is represented by

$$\kappa_n = \frac{1}{2}l^2\rho_n(\nu_n)^2(\dot{\boldsymbol{v}}_n)^2 + \frac{1}{2}\rho_n(\dot{\boldsymbol{u}}_n)^2 \tag{5.4}$$

where $\rho_n = \rho'\nu'_n + \rho''\nu''_n$. Under denotations (4.2), the discrete-continuum model is represented by equations

$$\rho_n\ddot{\boldsymbol{u}}^n - \overline{D}\cdot\boldsymbol{S}^n = \boldsymbol{0} \qquad l^2\rho_n\nu_n^2\ddot{\boldsymbol{v}}^n + \boldsymbol{h}^n = \boldsymbol{0} \qquad n = 2,\dots,N-1 \tag{5.5}$$

where

$$\boldsymbol{S}^n = \langle\mathbb{C}^n\rangle : D\boldsymbol{u}^n + [\mathbb{C}^n]\cdot\boldsymbol{v}^n \qquad \boldsymbol{h}^n = \{\boldsymbol{C}^n\}\cdot\boldsymbol{v}^n + [\mathbb{C}^n]^\top : D\boldsymbol{u}^n \qquad n = 1,\dots,N \tag{5.6}$$

In the framework of continuum models, one can be mention here the tolerance averaged model equations

$$\rho\ddot{\boldsymbol{u}} - \nabla\cdot\boldsymbol{S} = \boldsymbol{0} \qquad l^2\rho\nu^2\ddot{\boldsymbol{v}} + \boldsymbol{h} = \boldsymbol{0} \tag{5.7}$$

where

$$\rho = \rho'\nu' + \rho''\nu'' \qquad \boldsymbol{S} = \langle\mathbb{C}\rangle : \nabla\boldsymbol{u} + [\mathbb{C}]\cdot\boldsymbol{v} \qquad \boldsymbol{h} = \{\boldsymbol{C}\}\cdot\boldsymbol{v} + [\mathbb{C}]^\top : \nabla\boldsymbol{u} \tag{5.8}$$

and the asymptotic approximation model equations are

$$\rho\ddot{\boldsymbol{u}} - \nabla\cdot(\mathbb{C}^h : \nabla\boldsymbol{u}) = \boldsymbol{0} \qquad \boldsymbol{v} = -\{\boldsymbol{C}\}^{-1}\cdot[\mathbb{C}]^\top : \nabla\boldsymbol{u} \tag{5.9}$$

where $\mathbb{C}^h$ is the tensor of effective elastic moduli.

It can be easily observed that continuum model equations (4.6), (4.7) obtained in this paper have the same form like equations (5.9). It has to be emphasized that, contrary to discrete-continuum model equations (4.3), model equations (5.5) and (5.6) describe the microstructure length-scale effect on the overall behaviour of the FGL. It follows that also continuum tolerance averaged models take into account the effect of the layer thickness l on the dynamic behaviour of the FGL. The proposed continuum model neglects this effect. Equations (4.6), (4.7) represent the continuum model corresponding to that of the linear elasticity theory and described by equations obtained by the known homogenization approach. However, the form of equations (4.6), (4.7) is relatively simple and it can be applied to the analysis of special problems in which the length-scale effect can be neglected. An example of such a case will be shown in the subsequent section.

6. Example of applications

As an example of applications we shall investigate the problem of harmonic vibration along the x_3-axis of a laminated solid consisting of two isotropic homogeneous layers interconnected by a functionally graded layer, see Fig. 3. Let us denote by E', E'' the elastic moduli of component materials in the uniaxial extension and/or compression. By ρ', ρ'' mass densities of component materials will be denoted. The problem will be treated as independent of $\boldsymbol{x}$, and hence (4.6) implies that

$$\langle\rho\rangle\ddot{u} - (E^{eff}u_z)_z = 0 \tag{6.1}$$

where $u = u_3(z,t)$, $z \in [0,L]$, $t \in R$ and

$$\langle\rho\rangle = [1-\widetilde{\nu}(z)]\rho' + \widetilde{\nu}(z)\rho'' \qquad E^{eff} = \frac{E'E''}{\widetilde{\nu}(z)E' + [1-\widetilde{\nu}(z)]E''} \tag{6.2}$$

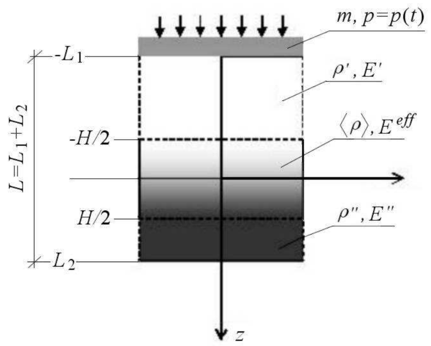

Fig. 3. The laminated solid consisting of two homogeneous layers and a graded interlayer with inertial loading of the mass m

The function $\widetilde{\nu}(\cdot)$ is defined on $[0,L]$ and determines the gradation of material properties for the component material with the superscript "bis". It is assumed that the graded layer has the thickness H and $L = L_1 + L_2$, where L_1, L_2 are thicknesses from the midplane of the transition zone to the boundary planes, see Fig. 3. The distribution of the volume fraction is shown in Fig. 4. It is postulated in the following form

$$\widetilde{\nu}(z) = \begin{cases} 0 & \text{if} \quad z \in \Big[-L_1, -\dfrac{H}{2}\Big] \\ \dfrac{1}{2} + \dfrac{z}{H} & \text{if} \quad z \in \Big[-\dfrac{H}{2}, \dfrac{H}{2}\Big] \\ 1 & \text{if} \quad z \in \Big[\dfrac{H}{2}, L_2\Big] \end{cases} \tag{6.3}$$

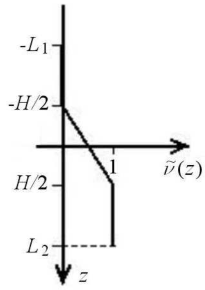

Fig. 4. The distribution of the volume fraction $\widetilde{\nu}(z)$, $z \in [-L_1, L_2]$

Let us denote

$$c^2 = \frac{E'}{\rho'} = \frac{E''}{\rho''} \qquad\qquad k = \frac{E'}{E''} - 1 = \frac{\rho'}{\rho''} - 1 \geqslant 0$$

The parameter k will be called the coefficient of inhomogeneity $(k \geqslant 0)$. Using this parameter, we obtain

$$\langle \rho \rangle = \rho' \Big(1 - \frac{k}{k+1} \widetilde{\nu}(z)\Big) \qquad\qquad E^{eff} = \frac{E'}{1 + k\widetilde{\nu}(z)} \tag{6.4}$$

For the sake of simplicity, let us restrict the considerations to the laminated solid with the inertial loading as shown in Fig. 3. Let us also assume that $L_1 = L_2 = L/2$. In this case, the governing equations have the form

$$\begin{aligned} &(E'u_z)_z = 0 && \text{if} \quad z \in \Big[-\frac{L}{2}, -\frac{H}{2}\Big] \\ &\Big(\frac{1}{1 + k\widetilde{\nu}(z)} u_z\Big)_z = 0 && \text{if} \quad z \in \Big[-\frac{H}{2}, \frac{H}{2}\Big] \\ &(E''u_z)_z = 0 && \text{if} \quad z \in \Big[\frac{H}{2}, \frac{L}{2}\Big] \end{aligned} \tag{6.5}$$

with boundary conditions

$$u\Big(\frac{L}{2}, t\Big) = 0 \qquad\qquad m\ddot{u}\Big(-\frac{L}{2}, t\Big) = p(t) - E'u_z\Big(-\frac{L}{2}, t\Big) \tag{6.6}$$

and jump (continuity) conditions

$$\begin{aligned} &u\Big(\frac{H}{2} + 0, t\Big) = u\Big(\frac{H}{2} - 0, t\Big) && u\Big(-\frac{H}{2} + 0, t\Big) = u\Big(-\frac{H}{2} - 0, t\Big) \\ &u_z\Big(\frac{H}{2} + 0, t\Big) = u_z\Big(\frac{H}{2} - 0, t\Big) && u_z\Big(-\frac{H}{2} + 0, t\Big) = u_z\Big(-\frac{H}{2} - 0, t\Big) \end{aligned} \tag{6.7}$$

We shall investigate the eigenvalue problem setting

$$p(t) = p_0 \cos \breve{\omega} t \qquad u(z,t) = w(z) \cos \breve{\omega} t$$

In the subsequent analysis, it is assumed that $p_0 = 0$. Then equations (6.5)-(6.7) are transformed to the form

$$\begin{aligned} & w_{zz} = 0 && \text{if} \quad z \in \Big[-\frac{L}{2}, -\frac{H}{2}\Big] \\ & \Big(\frac{1}{1 + k\widetilde{\nu}(z)} w_z\Big)_z = 0 && \text{if} \quad z \in \Big[-\frac{H}{2}, \frac{H}{2}\Big] \\ & w_{zz} = 0 && \text{if} \quad z \in \Big[\frac{H}{2}, \frac{L}{2}\Big] \end{aligned} \tag{6.8}$$

with boundary conditions

$$w\Big(\frac{L}{2}\Big) = 0 \qquad -\breve{\omega}^2 m w\Big(-\frac{L}{2}\Big) + E' w_z\Big(-\frac{L}{2}\Big) = 0 \tag{6.9}$$

and jump conditions

$$\begin{aligned} & w\Big(\frac{H}{2} + 0\Big) = w\Big(\frac{H}{2} - 0\Big) && w\Big(-\frac{H}{2} + 0\Big) = w\Big(-\frac{H}{2} - 0\Big) \\ & w_z\Big(\frac{H}{2} + 0\Big) = w_z\Big(\frac{H}{2} - 0\Big) && w_z\Big(-\frac{H}{2} + 0\Big) = w_z\Big(-\frac{H}{2} - 0\Big) \end{aligned} \tag{6.10}$$

Let us transform equations (6.8)-(6.10) to a dimensionless form by introducing the argument

$$\zeta = \frac{z}{L}$$

where $\zeta \in [-1/2, 1/2]$. Let us also denote

$$\delta = \frac{H}{L} \qquad \varOmega^2 = \frac{\breve{\omega}^2 L m}{E'}$$

Hence, we obtain equations (6.8)-(6.10) in the dimensionless form

$$\begin{aligned} & w_{\zeta\zeta} = 0 && \text{if} \quad \zeta \in \Big[-\frac{1}{2}, -\frac{\delta}{2}\Big] \\ & \Big(\frac{1}{1 + k\widetilde{\nu}(\zeta)} w_\zeta\Big)_\zeta = 0 && \text{if} \quad \zeta \in \Big[-\frac{\delta}{2}, \frac{\delta}{2}\Big] \\ & w_{\zeta\zeta} = 0 && \text{if} \quad \zeta \in \Big[\frac{\delta}{2}, \frac{1}{2}\Big] \end{aligned} \tag{6.11}$$

with boundary conditions

$$w\Big(\frac{1}{2}\Big) = 0 \qquad -\varOmega^2 w\Big(-\frac{1}{2}\Big) + w_\zeta\Big(-\frac{1}{2}\Big) = 0 \tag{6.12}$$

and jump conditions

$$\begin{aligned} w\Big(\frac{\delta}{2}+0\Big) &= w\Big(\frac{\delta}{2}-0\Big) & \qquad w\Big(-\frac{\delta}{2}+0\Big) &= w\Big(-\frac{\delta}{2}-0\Big) \\ w_\zeta\Big(\frac{\delta}{2}+0\Big) &= w_\zeta\Big(\frac{\delta}{2}-0) & \qquad w_\zeta\Big(-\frac{\delta}{2}+0\Big) &= w_\zeta\Big(-\frac{\delta}{2}-0\Big) \end{aligned} \tag{6.13}$$

We shall solve the optimization problem of finding the position of the graded layer. To this end, for the known k, $k > 0$, we shall look for $\lambda = \min \Omega^2$, $\lambda = \lambda_k(\delta)$, and finally we shall find $\delta_0 = \max \lambda_k(\delta)$, $\delta \in [0,1]$. Hence

$$\Omega^2 = \frac{1}{\frac{1}{2}(1-\delta)(k+2)+(k+1)\delta}$$

The analysis of the above optimization problem was carried out for $k = 1$, 10, 20. The results are shown in Table 1. The optimization result was obtained for $\delta = 1$ $(H = L)$ and $k = 1$ $(E' = 2E'')$.

Table 1. Results of the analysis of the optimization problem

k	δ									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
1	$\frac{20}{31}$	$\frac{5}{8}$	$\frac{20}{33}$	$\frac{10}{17}$	$\frac{4}{7}$	$\frac{5}{9}$	$\frac{20}{37}$	$\frac{10}{19}$	$\frac{20}{39}$	$\frac{1}{2}$
10	$\frac{2}{13}$	$\frac{1}{7}$	$\frac{2}{15}$	$\frac{1}{8}$	$\frac{2}{17}$	$\frac{1}{9}$	$\frac{2}{19}$	$\frac{1}{10}$	$\frac{2}{21}$	$\frac{1}{11}$
20	$\frac{1}{12}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$	$\frac{1}{16}$	$\frac{1}{17}$	$\frac{1}{18}$	$\frac{1}{19}$	$\frac{1}{20}$	$\frac{1}{21}$

7. Conclusions

The main results of this contribution are:

- An averaged mathematical model for analysis of dynamic behaviour of FGL is formulated. The obtained model equations are represented by a system of finite-difference/differential equations.
- It is shown that it is possible to eliminate the unknowns $\boldsymbol{v}^n$, $n = 1, \dots, N$, from the governing equations. Then, we arrive at model equations depending on certain effective smoothly varying coefficients.
- The possible applications of the proposed model are illustrated by analysis of an optimization problem for a FGL subjected to inertial loadings.

- It can be observed that for periodic laminated structures coefficients in continuum model equations (4.6), (4.7) are constant. In this case, the obtained results coincide with those derived by the asymptotic approximation, Woźniak and Wierzbicki (2000).

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Modelowanie i optymalizacja laminatów o strukturze gradientowej

Streszczenie

Przedmiot rozważań stanowi szczególna klasa materiałów gradientowych, tzw. laminatów o strukturze gradientowej, które na poziomie mikrostrukturalnym złożone są z dużej liczby bardzo cienkich warstewek. Celem pracy jest zaproponowanie modelu dyskretno-ciągłego i ciągłego zagadnień elastodynamiki takich laminatów. Sformułowany model ciągły został zastosowany do analizy drgań ośrodka obciążonego inercyjnie, złożonego z dwóch jednorodnych warstw, pomiędzy którymi znajduje się strefa przejściowa. Przedyskutowano zagadnienie optymalizacji położenia strefy przejściowej.

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