Fatigue "safe-life" criterion for metal elements under multiaxial static and dynamic random loads

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ABSTRACT

Random stress with Cartesian components of known statistical moments of their mean values and power spectral densities of stochastic stress processes is considered. To account for the mean stress effect the generalised Soderberg criterion for ductile materials is employed. An equivalent stress with periodic (in the mean-square sense) components is defined by means of the equivalence conditions based on the average strain distortion energy. Also, is formulated the fatigue "safe-life" design criterion which covers the conditions of both static strength and fatigue safety and includes yield strengths and fatigue limits which : have simple physical interpretation, can be determined by uniaxial tests, are directly related to the applied loads, and can reflect material anisotropy.

Key words : design criteria, multiaxial loading, random stress, mean stress effect

INTRODUCTION

The past decade has shown that fatigue is still a great challenge for the engineering community. The reasons of it , among other, may be the random or stochastic character of most loads that occur in nature. Therefore in this paper being direct continuation of the author's paper [1] devoted to fatigue safety of metal elements under deterministic loads, an attempt is made to formulate the fatigue "safe-life" design criterion for metal elements by using probabilistic approach. It is assumed that the considered stress is described by its Cartesian components as follows :

$$\sigma_i(t) = c_i + s_i(t) \quad , \quad i = x,\ y,\ z,\ xy,\ yz,\ zx \qquad (1)$$

where :

- c_i - random mean values of known 2[nd] statistical moments
- $s_i(t)$ - zero mean stochastic processes of known power spectral densities, which are stationary (in the wide sense), stationary correlated with each other and statistically independent of the mean values c_i .

For the sake of brevity the stress components $\sigma_z(t)$, $\sigma_{yz}(t)$ and $\sigma_{zx}(t)$ are dropped.

In view of practical calculations, the actual stress components should be modelled with equivalent stress components of a relatively simpler form by means of an appropriate fatigue strength theory. For the multiaxial stress when axial forces, bending moments and torsional loads vary in time none of the fatigue strength theories is universally accepted and all fatigue criteria usually demonstrate large scatter [2]. On the other hand, in certain dynamic cases the conventional strength theories are considered to be satisfactory [3÷5]. From some fatigue tests it has been concluded that also the criterion based on the average strain energy appears promising [6]. However, bearing in mind that for ductile metals the Huber-von Mises-Hencky strength theory is commonly accepted [7, 8], the average strain distortion energy is here, like in [1] , employed.

The presented paper is aimed at the design criterion which would cover both the static strength conditions and fatigue safety requirements under multiaxial stress. For this purpose use can be made of the generalised Soderberg criterion for non-zero mean in-phase stress [1] :

$$\left[\sum_i \left(\frac{c_i}{R_i}\right)^2 - \frac{c_x c_y}{R_x R_y}\right]^{1/2} + \left[\sum_i \left(\frac{a_i}{F_i}\right)^2 - \frac{a_x a_y}{F_x F_y}\right]^{1/2} \le 1 \quad , \quad i = x,\ y,\ xy \qquad (2)$$

where :

- R_i - yield strength relevant to the mean value of i-th stress component, c_i
- F_i - fatigue limit under fully reversed load relevant to the amplitude of i-th stress component, a_i .

As far as the computational effectiveness is concerned, spectral criteria can be very advantageous [9]. Therefore the applied equivalence conditions are transformed into the frequency domain.

The equation (2) suggests to model the stress components (1) by the equivalent stress components :

$$\sigma_i^{(eq)}(t) = c_i^{(eq)} + s_i^{(eq)}(t) \quad , \quad i = x,\ y,\ xy \tag{3}$$

where :

$c_i^{(eq)}$ - random mean value of i-th equivalent stress component

$s_i^{(eq)}(t)$ - i-th zero mean stochastic process.

It is assumed that the processes $s_i^{(eq)}(t)$ are periodically stationary (in the mean-square sense [10]), stationary correlated with each other and statistically independent of the mean values $c_i^{(eq)}$, which are sought in the form :

$$s_i^{(eq)}(t) = a_i^{(eq)} \sin\left(\omega_{eq} t + \varphi_i\right) = \\ = a_{i1} \exp\left(j\omega_{eq} t\right) + a_{i2} \exp\left(-j\omega_{eq} t\right) \ , \ i = x,\ y,\ xy \tag{4}$$

where :

$a_i^{(eq)}, \varphi_i$ - random amplitude and phase angle of i-th equivalent stress component.

$$a_{i1} = \frac{a_i^{(eq)}}{2j} \exp(j\varphi) \ , \quad a_{i2} = a_{i1}^* \tag{5}$$

$$\langle a_{i1} \rangle = \langle a_{i2} \rangle = \langle a_{i1}^* a_{i2} \rangle = \langle a_{i2}^* a_{i1} \rangle = 0 \tag{6}$$

$$\langle a_{x1}^* a_{y2} \rangle = \langle a_{x2}^* a_{y1} \rangle = 0 \tag{7}$$

ω_{eq} - equivalent circular frequency

$(\bullet)^*$ - complex conjugate

$\langle \bullet \rangle$ - expected value

In order to calculate the amplitudes of the equivalent stress components the following equivalence condition [1] is used at the beginning :

$$\frac{1}{T}\int_0^T \phi_{eq}(t)dt = \frac{1}{T}\int_0^T \phi(t)dt \tag{8}$$

where : T is the averaging time

$$\phi_{eq}(t) = \frac{1+\nu}{3E}\left[\begin{array}{c} \left(\sigma_x^{(eq)}\right)^2 + \left(\sigma_y^{(eq)}\right)^2 + \\ -\sigma_x^{(eq)}\sigma_y^{(eq)} + 3\left(\sigma_{xy}^{(eq)}\right)^2 \end{array}\right] \tag{9}$$

is the strain energy of distortion per unit volume in the equivalent stress state and

$$\phi(t) = \frac{1+\nu}{3E}\left(\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\sigma_{xy}^2\right) \tag{10}$$

is that in the actual stress state.

By applying (8) through (10), the following equivalence conditions can be written :

$$\frac{1}{T}\int_0^T \left[\sigma_i^{(eq)}(t)\right]^2 dt = \frac{1}{T}\int_0^T \sigma_i^2(t)dt \ , \ i = x, y, xy \tag{11}$$

$$\frac{1}{T}\int_0^T \sigma_x^{(eq)}(t)\sigma_y^{(eq)}(t)dt = \frac{1}{T}\int_0^T \sigma_x(t)\sigma_y(t)dt \tag{12}$$

Such equations of integral time averages are not convenient for evaluation of parameters of the equivalent stress components by using spectral data therefore they were replaced by the equations of ensemble averages :

$$\left\langle \left[\sigma_i^{(eq)}(t)\right]^2 \right\rangle = \left\langle \sigma_i^2(t) \right\rangle \ , \ i = x, y, xy \tag{13}$$

$$\left\langle \sigma_x^{(eq)}(t)\sigma_y^{(eq)}(t) \right\rangle = \left\langle \sigma_x(t)\sigma_y(t) \right\rangle \tag{14}$$

To obtain the frequency domain formulation of (13) they were rewritten in terms of correlation functions as follows :

$$\left\langle \begin{array}{l} \left[c_i^{(eq)} + a_{i1}^* \exp\left(-j\omega_{eq} t_1\right) + a_{i2}^* \exp\left(j\omega_{eq} t_1\right)\right] \cdot \\ \cdot \left[c_i^{(eq)} + a_{i1} \exp\left(j\omega_{eq} t_2\right) + a_{i2} \exp\left(-j\omega_{eq} t_2\right)\right] \end{array} \right\rangle = \\ = \left\langle \left[c_i + s_i^*(t_1)\right]\left[c_i + s_i(t_2)\right] \right\rangle \tag{15}$$

Stationarity, in the wide sense, of the processes $s_i(t)$ implies that :

$$\left\langle \left[c_i + s_i^*(t_1)\right]\left[c_i + s_i(t_2)\right] \right\rangle = \left\langle c_i^2 \right\rangle + K_i(\tau) \tag{16}$$

where :

$\tau = t_2 - t_1$

and :

$$K_i(\tau) = \left\langle s_i^*(t_1) s_i(t_2) \right\rangle$$

is the autocorrelation function of the process $s_i(t)$. In accordance with (3) through (6), and (16), one gets from (15) :

$$\left\langle \left(c_i^{(eq)}\right)^2 \right\rangle + \frac{1}{4}\left\langle \left(a_i^{(eq)}\right)^2 \right\rangle \left[\exp\left(j\omega_{eq}\tau\right) + \right. \\ \left. + \exp\left(-j\omega_{eq}\tau\right)\right] = \left\langle c_i^2 \right\rangle + K_i(\tau) \tag{17}$$

Thus :

$$\left\langle \left(c_i^{(eq)}\right)^2 \right\rangle = \left\langle c_i^2 \right\rangle \tag{18}$$

$$\frac{1}{4}\left\langle \left(a_i^{(eq)}\right)^2 \right\rangle \left[\exp\left(j\omega_{eq}\tau\right) + \exp\left(-j\omega_{eq}\tau\right)\right] = K_i(\tau) \tag{19}$$

Fourier transformation of (19) gives :

$$\frac{1}{4}\left\langle \left(a_i^{(eq)}\right)^2 \right\rangle \left[\delta\left(\omega - \omega_{eq}\right) + \delta\left(\omega + \omega_{eq}\right)\right] = S_i(\omega) \tag{20}$$

where :

δ - Dirac's delta function

$S_i(\omega)$ - power spectral density of the process $s_i(t)$.

The mean-square value of the amplitude of i-th equivalent stress component can be estimated with the use of (20) by its integration over the whole frequency range, which yields :

$$\frac{1}{2}\left\langle \left(a_i^{(eq)}\right)^2 \right\rangle = \int_{-\infty}^{\infty} S_i(\omega)d\omega \tag{21}$$

and :

$$\left\langle \left(a_i^{(eq)}\right)^2 \right\rangle = 2\int_{-\infty}^{\infty} S_i(\omega)d\omega \tag{22}$$

Similarly, stationary cross - correlation of the processes $\sigma_x(t)$ and $\sigma_y(t)$ requires that :

$$\left\langle \sigma_x^*(t_1)\sigma_y(t_2)\right\rangle = \left\langle c_x c_y\right\rangle + K_{x,y}(\tau) \qquad (23)$$

where :

$$K_{x,y}(\tau) = \left\langle s_x^*(t_1)s_y(t_2)\right\rangle$$

is the cross-correlation function of the processes $s_x(t)$ and $s_y(t)$.

By proceeding with (14) – in the same way as above – the following was obtained :

$$\left\langle c_x^{(eq)}c_y^{(eq)}\right\rangle + \frac{1}{4}\left\langle a_x^{(eq)}a_y^{(eq)} \cdot \right.$$
$$\cdot\left\{\exp\left[j(\varphi_y-\varphi_x)\right]\exp(j\omega_{eq}\tau) + \right. \qquad (24)$$
$$\left.\left. + \exp\left[-j(\varphi_y-\varphi_x)\right]\exp(-j\omega_{eq}\tau)\right\}\right\rangle =$$
$$= \left\langle c_x c_y\right\rangle + K_{x,y}(\tau)$$

So :

$$\left\langle c_x^{(eq)}c_y^{(eq)}\right\rangle = \left\langle c_x c_y\right\rangle \qquad (25)$$

$$\frac{1}{4}\left\langle a_x^{(eq)}a_y^{(eq)}\left\{\begin{array}{c}\exp\left[j(\varphi_y-\varphi_x)\right]\cdot \\ \cdot\exp(j\omega_{eq}\tau) + \\ + \exp\left[-j(\varphi_y-\varphi_x)\right]\cdot \\ \cdot\exp(-j\omega_{eq}\tau)\end{array}\right\}\right\rangle = K_{x,y}(\tau) \qquad (26)$$

After Fourier transformation of (26) one obtains :

$$\frac{1}{4}\left\langle a_x^{(eq)}a_y^{(eq)}\left\{\begin{array}{c}\exp\left[j(\varphi_y-\varphi_x)\right]\cdot \\ \cdot\delta(\omega-\omega_{eq}) + \\ + \exp\left[-j(\varphi_y-\varphi_x)\right]\cdot \\ \cdot\delta(\omega+\omega_{eq})\end{array}\right\}\right\rangle = S_{x,y}(\omega) \qquad (27)$$

where : $S_{x,y}(\omega)$ is the cross power spectral density of the processes $\sigma_x(t)$ and $\sigma_y(t)$. Integration of (27) over the whole frequency range yields :

$$\frac{1}{2}\left\langle a_x^{(eq)}a_y^{(eq)}\cos(\varphi_y-\varphi_x)\right\rangle = \int_{-\infty}^{\infty} S_{x,y}(\omega)d\omega \qquad (28)$$

i.e.:

$$\left\langle a_x^{(eq)}a_y^{(eq)}\cos(\varphi_y-\varphi_x)\right\rangle = 2\int_{-\infty}^{\infty} S_{x,y}(\omega)d\omega \qquad (29)$$

FATIGUE "SAFE-LIFE" CRITERION

With regard to (2) through (4), the criterion in question can be formulated as follows :

$$\left\langle f_s^{-1}\right\rangle + \left\langle f_d^{-1}\right\rangle < 1 \qquad (30)$$

where :

$$f_s^{-1} = \left[\sum_i\left(\frac{c_i^{(eq)}}{R_i}\right)^2 - \frac{c_x^{(eq)}c_y^{(eq)}}{R_x R_y}\right]^{1/2}, \quad i = x, y, xy \qquad (31)$$

$$f_d^{-1} = \left[\sum_i\left(\frac{a_i^{(eq)}}{F_i}\right)^2 - \frac{a_x^{(eq)}a_y^{(eq)}\cos(\varphi_y-\varphi_x)}{F_x F_y}\right]^{1/2} \qquad (32)$$

After expanding the functions f_s^{-1} and f_d^{-1} into Taylor series around expected values of their arguments and retaining the linear terms, one gets :

$$\left\langle f_s^{-1}\right\rangle = \left[\sum_i\frac{\left\langle\left(c_i^{(eq)}\right)^2\right\rangle}{R_i^2} - \frac{\left\langle c_x^{(eq)}c_y^{(eq)}\right\rangle}{R_x R_y}\right]^{1/2} \qquad (33)$$

$$\left\langle f_d^{-1}\right\rangle = \left[\sum_i\frac{\left\langle\left(a_i^{(eq)}\right)^2\right\rangle}{F_i^2} - \frac{\left\langle a_x^{(eq)}a_y^{(eq)}\cos(\varphi_y-\varphi_x)\right\rangle}{F_x F_y}\right]^{1/2} \qquad (34)$$

Hence the fatigue "safe-life" design criterion for anisotropic metal elements under multiaxial static and dynamic random loads becomes as follows :

$$\left(\sum_i\frac{\left\langle c_i^2\right\rangle}{R_i^2} - \frac{\left\langle c_x c_y\right\rangle}{R_x R_y}\right)^{1/2} + \left\{2\int_{-\infty}^{\infty}\left[\sum_i\frac{S_i(\omega)}{F_i^2} - \frac{S_{x,y}(\omega)}{F_x F_y}\right]d\omega\right\}^{1/2} < 1 \qquad (35)$$

CONCLUSIONS

- The fatigue "safe-life" design criterion which covers the conditions of both static strength and fatigue safety of metal elements under multiaxial static and dynamic random loads, was formulated.
- The presented criterion includes material constants which :
  - have simple physical interpretation
  - can be determined by uniaxial tests
  - are directly related to the applied load
  - and can reflect material anisotropy.

NOMENCLATURE

a_i - amplitude of i-th component of the in-phase stress (i = x, y, z, xy, yz, zx)
$a_i^{(eq)}$ - random amplitude of i-th equivalent stress component
a_{i1}, a_{i2} - quantities defined by (5)
c_i - mean value of i-th component of the in-phase stress, random mean value of i-th component of the actual stress
$c_i^{(eq)}$ - random mean value of i-th equivalent stress component
E - Young modulus
F_i - fatigue limit under fully reversed load relevant to the stress amplitude a_i
f_d, f_s - partial safety factors acc. to [1]
j - imaginary unit
K_i - autocorrelation function of the process $s_i(t)$
$K_{x,y}$ - cross-correlation function of the processes $s_x(t)$ and $s_y(t)$
R_i - yield strength relevant to the mean stress value c_i
s_i - time-variable part of i-th stress component
$s_i^{(eq)}$ - time-variable part of i-th equivalent stress component
S_i - power spectral density of the process $s_i(t)$
$S_{x,y}$ - cross power spectral density of the processes $s_x(t)$ and $s_y(t)$
t - time

T - averaging time
δ - Dirac's delta function
ν - Poisson's ratio
σ_i - i-th stress component
$\sigma_i^{(eq)}$ - i-th equivalent stress component
τ - time interval
φ_i - random phase angle of i-th equivalent stress component
ϕ, ϕ_{eq} - strain energy of distortion per unit volume in the actual and equivalent stress states
ω - circular frequency
ω_{eq} - equivalent circular frequency
$\langle \bullet \rangle$ - expected value
$(\bullet)^*$ - complex conjugate

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Miscellanea

TOP KORAB activity in 2003

In 2003 The Polish Society of Naval Architects and Marine Engineers, TOP KORAB, – in accordance with its custom of arranging topical meetings – carried out, in Gdańsk, 9 meetings devoted to the following topics :

- Achievements of POL-LEVANT shipping company in the light of situation of Polish shipping
- Development stategy of Pomeranian region
- Certification of management systems
- Dimension allowance system applied in shipbuilding industry
- Current problems of Central Maritime Museum in Gdańsk - presentation of permanent exhibitions
- Achievements and development prospects of Gdynia Naval Shipyard
- Prospects and future of Polish Shibuilding Industry Forum
- A proposal of topics of club meetings to be organized in 2004, and of forms of future activities of TOP KORAB
- Presentation of the current state of TOP KORAB chronicle.

A full-day coach excursion to Malbork to visit the historical Teutonian Knights' castle, was also carried out.

Whereas in Szczecin, apart from 4 organizational meetings, 2 topical meetings were held :

- Current problems of NOWA Szczecin Shipyard Co
- Problems associated with implementing a new IMO code : ISPS Code 2003.

Ministerial award

Some years ago a team of the Faculty of Ocean Engineering and Ship Technology, Gdańsk University of Technology, has commenced working on design and construction of a prototype fishing cutter for Baltic Sea, which has been hoped to be useful in renewal of the obsolete fishing vessels flying Polish flag. The undertaken task was sucessfully completed and the built cutter has already operated for a year, gathering flattering opinions from the side of its users.

Polish state authorities received with recognition that important achievement seeing in it a nucleus of a new generation of ecolgical family-operated fishing vessels. Giving voice to it, the authorities rewarded the team consisted of the following persons :

- K. Rosochowicz, Prof.D.Sc., Eng.
- A. Wołoszyn, M.Sc., Eng.
- J. Krępa, D.Sc., Eng
- Cz. Dymarski, D.Sc., Eng.
- T. Blekiewicz, Eng.
- E. Brzoska, D.Sc., Eng.
- G. Wendt, M.Sc., Eng.
- M. Stachowiak, Eng.

with the Ministerial prize which was solemnly handed over on 13 May 2004.