JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
43, 1, pp. 93-110, Warsaw 2005

ON A CERTAIN MODEL OF UNIPERIODIC MEDIUM THICKNESS PLATES SUBJECTED TO INITIAL STRESSES

Eugeniusz Baron

Department of Building Structures Theory, Silesian University of Technology
e-mail: ckibud@polsl.gliwice.pl

The aim of the paper is twofold. First, governing equations for medium thickness elastic plates which have a periodically non-homogeneous structure in one direction (uniperiodic) and subjected to initial in-plane stresses are derived. In order to obtain the aforementioned equations, the tolerance averaging technique is applied. This technique leads to equations with constant coefficients. Second, the above equations are applied to analysis of certain stability and dynamic problems. The stiffnesses of plates were calculated by treating them as structurally anisotrpic. An interesting result is that two values of the critical force can be obtained. This result can have a physical meaning for the stability of plates under compression in one direction and tension in the perpendicular direction.

Key words: uniperiodic plates, modelling, stability

1. Introduction

The subject of analysis are medium thickness rectangular uniperiodic elastic plates, i.e. plates with a periodic non-homogeneous structure in one direction. The above plates are composed of a large number of repeated elements having an identical form, dimensions and material properties. The geometry of a uniperiodic plate, apart from the global mid-plane length dimensions L_1, L_2, is characterized by the length l which determines the period of structure inhomogeneity. In general, in the direction perpendicular to the direction of periodicity, the material parameters may be not constant. However, in most cases "existing in engineering practice" uniperiodic plates have constant properties in that direction. Fragments of the aforementioned plates are shown in Fig. 1 and Fig. 2.

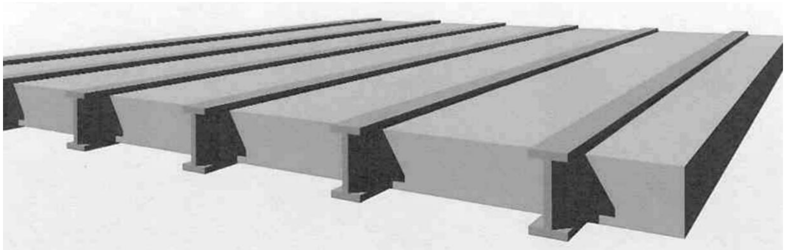

Fig. 1. Example of the plate with a uniperiodic structure

A formulation of different approximate models for these plates is a rather complicated problem. In most cases, homogeneous models of these plates are taken as a basis for analysis of special problems. The homogenized equations have constant coefficients and constitute a certain approximation of uniperiodic plate equations having highly oscillating and non-continuous coefficients, cf. Lewiński (1991). However, the homogenized equations cannot describe the effect of the periodicity length parameter l on the overall plate behaviour (the length-scale effect).

In the work by Baron (2002), a new approximate model of medium thickness uniperiodic plates was proposed. This model, obtained by using the tolerance averaging technique, cf. Woźniak and Wierzbicki (2000), includes the length scale effect.

The aim of this contribution is an extension and a certain generalization of the 2D model of a medium thickness plate derived by Baron (2002) and the analysis of a certain quasi-stationary and dynamic problem for a rectangular plate. In the above article, in the course of modelling in terms containing the initial stress, fluctuation of displacement were taken into consideration. The obtained model will be referred to as the length-scale model, since it includes the effect of the length period l on the overall plate behaviour. The general averaged model equations obtained in this paper will be transformed into a form which would enable investigation of dynamic and stability problems. A new expression for the critical force will be compared with those obtained from the homogenized model of uniperiodic plates. It will be shown that in some special cases related to compression in the mid-plane in the direction along a certain axis and tension in the perpendicular direction, the homogenized model leads to higher values of critical forces than in the length-scale model introduced in this paper.

Throughout the paper the subscripts $\alpha, \beta, \ldots$ run over $1, 2$, subscripts $i, j, \ldots$ over 1,2,3 and superscripts $A, B, \ldots$ over $1, 2, \ldots, N$; summation convention holds for all aforementioned indices.

2. Basic assumptions and notations

By $\boldsymbol{x} = (x_1, x_2)$ we denote Cartesian coordinates of a point on the plate mid-plane $\Pi = (0, L_1) \times (0, L_2)$, and by z a Cartesian coordinate in the direction normal to the mid-plane. By $z = \pm\delta(\boldsymbol{x})$, $\boldsymbol{x} \in \Pi$ we denote functions representing the upper and lower plate boundary, respectively; hence $2\delta(\boldsymbol{x})$ is the plate thickness in a point $\boldsymbol{x} \in \Pi$. By $\rho = \rho(\boldsymbol{x}, z)$ and $A_{ijkl}(\boldsymbol{x}, z)$ we denote mass density and the tensor of elastic moduli of the plate material and assume that every $z =$ const is an elastic symmetry plane. We also define $C_{\alpha\beta\gamma\delta} := A_{\alpha\beta\gamma\delta} - A_{\alpha\beta33}A_{33\gamma\delta}(A_{3333})^{-1}$, $B_{\alpha\beta} := A_{\alpha3\beta3}$. We shall assume that the functions $\delta(\cdot)$, $\rho(\cdot)$, $A_{ijkl}(\cdot)$ are l-periodic with respect to the x_1-coordinate, and are sufficiently regular with respect to z. Let p^+ and p^- be loadings (in the z-axis direction) on the upper and bottom surfaces of the plate, respectively. Let $\sigma^o_{\alpha\beta}$ be a tensor of the initial stress and b be a constant body force acting in the z-axis direction. Furthermore, let t be the time coordinate. The averaged value of an arbitrary integrable function $\varphi(x_1, x_2, t)$ in the periodicity interval $(x_1 - l/2, x_1 + l/2)$ will be denoted by

$$\langle\varphi\rangle(x, t) = \frac{1}{l}\int\limits_{x_1-l/2}^{x_1+l/2} \varphi(\xi, x_2, t)\; d\xi \qquad \boldsymbol{x} = (x_1, x_2) \tag{2.1}$$

For an uniperiodic function $\varphi(\cdot)$, the above averaged value is independent of x_1.

3. Modelling procedure. Model equations

Setting

$$\mu(\boldsymbol{x}) = \int\limits_{-\delta}^{\delta} \rho(\boldsymbol{x}, z)\; dz \qquad\qquad p(\boldsymbol{x}) = p^+(\boldsymbol{x}) + p^-(\boldsymbol{x}) + b\langle\mu\rangle(\boldsymbol{x})$$

$$J(\boldsymbol{x}) = \int\limits_{-\delta}^{\delta} z^2\rho(\boldsymbol{x}, z)\; dz \qquad\qquad G_{\alpha\beta\gamma\delta}(\boldsymbol{x}) = \int\limits_{-\delta}^{\delta} z^2 C_{\alpha\beta\gamma\delta}(\boldsymbol{x}, z)\; dz$$

$$N^o_{\alpha\beta\beta} = \int\limits_{-\delta}^{\delta} \sigma^o_{\alpha\beta}\; dz \qquad\qquad D_{\alpha\beta}(\boldsymbol{x}) = \int\limits_{-\delta}^{\delta} K_{\alpha\beta}B_{\alpha\beta}(\boldsymbol{x}, z)\; dz$$

where in the expression summation convention with respect to α and β does not hold for $D_{\alpha\beta}$, and $K_{\alpha\beta}$ is a shear coefficient (introduced by Jemielita (2001)), we obtain the system of equations

$$
\begin{aligned}
&(G_{\alpha\beta\gamma\delta}\vartheta_{(\gamma,\delta)})_{,\beta} - D_{\alpha\beta}\vartheta_{\beta} - J\ddot{\vartheta}_{\alpha} = 0 \\
&N^{o}_{\alpha\beta}w_{,\alpha\beta} + [D_{\alpha\beta}(\vartheta_{\beta} + w_{,\beta})]_{,\alpha} - \mu\ddot{w} + p = 0
\end{aligned}
\tag{3.1}
$$

in which the deflection w and rotation ϑ_{α} are basic unknowns.

The above equations represent the medium thickness 2D-plate model of the Hencky-Boole type. For an uniperiodic plate, the above system of equations has functional coefficients which are periodic with respect to the argument x_1. These coefficients are certain highly oscillating and non-continuous functions. The exact solution to boundary value problems formulated for these equations is, in most cases, rather complicated. That is why various approximate models leading to equations with constant coefficients are proposed. We can mention here a known homogenized model. However, this model is not able to describe the effect of the period length on the overall plate behaviour. In the paper by Baron (2002), a new non-asymptotic model was proposed. This model was obtained by using the tolerance averaging method summarized by Woźniak and Wierzbicki (2000).

In accordance with the tolerance averaging procedure, the unknown deflection w and rotations ϑ_{α} are assumed in the form

$$
\begin{aligned}
\vartheta_{\alpha}(\boldsymbol{x},t) &= \vartheta^{o}_{\alpha}(\boldsymbol{x},t) + \vartheta^{*}_{\alpha}(\boldsymbol{x},t) \\
w(\boldsymbol{x},t) &= w^{o}(\boldsymbol{x},t) + w^{*}(\boldsymbol{x},t)
\end{aligned}
$$

where $w^{o}(\cdot)$, $\vartheta^{o}_{\alpha}(\cdot)$ are the averaged deflection and rotations, and $w^{*}(\cdot)$, $\vartheta^{*}_{\alpha}(\cdot)$ describe fluctuations of the fields $\vartheta_{\alpha}(\boldsymbol{x},t)$, $w(\boldsymbol{x},t)$ caused by the inhomogeneity of the plate. At the same time, the functions $w^{o}(\cdot)$, $\vartheta^{o}_{\alpha}(\cdot)$ have to be slowly varying and $w^{*}(\cdot)$, $\vartheta^{*}_{\alpha}(\cdot)$ have to be periodic-like functions, cf. Woźniak and Wierzbicki (2000). We shall also assume that the fluctuations $w^{*}(\cdot)$, $\vartheta^{*}_{\alpha}(\cdot)$ can be approximated by

$$
\begin{aligned}
\vartheta^{*}_{\alpha}(\boldsymbol{x},t) &\cong h^{a}(x_1)\Theta^{a}_{\alpha}(\boldsymbol{x},t) \qquad && a = 1,2,\dots,n \\
w^{*}(\boldsymbol{x},t) &\cong g^{A}(x_1)W^{A}(\boldsymbol{x},t) \qquad && A = 1,2,\dots,N
\end{aligned}
\tag{3.2}
$$

where $W^{A}(\cdot)$, $\Theta^{a}_{\alpha}(\cdot)$ are new slowly varying unknowns. At the same time, $h^{a}(x_1)$, $g^{A}(x_1)$ represent two systems of linear independent periodic shape functions, postulated *a priori* in every special problem under consideration.

These functions are called *mode-shape functions* and they have to approximate the expected form of the oscillating part of free vibration modes of the periodicity cell. The above functions have to satisfy the conditions $\langle Jh^a \rangle = 0$, $\langle \mu g^A \rangle = 0$, $h^a(x_1) \in O(l)$, $g^A(x_1) \in O(l)$, $lh^a_{,1}(x_1) \in O(l)$, $lg^A_{,1}(x_1) \in O(l)$. Taking into account the aforementioned conditions, we shall also introduce functions

$$\overline{h}^a = l^{-1}h^a \qquad\qquad \overline{g}^A = l^{-1}g^A$$

which are of the order $O(1)$ when $l \to 0$.

In the subsequent considerations, slowly varying functions w^o, ϑ^o_α, W^A, Θ^a_α are basic kinematics unknowns. In order to obtain a system of equations for these unknowns, we shall apply a procedure similar to that discussed in Baron (2002), however, in terms containing the initial stress $N^o_{\alpha\beta}$, the fluctuation of displacement will not be neglected. That means that the assumption $N^o_{\alpha\beta}w_{,\alpha\beta} \approx N^o_{\alpha\beta}w^o_{,\alpha\beta}$ has been substituted by the relation

$$N^o_{\alpha\beta}w_{,\alpha\beta} = N^o_{\alpha\beta}(w^o + w^*)_{,\alpha\beta}$$

Setting aside all transformations, which are similar to those presented in Baron (2002), we arrive at the equations:
— equations of motion

$$\begin{aligned} &M_{\alpha\beta,\beta} - Q_\alpha - \langle J \rangle \ddot{\vartheta}^o_\alpha = 0 \\ &N^o_{\alpha\beta}w^o_{,\alpha\beta} + \underline{lN^o_{\alpha 2}\langle \overline{g}^A \rangle W^A_{,\alpha 2}} + Q_{\alpha,\alpha} - \langle \mu \rangle \ddot{w}^o + p = 0 \end{aligned} \tag{3.3}$$

— kinematic equations for Θ^a, W^A

$$\begin{aligned} &l^2\langle J\overline{h}^a\overline{h}^b \rangle \ddot{\Theta}^b_\alpha + M^a_\alpha - l\widetilde{M}^a_{\alpha,2} = 0 \\ &l^2\langle \mu\overline{g}^A\overline{g}^B \rangle \ddot{W}^B + Q^A - l\widetilde{Q}^A_{,2} + \\ &-N^o_{\alpha 2}(l\langle \overline{g}^A \rangle w^o_{,\alpha 2} + \underline{l^2\langle \overline{g}^A\overline{g}^B \rangle W^B_{,\alpha 2}}) + \underline{N^o_{11}\langle g^A_{,1}g^B_{,1} \rangle W^B} - l\langle \overline{g}^A p \rangle = 0 \end{aligned} \tag{3.4}$$

— constitutive equations

$$
\begin{aligned}
&M_{\alpha\beta} = \langle G_{\alpha\beta\gamma\delta}\rangle \vartheta^o_{(\gamma,\delta)} + \langle h^a_{,1} G_{\alpha\beta 1\delta}\rangle \Theta^a_\delta + l\langle \overline{h}^a G_{\alpha\beta 2\delta}\rangle \Theta^a_{\delta,2}\\
&Q_\alpha = \langle D_{\alpha\beta}\rangle (\vartheta^o_\beta + w^o_{,\beta}) + l\langle \overline{h}^a D_{\alpha\beta}\rangle \Theta^a_\beta + \langle g^A_{,1} D_{\alpha 1}\rangle W^A + l\langle \overline{g}^A D_{\alpha 2}\rangle W^A_{,2}\\
&M^a_\alpha = \langle h^a_{,1} h^b_{,1} G_{\alpha 11\delta}\rangle \Theta^b_\delta + \langle h_{,1} G_{\alpha 1\gamma\delta}\rangle \vartheta^o_{(\gamma,\delta)} + l\langle h^a_{,1}\overline{h}^b G_{\alpha 12\delta}\rangle \Theta^b_{\delta,2} +\\
&\qquad + l^2\langle \overline{h}^a \overline{h}^b D_{\alpha\beta}\rangle \Theta^b_\beta + l\langle \overline{h}^a D_{\alpha\beta}\rangle (\vartheta^o_\beta + w^o_{,\beta}) + l\langle \overline{h}^a g^A_{,1} D_{\alpha 1}\rangle W^A +\\
&\qquad + l^2\langle \overline{h}^a \overline{g}^A D_{\alpha 2}\rangle W^A_{,2} \qquad\qquad (3.5)\\
&\widetilde{M}^a_\alpha = \langle \overline{h}^a h^b_{,1} G_{\alpha 21\delta}\rangle \Theta^b_\delta + \langle \overline{h}^a G_{\alpha 2\gamma\delta}\rangle \vartheta^o_{(\gamma,\delta)} +\\
&\qquad + l\langle \overline{h}^a \overline{h}^b G_{\alpha 22\delta}\rangle \Theta^b_{\delta,2}\\
&Q^A = \langle g^A_{,1} g^B_{,1} D_{11}\rangle W^B + \langle g^A_{,1} D_{1\beta}\rangle (vt^o_\beta + w^o_{,\beta}) + l\langle g^A_{,1}\overline{h}^a D_{1\beta}\rangle \Theta^a_\beta +\\
&\qquad + l\langle g^A_{,1}\overline{g}^B D_{12}\rangle W^B_{,2}\\
&\widetilde{Q}^A = \langle \overline{g}^A g^B_{,1} D_{21}\rangle W^B + \langle \overline{g}^A D_{2\beta}\rangle (\vartheta^o_\beta + w^o_{,\beta}) + l\langle \overline{g}^A h^a D_{2\beta}\rangle \Theta^a_\beta +\\
&\qquad + l\langle \overline{g}^A \overline{g}^B D_{22}\rangle W^B_{,2}
\end{aligned}
$$

Averaged 2D-model equations (3.3)-(3.5) constitute the starting point for the subsequent analysis. The underlined terms in the above equations describe the influence of fluctuation displacement neglected in Baron (2002). In most cases, we deal with plates having a homogeneous structure in the x_2-axis direction (cf. Fig. 2). For such a type of uniperiodic plates, all coefficients in equations (3.3)-(3.5) are constant, and the subsequent considerations will be restricted to the aforementioned type of plates.

4. An orthotropic plate with stiffeners

Now let us assume that the plate is of constant thickness and is made of an orthotropic material, where the principal axis of orthotropy coincides with the Cartesian axis $(\boldsymbol{x}, z)$. Moreover, let us assume that the plate is reinforced by a certain system of periodically spaced stiffeners, cf. Fig. 2. We also assume that the torsional stiffness of the stiffeners in the plane normal to the x_2-axis is neglected. Let M be mass density of a stiffener and I be bending stiffness of the stiffener, respectively. Moreover, let

$$
\begin{aligned}
&G_{11} = G_{1111} && G_{22} = G_{2222}\\
&G_{12} = G_{1122} = G_{2211} && G = G_{1212} = G_{1221} = G_{2112} = G_{2121}\\
&D_1 = D_{11} && D_2 = D_{22}
\end{aligned}
$$

be stiffness of the orthotropic plate under consideration.

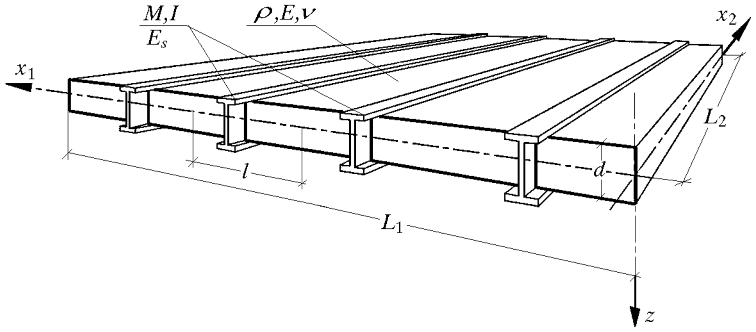

Fig. 2. A scheme of the uniperiodic plate under consideration

Let us take exclusively two modal shape functions

$$h(x_1) = h^1(x_1) = l\overline{h}(x_1) \qquad\qquad g(x_1) = g^1(x_1) = l\overline{g}(x_1)$$

as the first approximation of the plate fluctuations caused by the uniperiodic plate structure.

Let us consider the interval $\langle 0, l\rangle$ as a representative plate segment. We assume that $h(x_1)$ is an odd function and $g(x_1)$ is an even function of x_1. On the above assumptions, we obtain from (3.3)-(3.5) the following system of equations for the unknowns ϑ_α, $\Theta_1 = \Theta$, W, w

$$\begin{aligned}
&\langle G_{11}\rangle\vartheta^o_{1,11} + \langle G\rangle\vartheta^o_{1,22} + (\langle G_{12} + \langle G\rangle)\vartheta^o_{2,12} - \langle D_1\rangle(\vartheta^o_1 + w^o_{,1}) - \langle J\rangle\ddot{\vartheta}^o_1 = 0\\
&\langle G_{22}\rangle\vartheta^o_{2,22} + \langle G\rangle\vartheta^o_{2,11} + (\langle G_{12} + \langle G\rangle)\vartheta^o_{1,12} - \langle D_2\rangle(\vartheta^o_2 + w^o_{,2}) -\\
&-l\langle\overline{g}D_2\rangle W_{,2} - \langle J\rangle\ddot{\vartheta}^o_2 = 0\\
&N^o_{\alpha\beta}w^o_{,\alpha\beta} + \langle D_1\rangle(\vartheta^o_1 + w^o_{,1})_{,1} + \langle D_2\rangle(\vartheta^o_2 + w^o_{,2})_{,2} + lN^o_{\alpha 2}\langle\overline{g}\rangle W_{,\alpha 2} +\\
&+l\langle\overline{g}D_2\rangle W_{,22} - \langle\mu\rangle\ddot{w}^o + p = 0 \qquad\qquad (4.1)\\
&-l^2\langle\overline{h}^2 G\rangle\Theta_{,22} + (\langle h^2_{,1}G_{11}\rangle + l^2\langle\overline{h}^2 D_1\rangle\rangle)\Theta + l\langle\overline{h}g_{,1}D_1\rangle W + l^2\langle\overline{h}^2 J\rangle\ddot{\Theta} = 0\\
&-l^2N^o_{\alpha\beta}\langle\overline{g}^2\rangle W_{,\alpha\beta} - l^2\langle\overline{g}^2D_2\rangle W_{,22} + N^o_{11}\langle g^2_{,1}\rangle W + \langle g^2_{,1}D_1\rangle W - lN^o_{\alpha 2}\langle\overline{g}\rangle w^o_{,\alpha 2} +\\
&-l\langle\overline{g}D_2\rangle(\vartheta^o_2 + w^o_{,2}\rangle)_{,2} + l\langle\overline{h}g_{,1}D_1\rangle\Theta + l^2\langle\overline{g}^2\mu\rangle\ddot{W} - l\langle\overline{g}p\rangle = 0
\end{aligned}$$

and an independent equation for Θ_2

$$-l^2\langle\overline{h}^2 G_{22}\rangle\Theta_{2,22} + (\langle h^2_{,1}G\rangle + l^2\langle\overline{h}^2 D_2\rangle)\Theta_2 + l^2\langle\overline{h}^2 J\rangle\ddot{\Theta}_2 = 0 \qquad (4.2)$$

Equations (4.1) together with (4.2) have constant coefficients and will be examined together with appropriate boundary and initial conditions.

The stiffnesses of the plate will be calculated taking into account structural anisotropy. It means that the plate is made of a homogeneous and isotropic material and reinforced by a system of parallely spaced material inclusions. By means of a particular way of calculating the stiffnesses, cf. Sokołowski (1957), this composite plate can be treated as homogeneous but made of an anisotropic material. In this paper, in the inertial terms, the factual mass distribution is yet taken into consideration.

Material properties of structurally anisotropic (strictly: orthotropic) plate shown in Fig. 2 are represented by the Young modulae E_1, E_2 and by the Poisson ratios ν_1, ν_2. In this case, the plate stiffnesses are given by

$$G_{11} = \frac{E_1 d^3}{12(1-\nu_1\nu_2)} \qquad G_{22} = \frac{E_2 d^3}{12(1-\nu_1\nu_2)}$$
$$G_{12} = \nu_1 G_{22} = \nu_2 G_{11} \qquad G = \frac{\sqrt{G_{11}G_{22}}}{2(1+\nu_1)}$$

Setting for the plate material $E = E_1$ and $\nu = \nu_1$, it can be shown that

$$G_{11} = \langle G_{11} \rangle = \frac{Ed^3}{12(1-\nu^2)} = H_o$$

Similarly, taking into account averaging formula (2.1), we obtain

$$G_{22} = \langle G_{22} \rangle = H_o\Big(1 + \frac{E_s I}{H_o l}\Big)$$

where E_s is the Young modulus of the stiffener. Defining by $\psi = E_s I/(H_o l)$ a constant which will be called the coefficient of nonhomogeneity related to uniperiodic plate structure, we obtain

$$G_{22} = \langle G_{22} \rangle = H_o(1+\psi)$$

From the condition $\nu G_{22} = \nu_2 G_{11}$, cf. Sokołowski (1957), we conclude that $\nu_2 = \nu(1+\psi)$. Hence

$$G_{22} = \frac{E_2 d^3}{12(1-\nu_1\nu_2)} = \frac{E_2 d^3}{12[1-\nu^2(1+\psi)]} = \frac{Ed^3}{12(1-\nu^2)}(1+\psi)$$
$$E_2 = E\frac{1-\nu^2(1+\psi)}{1-\nu^2}(1+\psi)$$

One should pay attention that if $\nu_2 < 0.5$, we obtain an additional condition for the coefficient ψ

$$\psi < \frac{1-2\nu}{2\nu}$$

The shear stiffness will be calculated from the formula

$$\langle D_1 \rangle = \frac{Ed}{2(1+\nu)} K_{11}$$

$$\langle D_2 \rangle = \frac{E_2 d}{2(1+\nu_2)} K_{22} = \frac{Ed[1-\nu^2(1+\psi)]}{2(1-\nu^2)[1+\nu(1+\psi)]}(1+\psi)K_{22}$$

Following Jemielita (2001), for dynamic problems, we introduce the shear coefficients

$$K_{11} = \frac{5}{6-\nu} \qquad\qquad K_{22} = \frac{5}{6-\nu(1+\psi)}$$

From assumptions on structural anisotropy (the plate can be treated as homogeneous), we conclude that the stiffnesses, calculated by application of the mode-shape function $h(x_1)$, $g(x_1)$, are constant, i.e. $\langle h_{,1}^2 G_{11} \rangle = G_{11} \langle h_{,1}^2 \rangle$.

Equations (4.1), together with the aforementioned procedure of calculating the coefficients, are the starting point for the analysis of special problems, which will be explained in the next section.

5. Applications

We are going to apply the model equations obtained in the previous section to the analysis of stability and a dynamic problem for a rectangular uniperiodic plate. The plate is simply supported on its edges and subjected to the initial stress on the plate mid-plane, Fig. 3. Taking into account the boundary conditions, for a plate simply supported on all edges, we look for the solution to equations (4.1) in the form

$$\begin{aligned}
\vartheta_1 &= \mathrm{e}^{\mathrm{i}\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \vartheta_{1mn} \cos\alpha_m x_1 \sin\beta_n x_2 \\
\vartheta_2 &= \mathrm{e}^{\mathrm{i}\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \vartheta_{2mn} \sin\alpha_m x_1 \cos\beta_n x_2 \\
\Theta_1 &= \mathrm{e}^{\mathrm{i}\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Theta_{1mn} \cos\alpha_m x_1 \sin\beta_n x_2 \\
w &= \mathrm{e}^{\mathrm{i}\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin\alpha_m x_1 \sin\beta_n x_2 \\
W &= \mathrm{e}^{\mathrm{i}\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin\alpha_m x_1 \sin\beta_n x_2
\end{aligned} \tag{5.1}$$

where: $\alpha_m = m\pi/L_1$, $\beta_n = n\pi/L_2$, $m,n = 1,2,\dots$ and ϑ_{1mn}, ϑ_{2mn}, W_{mn}, w_{mn} are constant amplitudes, ω is a vibrations frequency.

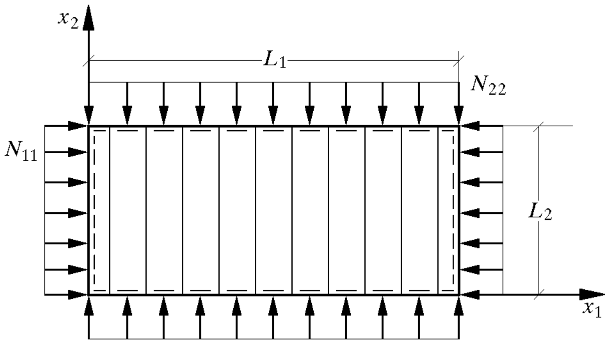

Fig. 3. A scheme of a uniperiodic plate subjected to an edge loading

Let us denote

$$
\begin{aligned}
H_1 &= \alpha_n^2\langle G_{11}\rangle + \beta_n^2\langle G\rangle + \langle D_1\rangle\\
H_2 &= \beta_n^2\langle G_{22}\rangle + \alpha_n^2\langle G\rangle + \langle D_2\rangle\\
\overline{G} &= \langle G_{12}\rangle + \langle G\rangle\\
\overline{B} &= \frac{\alpha_m^2\beta_n^2\langle\overline{g}^2\rangle\langle D_2\rangle\overline{G}}{H_1H_2 - \alpha_m^2\beta_n^2\overline{G}^2} + \frac{\langle hg_{,1}\rangle^2\langle D_1\rangle}{\langle g_{,1}^2G_{11}\rangle + l^2\langle\overline{h}^2D_1\rangle + l^2\beta_n^2\langle\overline{h}^2G\rangle}
\end{aligned}
$$

and introduce non-dimensional stiffness and forces

$$
\begin{aligned}
\overline{D}_1 &= 1 - \frac{H_2\langle D_1\rangle - \beta_n^2\langle D_2\rangle\overline{G}}{H_1H_2 - \alpha_m^2\beta_n^2\overline{G}^2}\\
\overline{D}_2 &= \frac{\langle D_2\rangle}{\langle D_1\rangle}\Big(1 - \frac{H_1\langle D_2\rangle - \alpha_m^2\langle D_1\rangle\overline{G}}{H_1H_2 - \alpha_m^2\beta_n^2\overline{G}^2}\Big)\\
\overline{N}_1 &= \frac{N_{11}}{\langle D_1\rangle} \qquad \overline{N}_2 = \frac{N_{22}}{\langle D_1\rangle} \qquad (N_{12} = 0)
\end{aligned}
$$

Substituting (5.1) into (4.1) and taking into account the aforementioned denotations, after some transformations we obtain the following equations for the unknowns w_{mn} and W_{mn}

$$
\begin{bmatrix} a_{11} & l\langle\overline{g}\rangle\beta_n^2(\overline{N}_2 + \overline{D}_2)\\ l\langle\overline{g}\rangle\beta_n^2(\overline{N}_2 + \overline{D}_2) & a_{22}\end{bmatrix}\begin{bmatrix} w_{mn}\\ W_{mn}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \tag{5.2}
$$

where

$$a_{11} = \alpha_m^2(\overline{N}_1 + \overline{D}_1) + \beta_n^2(\overline{N}_2 + \overline{D}_2) - \frac{\langle\mu\rangle}{\langle D_1\rangle}\omega^2$$

$$a_{22} = \langle g_{,1}^2\rangle(\overline{N}_1 + 1) + l^2\beta_n^2\langle\overline{g}^2\rangle(\overline{N}_2 + \overline{D}_2) - l^2\overline{B} - l^2\frac{\langle\overline{g}^2\mu\rangle}{\langle D_1\rangle}\omega^2$$

Equations (5.2) constitute the starting point for the subsequent examples.

Numerical calculations will be carried out for a constant-thickness concrete plate with $E = 29000\,\mathrm{MPa}$ (concrete B25), $\nu = 0.20$, reinforced by periodically spaced rolled steel sections (I-bar) with $E_s = 205000\,\mathrm{MPa}$, as it shown in Fig. 2. We assume mode-shape functions in the form

$$h(x_1) = l\sin\frac{2\pi}{l}x_1 \qquad\qquad g(x_1) = l\Big(c + \cos\frac{2\pi}{l}x_1\Big) \tag{5.3}$$

The constant c can be calculated from the condition $\langle\mu g\rangle = 0$

$$c = -\frac{\varphi_M}{1+\varphi_M} \qquad\qquad \varphi_M = \frac{M}{\rho dl}$$

For the above functions, we obtain

$$\langle\overline{g}\rangle = c \qquad\qquad \langle\overline{g}^2\rangle = \frac{1}{2} + c^2 \qquad\qquad \langle g_{,1}^2\rangle = 2\pi^2$$

$$\langle h_{,1}^2\rangle = 2\pi^2 \qquad\qquad \langle\overline{h}^2\rangle = \frac{1}{2}$$

and

$$\begin{aligned}\langle\mu\rangle &= \rho d(1+\varphi_M) = \mu_o\frac{1}{1+c} \qquad\qquad \mu_o = \rho d\\ \langle\overline{g}^2\mu\rangle &= \mu_o[\langle g^2\rangle + (1+\langle\overline{g}\rangle)^2\varphi_M] = \mu_o\Big(\frac{1}{2} - c\Big)\end{aligned} \tag{5.4}$$

In the course of calculations, the influence of slenderness ratio λ, the parameters $\varepsilon = l/L_2$, $\kappa = L_2/L_1$ and the coefficient of non-homogeneity $\psi = E_sI/H_0l$ have been taken into account.

5.1. Dynamic problem

In this subsection, free vibrations in the long-wave propagation problem will be discussed.

The system of two linear equations for amplitudes w_{mn}, W_{mn} (5.2) has nontrivial solutions provided that its determinant is equal to zero. In this way, we obtain the characteristic equation for free vibration frequencies

$$
\begin{aligned}
& l^2\frac{\langle\mu\rangle\langle\overline{g}^2\mu\rangle}{\langle D_1\rangle^2}\omega^4-\frac{1}{\langle D_1\rangle}\Big\{\langle\mu\rangle\langle g_{,1}^2\rangle(\overline{N}_1+1)+\\
&+l^2\beta_n^2\Big[\langle\overline{g}^2\mu\rangle\overline{N}_o+\langle\mu\rangle\langle\overline{g}^2\rangle(\overline{N}_2+\overline{D}_2)-\frac{\langle\mu\rangle}{\beta_n^2}\overline{B}\Big]\Big\}\omega^2+\beta_n^2\langle g_{,1}^2\rangle(\overline{N}_1+1)\overline{N}_o+\\
&+l^2\beta_n^2[\beta_n^2\langle\overline{g}^2\rangle(\overline{N}_2+\overline{D}_2)N_o-\beta_n^2\langle\overline{g}\rangle^2(\overline{N}_2+\overline{D}_2)^2-\overline{B}\overline{N}_o]=0
\end{aligned}
\tag{5.5}
$$

where

$$
\beta_n^2\overline{N}_o=\alpha_m^2(\overline{N}_1+\overline{D}_1)+\beta_n^2(\overline{N}_2+\overline{D}_2)
$$

From (5.5), we arrive at the following approximate formulae for the lower ω_1 and higher ω_2 free vibration frequencies

$$
\begin{aligned}
\omega_1^2&=\frac{\beta_n^2\langle D_1\rangle\overline{N}_o}{\langle\mu\rangle}-l^2\frac{\beta_n^4\langle\overline{g}\rangle^2\langle D_1\rangle(\overline{N}_2+\overline{D}_2)^2}{\langle g_{,1}^2\rangle\langle\mu\rangle(\overline{N}_1+1)}\\
\omega_2^2&=\frac{\langle g_{,1}^2\rangle\langle D_1\rangle(\overline{N}_1+1)}{l^2\langle\overline{g}^2\mu\rangle}+\frac{\langle D_1\rangle[\beta_n^2\langle\overline{g}^2\rangle(\overline{N}_2+\overline{D}_2)-\overline{B}]}{\langle\overline{g}^2\mu\rangle}
\end{aligned}
\tag{5.6}
$$

Commenting on the obtained results, it should be admitted that, contrary to the asymptotic homogenisation method, two basic free vibration frequencies have been obtained. The higher frequency ω_2 depends on the period-length l and cannot be derived from the homogenized model.

In further analysis, formulae for frequencies (5.6) will be transformed into a dimensionless form. To this end, we will introduce the denotations

$$
\begin{aligned}
a_1&=\lambda^2\frac{30(1-\nu)}{6-\nu} & a_2&=\lambda^2\frac{1-\nu^2(1+\psi)}{1+\nu(1+\psi)}\,\frac{30(1+\psi)}{6-\nu(1+\psi)}\\
e&=\nu(1+\nu)+\frac{\sqrt{1+\psi}}{2(1+\nu)} & h_1&=\pi^2\Big[m^2\kappa^2+n^2\frac{\sqrt{1+\psi}}{2(1+\nu)}\Big]+a_1\\
h_2&=\pi^2\Big[n^2(1+\psi)+m^2\kappa^2\frac{\sqrt{1+\psi}}{2(1+\nu)}\Big]+a_2
\end{aligned}
$$

$$\overline{D}_1 = 1 - \frac{h_2 a_1 - \pi^2 n^2 a_2 e}{h_1 h_2 - \pi^4 m^2 n^2 \kappa^2 e^2} \tag{5.7}$$

$$\overline{D}_2 = \frac{a_2}{a_1}\Big(1 - \frac{h_1 a_2 - \pi^2 m^2 \kappa^2 a_1 e}{h_1 h_2 - \pi^4 m^2 n^2 \kappa^2 e^2}\Big)$$

$$\overline{B} = \frac{n^2 \frac{a_2^2}{a_1} h_1 c^2}{h_1 h_2 - \pi^4 m^2 n^2 \kappa^2 e^2} + \frac{m^2 n^2 \kappa^2 e\Big(\frac{1}{2} + c^2\Big)}{2\pi^2 + \frac{1}{2}\varepsilon^2\Big[a_1 + \pi^2 n^2 \frac{\sqrt{1+\psi}}{2(1+\nu)}\Big]}$$

Multiplying both relations (5.6) by $L_2^2 \mu_o \langle D_1 \rangle^{-1}$, and taking into account (5.4) and (5.7), we obtain the following formulae for the non-dimensional frequencies

$$\varOmega_1^2 = n^2 \pi^2 (1+c)\Big[m^2 \kappa^2 (\overline{N}_1 + \overline{D}_1) + \overline{N}_2 + \overline{D}_2 + \varepsilon^2 c^2 \frac{(\overline{N}_2 + \overline{D}_2)^2}{2(\overline{N}_1 + 1)}\Big]$$
$$\varOmega_2^2 = \frac{2\pi^2}{1-2c}\Big[\frac{2}{\varepsilon^2}(\overline{N}_1 + 1) + n^2\Big(\frac{1}{2} + c^2\Big)(\overline{N}_2 + \overline{D}_2) - \overline{B}\Big] \tag{5.8}$$

In the course of numerical calculations, it has been assumed that the concrete plate has mass density $\rho = 2200\,\mathrm{kg/m^3}$, thickness $d = 0.15\,\mathrm{m}$ and span $L_2 = 6.00\,\mathrm{m}$. Three variants of reinforcing by rolled steel sections: **I**180, **I**220, **I**240 are taken into account. The **I**-bar is spaced every 0.75 m; also $l = 0.75\,\mathrm{m}$ and $\varepsilon = 0.125$. The shape of the mid-plane is characterized by the ratio $\kappa = L_2/L_1$, $\kappa = 0.5$; 1.0; 2.0. The values of parameters ψ and c are placed in Table 1.

Table 1

I-bar	I $[10^{-8}\mathrm{m}^4]$	M [kg/m]	φ	ψ	c
I180	1450	21.9	0.0885	**0.490**	**−0.0813**
I220	3060	33.1	0.1337	**1.030**	**−0.1179**
I240	4250	36.2	0.1463	**1.420**	**−0.1276**

Diagrams representing the interrelation between non-dimensional free vibration frequencies $\varOmega$ and forces $\overline{N}_1$, $\overline{N}_2$ are presented in Fig. 4 and Fig. 5. In these diagrams, the values of $\overline{N}_1$, $\overline{N}_2$, $\varOmega$ should be multiplied by 10^{-3}.

Numerical calculations were carried out for existing engineering structures. We into account a concrete plate reinforced by a system of periodically spaced **I**-bars. Thus, we dealt with a structure which has practical meaning in civil engineering. The values of the in-plane stresses N_{11}, N_{22} are restricted to those

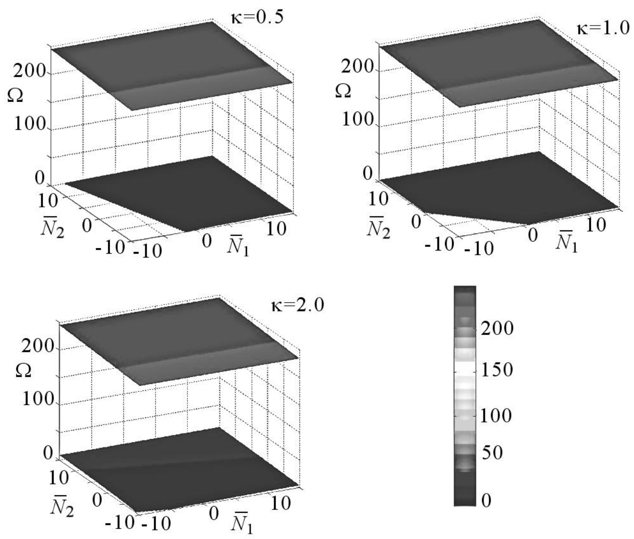

Fig. 4. Diagrams of interrelations between lower and higher free vibration frequencies Ω_1, Ω_2 and forces $\overline{N}_1$ and $\overline{N}_2$

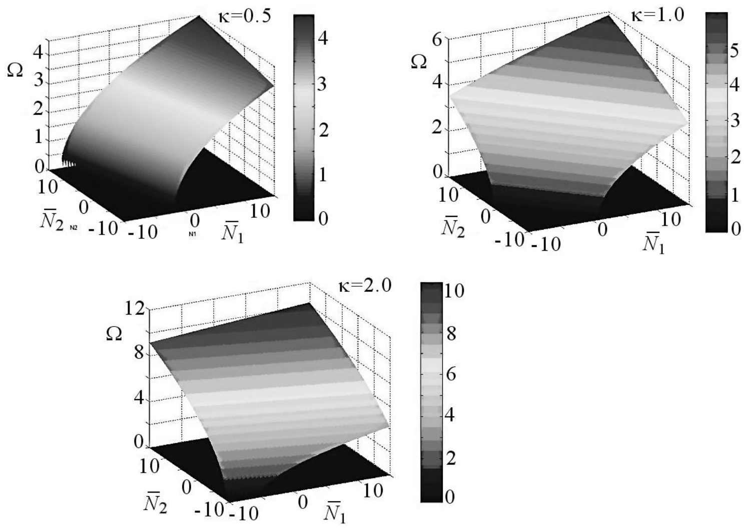

Fig. 5. Interrelation between the lower and higher free vibration frequency Ω_1 and stresses $\overline{N}_1$ and $\overline{N}_2$

which do not exceed the permissible stress. It has to be mentioned that the plates under consideration satisfy, in the exact manner, all assumptions of the theory proposed in this contribution.

Under the aforementioned restrictions, lower free vibration frequencies nearly coincide with those resulting from the homogenisation theory. Higher free vibration frequencies, which cannot be calculated by the homogenisation method, do not have any meaning from the engineering point of view. However, discussion on formulae (5.6) leads to the conclusion that higher vibration frequencies can be calculated and applied provided that we shall deal with some new composite material having suitable material properties.

5.2. Stability problem

Let us restrict the considerations to quasi-stationary processes and assume that the plate is subjected to compression N_{11} along the x_1 axis. This compression have to be proportional to the stress N_{22} along the x_2 axis; denote then $\gamma = N_{22}/N_{11}$. We conclude that nontrivial solutions to (5.2) exist provided that

$$\begin{aligned}
&[\langle g_{,1}^2\rangle k + l^2\beta_n^2\gamma(k\langle\overline{g}^2\rangle - \beta_n^2\langle\overline{g}\rangle^2\gamma)]\overline{N}_1^2 + \\
&-\{\langle g_{,1}^2\rangle(D_o + k) + l^2[\beta_n^2\langle\overline{g}^2\rangle(D_o + k\overline{D}_2) - 2\beta_n^4\langle\overline{g}\rangle^2\overline{D}_2\gamma - kB]\}\overline{N}_1 + \\
&+\langle g_{,1}^2\rangle D_o + l^2[\beta_n^2\langle\overline{g}^2\rangle D_o\overline{D}_2 - \beta_n^4\langle\overline{g}\rangle^2\overline{D}_2^2 - D_oB] = 0
\end{aligned} \tag{5.9}$$

where

$$D_o = \alpha_m^2\overline{D}_1 + \beta_n^2\overline{D}_2 \qquad\qquad k = \alpha_m^2 + \beta_n^2\gamma$$

Real roots of Eqs (5.9) represent critical values of the edge in-plane loadings for the stability problem under consideration. It can be observed that in the framework of the proposed model we deal with two values of the critical force $N_{11,kr}$. This situation is quite different from those resulting from the well-known typical procedures leading to the evaluation of the critical force. Generalization of the well known analysis of a typical stability problem leads in the considered case to the following results

- $N_{11,kr} = \langle D_1\rangle D_o/k$ for the homogenized model
- $N_{11,kr} = \langle D_1\rangle D_o/k + O(l)^2$ for the model describing the length-scale effect.

The aim of the foregoing numerical analysis of equation (5.9) is to determine the interrelation between the non-dimensional critical force $\overline{N}_{11,kr}$ and parameter $\gamma = N_{22}/N_{11}$. It is easy to verify that this interrelation depends on parameters: ν, $\kappa = L_2/L_1$, $\lambda = L_2/d$, $\varepsilon = l/L_2$. Having introduced (5.7) and bearing in mind (5.9), we arrive at the following formula

$$\begin{aligned}
&\Big\{2\overline{k} + \varepsilon^2 n^2\gamma\Big[\overline{k}\Big(\frac{1}{2}+c^2\Big) - n^2c^2\gamma\Big]\Big\}\overline{N}_1^2 + \\
&-\Big\{2(\overline{D}_o+\overline{k}) + \varepsilon^2\Big[n^2\Big(\frac{1}{2}+c^2\Big)\Big(\overline{D}_o\gamma + \overline{k}\frac{a_2}{a_1}\Big) - 2n^4c^2\overline{D}_2\gamma - \overline{kB}\Big]\Big\}\overline{N}_1 + \\
&+2\overline{D}_o + \varepsilon^2\Big[n^2\Big(\frac{1}{2}+c^2\Big)\overline{D}_o\frac{a_2}{a_1} - n^4c^2\overline{D}_2^2 - \overline{D}_o\overline{B}\Big] = 0
\end{aligned} \tag{5.10}$$

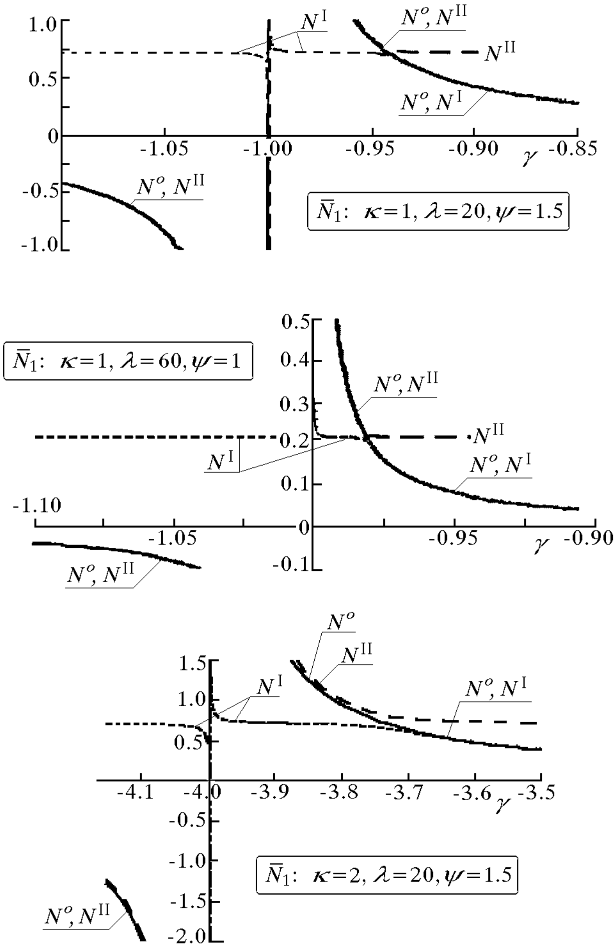

Fig. 6. Diagrams of interrelations between the non-dimensional critical force and the parameter $\gamma = N_{11}/N_{22}$

The shape of the mid-plane is characterized here by the ratio $\kappa = L_2/L_1$, $\kappa = 1$; 0.5; 2 for two cases of the slenderness ratio $\lambda = L_2/d$, $\lambda = 20$ and $\lambda = 60$. At the same time, the parameter $\varepsilon = l/L$ is equal to 0.10. In both cases, the ratio $\psi = E_S I/(H_o l)$ is 0.5; 1,0; 1,5. Subsequent calculations will be carried out for the aforementioned values of parameters.

In Fig. 6, diagrams representing the interrelation between N_{11} and $\gamma = N_{22}/N_{11}$ is presented. In these diagrams N^{I} and N^{II} denote solutions to equation (5.10), and $N^o = D_o/k$ is a non-dimensional critical force which can be derived also from the homogenized plate model.

The diagrams presented in Fig. 6 indicate that for the plate compressed in both directions, i.e. for $\gamma > 0$, the critical force is equal to N^{I}, but the above value is close to N^o obtained from the homogenized model. In this case, the stability analysis based on the proposed model leads to similar results found from classical analysis. The above remark applies to a certain domain of $\gamma \leqslant 0$ as well. Remarkable differences between the critical forces N^0 and N^{I}, N^{II} appear for the parameter γ tending to $-1/\kappa^2$. For example, for a square plate, if the value of the tensile force N_{22} tends to the value of the compressive force N_{11}, we obtain $N^{\mathrm{I}} < N^0$. Thus, the critical force should be calculated from relations obtained within the proposed model, not from the homogenized one.

6. Conclusions

In this contribution, a new averaged 2D-model of uniperiodic medium-thickness elastic plates is proposed. The model is described by a system of equations with constant coefficients. In contrast to the homogenized model, cf. Lewiński (1991), the proposed model is derived by using a tolerance averaging technique and describes the effect of the period-length on the overall plate behaviour. Moreover, this model takes into account the new effect caused by the interrelation between the in-plane forces and displacement fluctuations due to uniperiodicity of the plate structure, and is a certain generalization of that introduced in Baron (2002), where the above effect was neglected. The obtained theoretical results were applied to stability analysis of a rectangular uniperiodic plate. It was shown that, for some special cases, the value of the critical force obtained from the proposed model were smaller than values derived from the homogenized plate model. However, the specification of those special cases has rather a qualitative than quantitative significance. At the same time, the effect of the coupling between the in-plane forces and displacement fluctuations due to uniperiodicity of the plate structure does not play any role as far as the plate stability is concerned.

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Pewien model wstępnie napiętych uniperidycznych płyt średniej grubości

Streszczenie

Celem pracy jest rozszerzenie i pewne uogólnienie dwywymiarowego modelu średniej grubości (typu Reissnera) sprężystych płyt o jednokierunkowej strukturze periodycznej zaproponowanego przez Barona (2002). Zastosowano metodę uśredniania tolerancyjnego opisanego np. przez Woźniaka i Wierzbickiego (2000). Metoda ta pozwala uwzględnić wpływ wymiaru powtarzalnego segmentu płyty (okresu powtarzalności) na jej makromechaniczne własności, czyli tzw. efekt skali. Uzyskane ogólne równania przekształcono do postaci dogodnej do analizy płyt technicznie anizotropowych. Przeprowadzono analizę zagadnienia drgań swobodnych oraz stateczności płyt stalowo-betonowych stosowanych w budownictwie. Uzyskano dodatkową relację do obliczania sił krytycznych.

Manuscript received April 27, 2004; accepted for print September 1, 2004