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STABILIZATION OF PLATE PARAMETRIC VIBRATION VIA DISTRIBUTED CONTROL

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A theoretical investigation of vibration control for linear laminated plate due to uniform, harmonically or arbitrarily varying in-plane forces is presented. A distributed controller in an active system consisting of electroded piezoelectric sensors/actuators with suitable polarization profiles is considered. To satisfy the Maxwell electrostatics equation in the actuator, a constant electrical potential distribution in the in-plane directions and linear distribution in transverse direction cannot be assumed but is rather obtained by solving the coupled governing equations by assuming a certain theoretically advisable distribution in the thickness direction. Coupled dynamics equations with respect to a plate displacement and an electric field are derived using the Hamilton principle. The rate velocity feedback is applied to stabilize the plate parametric vibration. The almost sure stability of the trivial solution is analysed using the appropriate Liapunov functional.

Key words: piezoelectric layers, coupled dynamics equations, parametric excitation, stability analysis

1. Introduction

Distributed piezoelectric layers can be used as distributed sensors and actuators for structural monitoring and control of elastic structures. In most of the published literature on active systems consisting of continuous mechanical systems with piezoelectric sensors and actuators, the actuator equations were simplified by assuming that the generated electric fields depend only on the applied external voltages (Lee, 1990; Tylikowski, 2001). In addition, the distribution of the electric potential φ is assumed to be uniform in the in-plane directions of the piezoelectric actuator and linear in its thickness direction.

The approach was criticized since the Maxwell electrostatics equation is not satisfied for the actuator layers (Gopinathan *et al.*, 2000; Krommer and Irschik, 1999, 2000; Tylikowski, 2005; Wang and Quek, 2000). In this paper, the coupled partial differential equations describing the transverse plate motion w and the electric potential are derived. The two piezoelectric layers are placed symmetrically with the poling direction oriented in the transverse direction of the plate. When voltages of equal magnitude but opposite phase are applied to the upper and the lower piezoelectric layers of the plate, the induced strains are resulting in flexure action. The sensor layer is made of a thin piezoelectric foil PVDF with a negligible stiffness as compared with the plate and the piezoelectric actuators stiffnesses. The derived plate dynamics equation which contains the component dependent on the second derivative of electric field is derived. Similarly, the Maxwell equation contains an additional component dependent on the plate curvatures. The plate is supposed to be bi-axially loaded by time-dependent forces, which can excite parametric vibration and destabilize the equilibrium state. The present paper is devoted to formulating a control law without the necessity of modeling the plate in terms of its vibration modes. The voltage applied to the actuators is calculated from the rate feedback. The derived coupled equations are applicable to the stability analysis of plate parametric vibration. The control law is derived using the Liapunov approach, with a functional as a sum of the modified mechanical plate energy and the energy of electric field. The results indicate that the transverse plate vibrations can be effectively stabilized using the distributed piezoelectric elements.

2. Derivation of dynamic equations

Consider the symmetrically laminated rectangular plate of length a, width b and the total thickness $2h$ with two piezoelectric actuators of thickness h_1, embedded symmetrically at the distance e from the plate middle plain, with poling direction oriented in the transverse direction of the plate. When voltages of equal magnitude but opposite phase are applied to the upper and the lower piezoelectric layers of the plate, the induced strains are resulting in flexure action. The sensor layer is made of a thin piezoelectric foil PVDF with a negligible stiffness as compared with the plate and the piezoelectric actuators stiffnesses and with the appropriate polarization function. The analysis will use the Kirchhoff kinematic assumptions to describe the plate strains. The plate is assumed to be simply supported on all edges.

Strains and stresses in a laminated plate according to the Kirchhoff theory are

$$\varepsilon_x = -zw_{,xx} \qquad \varepsilon_y = -zw_{,y} \qquad \gamma_{xy} = -2zw_{,xy} \tag{2.1}$$

$$\begin{aligned}
\sigma_x^2 &= \frac{Y_p}{1-\nu_P^2}(\varepsilon_x + \nu_p\varepsilon_y) + e_{31}E_z & \sigma_y^2 &= \frac{Y_p}{1-\nu_P^2}(\varepsilon_y + \nu_p\varepsilon_x) + e_{32}E_z \\
\sigma_{xy}^2 &= \frac{2Y_p}{1+\nu_p}\gamma_{xy} & \sigma_x^{1(k)} &= Q_{11}^{(k)}\varepsilon_x + Q_{12}^{(k)}\varepsilon_y \\
\sigma_y^{1(k)} &= Q_{12}^{(k)}\varepsilon_x + Q_{22}^{(k)}\varepsilon_y & \sigma_{xy}^{1(k)} &= Q_{66}^{(k)}\gamma_{xy}
\end{aligned} \tag{2.2}$$

where the superscripts 1 and 2 represent the laminated plate and the piezoelectric material, respectively, w is the plate transverse displacement, z denotes the distance from the plate middle plain, Y_p and ν_p denote the Young modulus and the Poisson ratio of the piezoelectric material, respectively. $\{Q_{ij}^{(k)}\}$ denotes the stiffness matrix of the kth lamina, e_{31} and e_{32} denote the piezoelectric constants and E_z denotes the component of electric field in z-direction. The electric potential $\overline{\varphi}$ is assumed in the form proposed by Wang and Quek (2000) as a combination of an approximate half-cosine and linear variation

$$\overline{\varphi} = \overline{\varphi}(x,y,z,t) = -\cos\Big(\frac{\pi z_l}{h_1}\Big)\varphi(x,y,t) + \frac{2z_l}{h_1}\varphi_a \tag{2.3}$$

where z_l is a local co-ordinate measured from the center of the piezoelectric actuator in the global z-direction, φ_a is the value of external electric voltage applied to the actuator electrodes and $\varphi(x,y,t)$ is the time and spatial variation of the electric potential in x-direction. The components of electric field $\boldsymbol{E}$ are given as follows

$$\begin{aligned}
E_x &= -\cos\Big(\frac{\pi z_l}{h_1}\Big)\varphi_{,x} & E_y &= -\cos\Big(\frac{\pi z_l}{h_1}\Big)\varphi_{,y} \\
E_z &= -\sin\Big(\frac{\pi z_l}{h_1}\Big)\varphi\frac{\pi}{h_1} + \frac{2}{h_1}\varphi_a
\end{aligned} \tag{2.4}$$

Components of the electric displacement in a piezoelectric layer are given by

$$D_x = \varXi_{11}E_x \qquad D_y = \varXi_{22}E_y \qquad D_z = \varXi_{33}E_z + e_{31}\varepsilon_x + e_{32}\varepsilon_y \tag{2.5}$$

The system Lagrangian is written as the volume integral of kinetic energy and electric enthalpy

$$L = \int\limits_{\varOmega} \Big(\frac{1}{2}\rho w_{,T}^2 - U + D_xE_x + D_yE_y + D_zE_z\Big)\,d\varOmega \tag{2.6}$$

Substituting the components of electric field and strains we have

$$
\begin{aligned}
L = \frac{1}{2}\int\limits_{\Omega} \Big\{ & \rho(z)w_{,t}^2 - Q_{11}^{(k)}z^2w_{,xx}^2 - 2Q_{12}^{(k)}z^2w_{,xx}w_{,yy} - Q_{22}^{(k)}z^2w_{,yy}^2 + \\
& -4Q_{66}^{(k)}z^2w_{,xy}^2 + \frac{S_{0x}+S_x(t)}{2h}w_{,x}^2 + \frac{S_{0y}+S_y(t)}{2h}w_{,y}^2 + \\
& +\varXi_{11}\cos^2\Big(\frac{\pi z_l}{h_1}\Big)\varphi_{,x}^2 + \varXi_{22}\cos^2\Big(\frac{\pi z_l}{h_1}\Big)\varphi_{,y}^2 + \varXi_{33}\Big[-\sin\Big(\frac{\pi z_l}{h_1}\Big)\varphi\frac{\pi}{h_1} + \frac{2}{h_1}\varphi_a\Big]^2 + \\
& +2\Big[\sin\Big(\frac{\pi z_l}{h_1}\Big)\varphi\frac{\pi}{h_1} - \frac{2}{h_1}\varphi_a\Big](e_{31}zw_{,xx} + e_{32}zw_{,yy})\Big\}\, d\Omega
\end{aligned}
\tag{2.7}
$$

Integrating with respect to z over the thickness $2h$ we obtain the Lagrangian of the system as a function of the transverse displacement $w(x,y)$ and electric potential $\varphi(x,y)$ of the plate

$$
\begin{aligned}
L = \frac{1}{2}\int\limits_0^a\int\limits_0^b \Big\{ & 2\overline{\rho}hw_{,t}^2 - D_{11}w_{,xx}^2 - 2D_{12}w_{,xx}w_{,yy} - D_{22}w_{,yy}^2 - 4D_{66}w_{,xy}^2 + \\
& +\varXi_{11}h_1\varphi_{,x}^2 + \varXi_{22}h_1\varphi_{,y}^2 + [S_{0x}+S_x(t)]w_{,x}^2 + [S_{0y}+S_y(t)]w_{,y}^2 + \\
& +\varXi_{33}\Big(\frac{\pi^2}{h_1}\varphi^2 + \frac{4\varphi_a^2}{h_1}\Big) + 4(e_{31}w_{,xx} + e_{32}w_{,yy})\Big[\frac{2h_1}{\pi}\varphi - \Big(e + \frac{h_1}{2}\Big)\varphi_a\Big]\Big\}\, dxdy
\end{aligned}
\tag{2.8}
$$

Remembering that the variation of external potential φ_a is equal to zero we apply the Hamilton principle and finally we obtain

$$
\begin{aligned}
& \int\limits_{t_0}^{t}\int\limits_0^a\int\limits_0^b \Big\{\Big[\Big(-2\overline{\rho}hw_{,tt} - D_{11}w_{,xxxx} - 2(D_{12}+2D_{66})w_{,xxyy} - D_{22}w_{,yyyy} + \\
& -[S_{0x}+S_x(t)]w_{,xx} - [S_{0y}+S_y(t)]w_{,yy} + \frac{4h_1}{\pi}(e_{31}\varphi_{,xx} + e_{32}\varphi_{,yy}) + \\
& -2\Big(e+\frac{h_1}{2}\Big)(e_{31}\varphi_{a,xx} + e_{32}\varphi_{a,yy})\Big)\delta w + \\
& +\Big(\varXi_{33}h_1\Big(\frac{\pi}{h_1}\Big)^2\varphi - \varXi_{11}h_1\varphi_{,xx} - \varXi_{22}h_1\varphi_{,yy} + \frac{4h_1}{\pi}(e_{31}w_{,xx} + e_{32}w_{,yy})\Big)\delta\varphi\Big]\, dxdy + \\
& +\int\limits_0^b\Big(-D_{11}w_{,xx} - D_{12}w_{,yy} + \frac{4e_{31}h_1}{\pi}\varphi - 2\Big(e+\frac{h_1}{2}\Big)e_{31}\varphi_a\Big)\Big|_0^a\delta w_{,x}\, dy +
\end{aligned}
\tag{2.9}
$$

$$+\int\limits_0^a \Big(-D_{12}w_{,xx} - D_{22}w_{,yy} + \frac{4e_{32}h_1}{\pi}\varphi - 2\Big(e + \frac{h_1}{2}\Big)e_{32}\varphi_a\Big)\Big|_0^b \delta w_{,y}\;dx +$$
$$-\int\limits_0^b \varXi_{11}h_1\varphi_{,x}\Big|_0^a \delta\varphi\;dy - \int\limits_0^a \varXi_{22}h_1\varphi_{,y}\Big|_0^b \delta\varphi\;dx\Big\}\;dt = 0$$

The Euler equations which are obtained from the condition that δw and $\delta\varphi$ are independent of the transverse plate displacement w and the in-plane electric potential vp are as follows

$$\begin{aligned}
&2\overline{\rho}hw_{,tt} + D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + \\
&+[S_{0x} + S(t)]w_{,xx} + [S_{0y} + S_y(t)]w_{,yy} + \frac{4h_1}{\pi}(e_{31}\varphi_{,xx} + e_{32}\varphi_{,yy}) + \\
&-2\Big(e + \frac{h_1}{2}\Big)(e_{31}\varphi_{a,xx} + e_{32}\varphi_{a,yy}) = 0 \\
&\varXi_{33}\frac{\pi^2}{h_1}\varphi - \varXi_{11}h_1\varphi_{,xx} - \varXi_{22}h_1\varphi_{,yy} + \frac{4h_1}{\pi}(e_{31}w_{,xx} + e_{32}w_{,yy}) = 0 \\
&(x, y) \in (0, a) \times (0, b)
\end{aligned} \tag{2.10}$$

From Eq. (2.9) we also obtain the natural boundary conditions for $x = 0$ and $x = a$

$$\begin{aligned}
&w = 0 \qquad D_{11}w_{,xx} + D_{12}w_{,yy} - \frac{4h_1}{\pi}e_{31}\varphi + 2\Big(e + \frac{h_1}{2}\Big)e_{31}\varphi_a = 0 \\
&\varphi = 0 \qquad or \quad \varphi_{,x} = 0
\end{aligned} \tag{2.11}$$

and for $y = 0$ and $y = b$

$$\begin{aligned}
&w = 0 \qquad D_{12}w_{,xx} + D_{22}w_{,yy} - \frac{4h_1}{\pi}e_{32}\varphi + 2\Big(e + \frac{h_1}{2}\Big)e_{32}\varphi_a = 0 \\
&\varphi = 0 \qquad or \quad \varphi_{,y} = 0
\end{aligned} \tag{2.12}$$

Introducing the passive damping viscous term with coefficient $\alpha = 4\overline{\rho}h\beta$, and the active damping feedback term (Tylikowski, 2005)

$$g_a(x, y) = G(e_{31}\chi_{a,xx} + e_{32}\chi_{a,yy})\int\limits_0^a\int\limits_0^b \chi_s(x, y)(e^s_{31}w_{,xx} + e^s_{32}w_{,yy})_{,t}\;dxdy \tag{2.13}$$

where χ_a is the actuator polarization function and χ_s is the sensor sensitivity function (Gardonio and Elliott, 2004) with piezoelectric constants e^s_{31}, e^s_{32}, we obtain the following basic system of partial differential equations with respect to w, φ

$$
\begin{aligned}
&2\overline{\rho}hw_{,tt} + 4\overline{\rho}h\beta w_{,t} + D_{11}w_{,xxxx} + 2(D_{12} + 2D_{66})w_{,xxyy} + D_{22}w_{,yyyy} + \\
&+[S_{0x} + S_x(t)]w_{,xx} + [S_{0y} + S_y(t)]w_{,yy} + \frac{4h_1}{\pi}(e_{31}\varphi_{,xx} + e_{32}\varphi_{,yy}) + \\
&+g_a(x,y) = 0 \\
&\Xi_{33}\frac{\pi^2}{h_1}\varphi - \Xi_{11}h_1\varphi_{,xx} - \Xi_{22}h_1\varphi_{,yy} + \frac{4h_1}{\pi}(e_{31}w_{,xx} + e_{32}w_{,yy}) = 0 \\
&(x,y) \in (0,a) \times (0,b)
\end{aligned}
\tag{2.14}
$$

The sensor voltage is calculated from an elementary formula relating the charge, the generated voltage and the sensor capacity. Distributed piezoelectric elements are implemented to suppress the motion caused by parametric disturbances. A proportional controller in an active system consisting of electroded piezoelectric sensors/actuators with a suitable polarization profile is considered. The active stabilizing effect with velocity feedback is described by term $g_a(x,y)$ with the gain G.

3. Energy extraction

Vibration damping of the plate with parametric excitation can be examined by means of the total energy considerations. The method can be applied without earlier modal or finite-dimensional approximations. The energy consists of the kinetic energy, the bending energy, the elastic energy of compression due to constant components of the in-plane forces S_{0x} and S_{0y}, and the energy of electric field

$$
\begin{aligned}
E = \frac{1}{2}\int_0^a\int_0^b \Big(&2\overline{\rho}hw_{,t}^2 + D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2 + \\
&-S_{0x}w_{,x}^2 - S_{0y}w_{,y}^2 + \Xi_{11}h_1\varphi_{,x}^2 + \Xi_{22}h_1\varphi_{,y}^2 + \Xi_{33}\frac{\pi^2}{h_1}\varphi^2\Big)\,dx
\end{aligned}
\tag{3.1}
$$

The energy is positive definite if the clasic buckling condition is fulfilled by the constant components of in-plane forces. By differentiating Eq. (3.1) with respect to time, the rate of energy extraction is given by

$$\begin{aligned}\frac{dE}{dt} &= \int\limits_0^a \int\limits_0^b \Big(2\overline{\rho}hw_{,tt}w_{,t} + D_{11}w_{,xxt}w_{,xx} + D_{12}w_{,xxt}w_{,yy} + D_{12}w_{,xx}w_{,yyt} + \\ &+D_{22}w_{,yy}w_{,yyt} + 4D_{66}w_{,xy}w_{,xyt} - S_{0x}w_{,x}w_{,xt} - S_{0y}w_{,y}w_{,yt} + \\ &+\varXi_{11}h_1\varphi_{,x}\varphi_{,xt} + \varXi_{22}h_1\varphi_{,y}\varphi_{,yt} + \varXi_{33}\frac{\pi^2}{h_1}\varphi\varphi_{,t}\Big)\; dx\end{aligned} \tag{3.2}$$

Integrating by parts we have

$$\begin{aligned}I_1 &= \int\limits_0^a \int\limits_0^b \Big(\varXi_{11}h_1\varphi_{,x}\varphi_{,xt} + \varXi_{22}h_1\varphi_{,y}\varphi_{,yt} + \varXi_{33}\frac{\pi^2}{h_1}\varphi\varphi_{,t}\Big)\; dxdy = \\ &= \int\limits_0^a \int\limits_0^b \Big(-\varXi_{11}h_1\varphi_{,xxt} - \varXi_{22}h_1\varphi_{,yyt} + \varXi_{33}\frac{\pi^2}{h_1}\varphi_{,t}\Big)\varphi\; dxdy\end{aligned} \tag{3.3}$$

Substituting the electrostatic equation $(2.14)_2$ we have

$$I_1 = -\frac{4h_1}{\pi}\int\limits_0^a \int\limits_0^b \Big(e_{31}w_{,xxt} + e_{32}w_{,yyt}\Big)\varphi\; dxdy$$

$$\begin{aligned}&\int\limits_0^a \int\limits_0^b \Big[D_{11}w_{,xxxx} + D_{12}w_{,xxyy} - \frac{4h_1}{\pi}\varphi_{,xx} + 2\Big(b+\frac{h_1}{2}\Big)e_{31}\varphi_{a,xx}\Big)w_{,t}\; dxdy = \\ &= \int\limits_0^a \int\limits_0^b \Big(D_{11}w_{,xx} + D_{12}w_{,yy} - \frac{4h_1}{\pi}\varphi + 2\Big(b+\frac{h_1}{2}\Big)e_{31}\varphi_a\Big]w_{,xxt}\; dxdy \\ &\int\limits_0^a \int\limits_0^b \Big[D_{12}w_{,xxyy} + D_{22}w_{,yyyy} - \frac{4e_{31}h_1}{\pi}\varphi_{,yy} + 2\Big(b+\frac{h_1}{2}\Big)e_{31}\varphi_{a,yy}\Big]w_{,t}\; dxdy = \\ &= \int\limits_0^a \int\limits_0^b \Big[D_{12}w_{,xx} + D_{22}w_{,yy} - \frac{4e_{32}h_1}{\pi}\varphi + 2\Big(b+\frac{h_1}{2}\Big)e_{31}\varphi_a\Big]w_{,yyt}\; dxdy\end{aligned} \tag{3.4}$$

Eliminating the acceleration in the first part of integrand of Eq. (3.2) by means of dynamic equation $(2.14)_1$ and using Eqs. (3.4) gives

$$\frac{dE}{dt} = -\int_0^a\int_0^b 4\overline{\rho}h\beta w_{,t}^2\, dxdy + \int_0^a\int_0^b [S_x(t)w_{,xx} + S_y(t)w_{,yy}]w_{,t}\, dxdy + \\ -2\Big(b+\frac{h_1}{2}\Big)\int_0^a\int_0^b \varphi_a(e_{31}w_{,xxt} + e_{32}w_{,yyt})\, dxdy \tag{3.5}$$

The first negative component in Eq. (3.5) represents the rate at which the energy is extracted from the plate by the passive viscous damping. The second component with an undefined sign is the power flow due to the parametric excitation. The third term in Eq. (3.5) represents the active damping. Rewriting the active damping feedback term in the form (2.13) and assuming the same polarization functions of the sensor and actuator $\chi_a = \chi_s$, it can be shown that the third term is also negative

$$\frac{dE}{dt} = -\int_0^a\int_0^b 4\overline{\rho}h\beta w_{,t}^2\, dxdy + \int_0^a\int_0^b [S_x(t)w_{,xx} + S_y(t)w_{,yy}]w_{,t}\, dxdy + \\ -G\Big[\int_0^a\int_0^b \chi_a(x,y)(e_{31}w_{,xxt} + e_{32}w_{,yyt})\, dxdy\Big]^2 \tag{3.6}$$

Therefore, for a sufficiently large gain factor G it is possible to stabilize the parametric vibration excited by the time-dependent in-plane forces.

4. Stability analysis

The derived Eq. (3.6) does not provide an effective quantitative estimation of the minimal active damping coefficient or the gain factor stabilizing the parametric vibration. In order to derive an analytical relation involving characteristics of the parametric excitation, and parameters of the passive damping and the active damping, it is necessary to define precisely the class of the parametric excitation. The derived equations are applicable to stability analysis of the parametric plate vibration due to the action of time-dependent in-plane forces. As the equations (2.14) are linear, it is sufficient to examine the asymptotic stability of the trivial solution. We look for conditions imposed on the plate geometry, the viscous damping, the gain factor and the in-plane force characteristics which imply tending to zero of the distance of the disturbed solutions from the trivial one. If the in-plane forces are stochastic ergodic processes with physically realizable trajectories, we examine the almost sure stochastic stability (cf. Kozin, 1972)

$$P\Big\{\lim_{t\to\infty} \|w(t), \varphi(t)\| = 0\Big\} = 1 \tag{4.1}$$

where $\|\cdot,\cdot\|$ denotes the distance between solutions. The energy-like Liapunov functional containing among others the kinetic energy, the elastic energy and the energy of electric field, is introduced to examine the asymptotic stability

$$\begin{aligned} V = \frac{1}{2}\int\limits_0^a\int\limits_0^b \Big(& 2\overline{\rho}hw_{,t}^2 + 4\overline{\rho}h\beta ww_{,t} + 4\overline{\rho}h\beta^2 w^2 + D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + \\ & + D_{22}w_{,yy}^2 + 4D_{66}w_{,xy}^2 - S_{0x}w_{,x}^2 - S_{0y}w_{,y}^2 + \varXi_{11}h_1\varphi_{,x}^2 + \\ & + \varXi_{22}h_1\varphi_{,y}^2 + \varXi_{33}\frac{\pi^2}{h_1}\varphi^2\Big)\, dx \end{aligned} \tag{4.2}$$

For sufficiently small in-plane forces S_{0x} and S_{0y} functional (3.1) is positive definite, and its square root can be chosen as the distance between the disturbed solution and the trivial one $\|\cdot\| = \sqrt{V}$. If time-dependent components of the in-plane forces are continuous and physically realizable, the functional can be differentiated in a classical way

$$\begin{aligned} \frac{dV}{dt} = \int\limits_0^a\int\limits_0^b \Big[& 2\overline{\rho}hw_{,tt}(w_{,t} + \beta w) + 4\overline{\rho}h\beta^2 ww_{,t} + 2\overline{\rho}h\beta w_{,t}^2 + D_{11}w_{,xxt}w_{,xx} + \\ & + D_{12}w_{,xxt}w_{,yy} + D_{12}w_{,xx}w_{,yyt} + D_{22}w_{,yy}w_{,yyt} + 4D_{66}w_{,xy}w_{,xyt} + \\ & - S_{0x}w_{,x}w_{,xt} - S_{0y}w_{,y}w_{,yt} + \varXi_{11}h_1\varphi_{,x}\varphi_{,xt} + \varXi_{22}h_1\varphi_{,y}\varphi_{,yt} + \varXi_{33}\frac{\pi^2}{h_1}\varphi\varphi_{,t}\Big]\, dx \end{aligned} \tag{4.3}$$

Eliminating the acceleration of transverse motion by means of Eq. $(2.14)_1$, using Eqs. (3.3) and (3.4) and integrating by parts using the boundary conditions (2.11) and (2.12), we rewrite the time derivative of functional in the form

$$\frac{dV}{dt} = -2\beta V + 2U \tag{4.4}$$

where U denotes the auxiliary functional as follows

$$\begin{aligned} U = \frac{1}{2}\int\limits_0^a\int\limits_0^b \Big\{& 4\overline{\rho}h\beta^2 ww_{,t} + 4\overline{\rho}h\beta w^3 + \\ & + [S_x(t)w_{,yy} + S_y(t)w_{,yy}](w_{,t} + \beta w) - g_a(x,y)(w_{,t} + \beta w)\Big\}\, dx \end{aligned} \tag{4.5}$$

Solving the variational inequality

$$U \leqslant \lambda V \tag{4.6}$$

in the set of functions satisfying boundary conditions (2.11)-(2.13), we obtain function λ as follows

$$\lambda = \max_{m,n=1,2,\dots} \left\{ \frac{1}{2} \sqrt{G^2 + \frac{4\Big[(2\overline{\rho}h\beta)^2 + \overline{\rho}h\beta G + \frac{1}{2}S_x(t)\Big(\frac{m\pi}{a}\Big)^2 + \frac{1}{2}S_y(t)\Big(\frac{n\pi}{a}\Big)^2\Big]^2}{(2\overline{\rho}h\beta)^2(1+\omega_{mn}^2) + \kappa_{mn} - S_{0x}\Big(\frac{m\pi}{a}\Big)^2 - S_{0y}\Big(\frac{n\pi}{a}\Big)^2}} - \frac{G}{2} \right\} \tag{4.7}$$

where ω_{mn} is the frequency of plate free vibrations without in-plane forces

$$\omega_{mn}^2 = \frac{D_{11}\Big(\frac{m\pi}{a}\Big)^4 + 2(D_{12}+2D_{66})\Big(\frac{m\pi}{a}\Big)^2\Big(\frac{n\pi}{b}\Big)^2 + D_{22}\Big(\frac{n\pi}{b}\Big)^4}{2\overline{\rho}h} \tag{4.8}$$

κ_{mn} is a coefficient dependent on electric properties of the embedded actuator

$$\kappa_{mn} = \frac{16h_1^2}{\pi^2} \frac{\Big[e_{31}\Big(\frac{m\pi}{a}\Big)^2 + e_{32}\Big(\frac{n\pi}{b}\Big)^2\Big]^2}{\varXi_{11}h_1\Big(\frac{m\pi}{a}\Big)^2 + \varXi_{22}h_1\Big(\frac{n\pi}{b}\Big)^2 + \varXi_{33}\frac{\pi^2}{h_1}} \tag{4.9}$$

and G is the gain factor for the mn mode.

The almost sure stability condition has the form

$$\langle\lambda\rangle < 2\overline{\rho}h\beta \tag{4.10}$$

where $\langle\cdot\rangle$ denotes the mathematical expectation.

Analyzing Eq. (4.7) it is evident that as the gain factor G increases, we observe a decrease of λ_{mn} resulting in a saturation effect. The presence of a positive coefficient κ_{mn} in the denominator of Eq. (4.7) increases the stability domain described by Eq. (4.10). Therefore, the results are more conservative when the Maxwell electrostatic equation is neglected.

5. Conclusions

Coupled dynamics equations with respect to the plate displacement and the electric field are derived using Hamilton's principle. The rate velocity feedback is applied to stabilize the parametric plate vibration. A new form of

Liapunov functional suitable for the stability analysis of a coupled problem is proposed. The almost sure stability of the trivial solution is analyzed using the appropriate Liapunov functional. It is shown that stability domains are increased when the stability problem is solved for coupled equations, i.e. when the electric potential is taken into account. Therefore, the results are more conservative when the Maxwell electrostatic equation is neglected. A saturation effect is observed during increasing of the feedback gain factor. An unlimited increase of the gain factor does not lead to increasing of the critical variance of axial force. A further increase of the critical variance can be obtained applying multimode control.

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Stabilizacja drgań parametrycznych płyty za pomocą rozłożonego sterowania

Streszczenie

W dotychczasowym opisie układów aktywnych składających się z ciągłego układu mechanicznego z warstwami piezoelektrycznymi upraszczano równanie aktuatora zakładając, że generowane w nim pole elektryczne zależy jedynie od przyłożonego zewnętrznego napięcia generowanego przez układ sterowania. Zakładano również arbitralnie liniowy rozkład napięcia na grubości warstwy. Nie spełniano w ten sposób równania elektrostatyki warstwy wykonanej z materiału piezoelektrycznego. W niniejszej pracy wyprowadzono sprzężony układ równań opisujący dynamikę płyty prostokątnej. W płycie zanurzone są dwie symetryczne warstwy piezoelektryczne o polaryzacji prostopadłej do powierzchni płyty. Sensor wykonany jest z cienkiej folii piezoelektrycznej PVDF o pomijalnie małej w porównaniu z belką i piezoceramicznymi aktuatorami sztywności. Odkształcenie w płycie i warstwach piezoelektrycznych opisano zgodnie z teorią Kirchhoffa. Za pomocą zasady Hamiltona otrzymano zmodyfikowane równanie dynamiki płyty i równanie elektrostatyki zawierające składniki zależne od krzywizn i torsji oraz pochodnych potencjału elektrycznego. Wyprowadzono również zmodyfikowane warunki brzegowe odpowiadające swobodnemu podparciu. Napięcie działające na piezoceramiczne aktuatory wyznaczono przy założeniu prędkościowego sprzężenia zwrotnego na podstawie zmierzonego napięcia.

Otrzymane równania posłużyły do analizy stateczności i stabilizacji drgań parametrycznych płyty poddanej działaniu sił jawnie zależnych od czasu działających w płaszczyźnie środkowej. Wprowadzono nowy funkcjonał Lapunowa, zawierający obok składników mechanicznych składniki będące energią pola elektrycznego. Po założeniu rozkładu gęstości prawdopodobieństwa sił błonowych możliwe jest wyznaczenie obszaru stateczności w funkcji parametrów, to jest współczynnika tłumienia, współczynnika wzmocnienia sprzężenia zwrotnego, średnich wartości i wariancji sił. Z analizy wzorów wynika, że obszar stateczności jest większy przy uwzględnieniu działania pola elektrycznego opisanego równaniem elektrostatyki podczas badania stateczności. Pominięcie równania elektrostatyki prowadzi do zbyt konserwatywnych wyników stabilizacji. Występuje tu zjawisko nasycenia podczas wzrostu współczynnika wzmocnienia.

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