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THE TRANSIENT TEMPERATURE FIELD IN A RECTANGULAR AREA WITH MOVABLE HEAT SOURCES AT ITS EDGE

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The paper provides an exact solution to a nonstationary two-dimensional heat transfer problem where heat sources move along the edge of the area. Finite Fourier transforms are applied to find the solution. It is given as a sum of four parts. The investigations aim at the determination of the temperature distribution in a brake drum while the vehicle rolls down a slope at a constant velocity. Brake linings, brought into frictional contact with the drum in braking, constitute moving heat sources. Due to the nature of the process under examination, it is possible to assume that the heat transfer is two-dimensional. The dimensions of the brake drum (the internal radius to external radius ratio is approx. 0.95) and simplifications allow one to model it as a rectangular area.

Key words: moving heat source, 2D heat conduction, finite Fourier transform, brake drum

1. Introduction

The problem of the identification of a temperature field generated by a moving heat source has been investigated in numerous papers. In a paper by Grysa (1977a), the author considered the temperature distribution in a long circular cylinder whose lateral surface was affected by temperature being a function of the angular coordinate. The cylinder itself rotated around its axis with a constant angular velocity ω. The problem was analysed in the cylindrical coordinate system r, φ, z. Because points were regarded to be located at a sufficient distance from both ends of the cylinder, it was assumed that the temperature distribution was a function of time t and spatial variables r and φ. The problem was solved by applying Hankel transformations. In a paper by

Drzewicki *et al.* (1977), the authors investigated the temperature field in an infinite circular cylinder when its lateral surface was affected by a temperature field distributed according to a function $T = T(\varphi, t)$. The cylinder rotated around its axis with a constant angular velocity ω. The problem was solved by applying Green's functions. The solution was presented for certain special cases by appropriately specifying the form of the function $T = T(\varphi, t)$. The calculations were made for three forms of the boundary condition: a constant function; a square function of $\varphi - \omega t$ at a part of the boundary and zero at the remaining boundary part; a cosine function at a part of the boundary with the argument $\varphi - \omega t$ and zero elsewhere. In a paper by Grysa and Legutko (1981), the authors determined the intensity of a moving heat source in the contact area between the blade and the grinding detail. The temperature was calculated by applying an analytic method based on inverse heat conduction problems. Two shapes of a grinding object were taken into account: a circular disk and a cubicoid.

2. Mathematical model

The aim of the present paper is to provide an exact solution to a transient heat transfer problem when a part of the boundary of the area under analysis is heated with a moving heat source. For the sake of calculations, a boundary condition of the second kind is adopted. The condition can be expressed in a form $-\lambda \partial T/\partial n = q_n$, where λ denotes the thermal conductivity coefficient [W/mK], $\partial T/\partial n$ – derivative in the direction perpendicular to the surface of the body and directed outwards, $-q_n = \overline{q}_n f(x,t)$ – normal component of the heat flux density [W/m^2], $\overline{q}_n$ – its extreme value at the contact of the source with the body $(\overline{q}_n > 0)$, $f(x,t)$ – polynomial function characterizing the type of the heat flux density distribution $(f(x,t) > 0)$.

The determination of the temperature distribution in the brake drum is carried out when velocity of the vehicle riding down a slope is being reduced. It is assumed that the road inclination angle α is constant and velocity of the vehicle is also constant. Brake linings fixed to brake shoes that come into frictional contact with the brake drum while braking are treated as moving heat sources. For the purpose of formulating a mathematical model, the following simplifying assumptions were made:

1. The temperature is constant along the entire width of the drum. It means that the direction of the heat transfer is also assumed to be two-dimensional.

2. Due to a large diameter of the drum in relation to its thickness, the annulus is thought to be unrolled into a flat rectangular area. What is investigated is the temperature distribution within a rectangular area of the length $\bar{l}$ and width $\bar{b}$.
3. Two heat sources, each of the length $\bar{a}$ equal to the length of the brake lining and the width equal to the drum width, move at a constant velocity $\bar{v}$ in a periodic manner. The density distribution of the heat flux at the contact of the brake lining with the drum is modelled with a polynomial function. Figure 1 presents the graph of the function $-q_n$ at a fixed time instant.

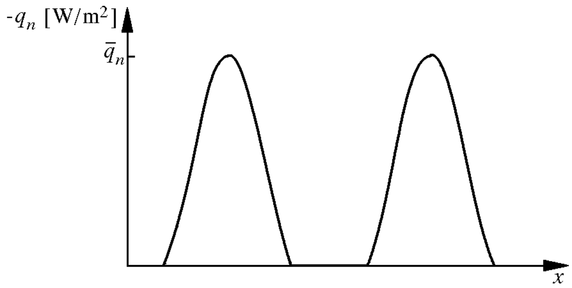

Fig. 1. Distribution of the heat flux on the heated body surface at a fixed time instant

4. Except for the brake lining contact with the brake drum (also on the opposite side), the area is assumed to be thermally insulated.
5. Repeatability of the process resulting from the vehicle wheel rotation is achieved by the adoption of boundary conditions of the fourth kind at the edges of the rectangle, which are perpendicular to the side affected by the heat source.
6. It is assumed that at the initial time instant, the temperature of the brake drum and the environment is constant and equals Θ_0.
7. The brake drum is made of a homogenous and isotropic material.
8. It is assumed that the thermal conductivity coefficient κ and thermal diffusivity k of the brake drum do not depend on temperature.

The problem will be formulated mathematically in a dimensionless form. Dimensionless (reduced) temperature is defined as follows: $T = (\Theta - \Theta_0)/\frac{\bar{q}_n \bar{b}}{\lambda}$, where Θ denotes actual temperature [K], Θ_0 – actual temperature at the initial moment [K], $\bar{q}_n$ – maximum value of the heat flux density resulting from the action of the moving heat source [W/m^2], $\bar{b} = r_z - r_w$ – thickness of the body [m], r_z - the external radius of the brake drum, r_w – internal radius of the brake drum, λ – thermal conductivity coefficient [W/mK]. Dimensionless coordinates are expressed in the following way: $x = \overline{x}/\overline{b}$, $y = \overline{y}/\overline{b}$, $t = \kappa\overline{t}/\overline{b}^2$ where κ is the thermal diffusivity coefficient [m^2/s]. We define dimensionless parameters: $l = \overline{l}/\overline{b}$, $a = \overline{a}/\overline{b}$, $b = 1$, $v = \overline{v}\overline{b}/\kappa$ ($\overline{l} = \pi(r_z - r_w)$ is length of the body [m], $\overline{a}$ – length of the source [m], $\overline{v}$ – actual velocity of the source [m/s]).

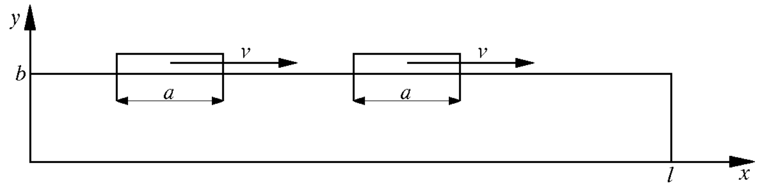

Fig. 2. A model of the system for temperature identification in heating with moving heat sources

The following dimensionless form of the problem under analysis is achieved

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} - \frac{\partial T}{\partial t} = 0 \tag{2.1}$$

for $(x,y) \in \Omega$, $t > 0$, $\Omega = \{(x,y) \in R^2 : 0 < x < l,\ 0 < y < b\}$.

The initial and boundary conditions are

$$\begin{gathered} T(x,y,0) = 0 \qquad\qquad \frac{\partial T}{\partial y}(x,0,t) = 0 \\ \frac{\partial T}{\partial y}(x,b,t) = f(x,t) = \\ = \begin{cases} 0 & \text{for} \quad X < \frac{l}{4} - \frac{a}{2} \\ \Big(\frac{2}{a}\Big)^4\Big(-\frac{l}{4} - \frac{a}{2} + X\Big)^2\Big(-\frac{l}{4} + \frac{a}{2} + X\Big)^2 & \text{for} \quad \frac{l}{4} - \frac{a}{2} \leqslant X < \frac{l}{4} + \frac{a}{2} \\ 0 & \text{for} \quad \frac{l}{4} + \frac{a}{2} \leqslant X < \frac{3l}{4} - \frac{a}{2} \\ \Big(\frac{2}{a}\Big)^4\Big(-\frac{3l}{4} - \frac{a}{2} + X\Big)^2\Big(-\frac{3l}{4} + \frac{a}{2} + X\Big)^2 & \text{for} \quad \frac{3l}{4} - \frac{a}{2} \leqslant X < \frac{3l}{4} + \frac{a}{2} \\ 0 & \text{for} \quad \frac{3l}{4} + \frac{a}{2} \leqslant X < l \end{cases} \end{gathered} \tag{2.2}$$

where $X = (x - vt) mod.l$ gives the remainder of division of $(x - vt)$ by l.

Additionally, consistency conditions are required in the following form

$$T(0,y,t) = T(l,y,t) \qquad\qquad \frac{\partial T}{\partial x}(0,y,t) = \frac{\partial T}{\partial x}(l,y,t) \tag{2.3}$$

Condition $(2.2)_3$ describes the distribution of the heat transfer generated by the moving source. The condition can be presented in a form better suited for calculations, that is, the one in which the function f is expanded into a Fourier series

$$\begin{aligned}\frac{\partial T}{\partial y}(x,b,t) = \frac{16a}{15l} - \sum_{n=1}^{\infty}\Big\{\frac{16l^2}{a^4(n\pi)^5}\Big(\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{2}\Big)\cdot\\ \cdot\Big[3an\pi l\cos\frac{an\pi}{l} + (-3l^2+(an\pi)^2)\sin\frac{an\pi}{l}\Big]\cos[\lambda_n(x-vt)]\Big\}\end{aligned}$$

where $\lambda_n = 2\pi n/l$.

3. Solution of the heat conduction equation

A solution to equation (2.1) is sought in a form of a Fourier series with respect to the variable $x - vt$

$$T(x,y,t) = \frac{1}{l}T_0(y,t) + \frac{2}{l}\sum_{n=1}^{\infty}\{T_{n1}(y,t)\cos[\lambda_n(x-vt)] + T_{n2}(y,t)\sin[\lambda_n(x-vt)]\} \tag{3.1}$$

Substitution of (3.1) in (2.1) results in a system of equations with unknown coefficients T_0, T_{n1}, T_{n2} in the form

$$\begin{aligned}&\frac{\partial^2 T_0}{\partial y^2} - \frac{\partial T_0}{\partial t} = 0\\ &\frac{\partial^2 T_{n1}}{\partial y^2} - \lambda_n^2 T_{n1} - \frac{\partial T_{n1}}{\partial t} + \lambda_n v T_{n2} = 0\\ &\frac{\partial^2 T_{n2}}{\partial y^2} - \lambda_n^2 T_{n2} - \frac{\partial T_{n2}}{\partial t} - \lambda_n v T_{n1} = 0\end{aligned} \tag{3.2}$$

with conditions

$$\begin{aligned}&T_0(y,0) = 0 \qquad T_{n1}(y,0) = 0 \qquad T_{n2}(y,0) = 0\\ &\frac{\partial T_0}{\partial y}(b,t) = q_0 = \frac{16a}{15}\\ &\frac{\partial T_{n1}}{\partial y}(b,t) = q_{n1} =\\ &= \frac{8l^3}{a^4(n\pi)^5}\Big(\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{2}\Big)\Big\{3an\pi l\cos\frac{an\pi}{l} + [-3l^2+(an\pi)^2]\sin\frac{an\pi}{l}\Big\}\end{aligned} \tag{3.3}$$

$$\frac{\partial T_{n2}}{\partial y}(b,t)=0 \qquad \frac{\partial T_0}{\partial y}(0,t)=0$$
$$\frac{\partial T_{n1}}{\partial y}(0,t)=0 \qquad \frac{\partial T_{n2}}{\partial y}(0,t)=0$$

In order to solve the system of equations (3.2) with conditions (3.3), a finite Fourier transformation is applied (Sneddon, 1951). Therefore the functions T_0, T_{n1}, T_{n2} are assumed to satisfy Dirichlets conditions for a fixed t. With the aforementioned assumptions, the Fourier transformation has the form

$$\overline{T}(k,t)=\int\limits_0^b T(y,t)\cos\alpha_k\,dy \tag{3.4}$$

where $\alpha_k=k\pi/b$, $k=0,1,2,...$ and the function is expressed through its transform by the formula

$$T(y,t)=\frac{1}{b}\overline{T}(0,t)+\frac{2}{b}\sum_{k=1}^{\infty}\overline{T}(k,t)\cos\alpha_k y \tag{3.5}$$

Application of the transformation to the system of equations (3.2) and conditions (3.3) results in a system of differential equations for the variable t

$$\begin{aligned}
&(-1)^k q_0-\alpha_k^2\overline{T}_0(k,t)-\frac{d\overline{T}_0}{dt}(k,t)=0\\
&(-1)^k q_{n1}-(\alpha_k^2+\lambda_n^2)\overline{T}_{n1}(k,t)-\frac{d\overline{T}_{n1}}{dt}(k,t)+\lambda_n v\overline{T}_{n2}(k,t)=0\\
&-(\alpha_k^2+\lambda_n^2)\overline{T}_{n2}(k,t)-\frac{d\overline{T}_{n2}}{dt}(k,t)-\lambda_n v\overline{T}_{n1}(k,t)=0
\end{aligned} \tag{3.6}$$

Solutions to the system must be equal to zero for $t=0$. The system of equations (3.6) has the following solutions

$$\begin{aligned}
&\overline{T}_0(0,t)=q_0t\\
&\overline{T}_0(k,t)=\frac{(-1)^k q_0}{\alpha_k^2}(1-\mathrm{e}^{-\alpha_k^2 t}) && k>0\\
&\overline{T}_{n1}(k,t)=-\mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}(B_{kn}\cos\lambda_n vt-A_{kn}\sin\lambda_n vt)+B_{kn} && k\geqslant 0\\
&\overline{T}_{n2}(k,t)=-\mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}(A_{kn}\cos\lambda_n vt+B_{kn}\sin\lambda_n vt)-A_{kn} && k\geqslant 0
\end{aligned} \tag{3.7}$$

where

$$A_{kn}=\frac{(-1)^k q_{n1}v\lambda_n}{(v\lambda_n)^2+(\lambda_n^2+\alpha_k^2)^2} \qquad B_{kn}=\frac{(-1)^k q_{n1}(\lambda_n^2+\alpha_k^2)}{(v\lambda_n)^2+(\lambda_n^2+\alpha_k^2)^2}$$

By virtue of (3.5), we arrive at the following solution to the system of equations (3.2)

$$
\begin{aligned}
T_0(y,t) &= \frac{q_0 t}{b} + \frac{2}{b}\sum_{k=1}^{\infty}\frac{(-1)^k q_0}{\alpha_k^2}(1-\mathrm{e}^{-\alpha_k^2 t})\cos\alpha_k y\\
T_{n1}(y,t) &= \frac{1}{b}[(-\mathrm{e}^{-\lambda_n^2 t}(B_{0n}\cos\lambda_n vt - A_{0n}\sin\lambda_n vt) + B_{0n}] +\\
&+\frac{2}{b}\sum_{k=1}^{\infty}(-\mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}(B_{kn}\cos\lambda_n vt - A_{kn}\sin\lambda_n vt) + B_{kn})\cos\alpha_k y\\
T_{n2}(y,t) &= \frac{1}{b}[(\mathrm{e}^{-\lambda_n^2 t}(A_{0n}\cos\lambda_n vt + B_{0n}\sin\lambda_n vt) - A_{0n}] +\\
&+\frac{2}{b}\sum_{k=1}^{\infty}(\mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}(A_{kn}\cos\lambda_n vt + B_{kn}\sin\lambda_n vt) - A_{kn}]\cos\alpha_k y
\end{aligned}
\tag{3.8}
$$

Then, the problem described by equations (2.1)-(2.3) has the following solution

$$
\begin{aligned}
T(x,y,t) &= \frac{q_0 t}{bl} + \frac{2}{bl}\sum_{k=1}^{\infty}\frac{(-1)^k q_0\cos\alpha_k y}{\alpha_k^2}(1-\mathrm{e}^{-\alpha_k^2 t}) +\\
&+\frac{2}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}\cos\gamma_{n0}}{\lambda_n^2}\Big\{\cos[\lambda_n(x-vt)+\gamma_{n0}] - \mathrm{e}^{-\lambda_n^2 t}\cos[\lambda_n x+\gamma_{n0}]\Big\} +\\
&+\frac{4}{bl}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k q_{n1}\cos\gamma_{nk}\cos\alpha_k y}{\lambda_n^2+\alpha_k^2}\cdot\\
&\cdot\Big\{\cos[\lambda_n(x-vt)+\gamma_{nk}] - \mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}\cos(\lambda_n x+\gamma_{nk})\Big\}
\end{aligned}
\tag{3.9}
$$

where

$$
\cos\gamma_{nk} = \frac{\lambda_n^2+\alpha_k^2}{\sqrt{(v\lambda_n^2)^2+(\lambda_n^2+\alpha_k^2)^2}}
$$

4. Analysis of the obtained solution

Due to the nature of the thermal field changes in time, a division of the transient state is introduced following Kondratiev (1954) into:

- a purely nonstationary heating process $(t < 0.5)$
- a regular heating process $(t > 0.5)$.

In the purely nonstationary heating process, the temperature field depends on physical properties of the body, its geometry, dimensions and also initial and boundary conditions. The regular heating process is already a well-established one, in which the time-space temperature distribution depends on the body geometry, its dimensions, physical properties and boundary conditions. The impact of initial conditions on the temperature, however, is negligibly small. For the sake of analysis, relation (3.9) is presented in a form of the sum

$$T(x,y,t) = T^K(x,y,t) + T^N(x,y,t) + T^B(x,y,t) + T^S(x,y,t) \tag{4.1}$$

where

$$T^K(x,y,t) = \frac{q_0 t}{bl} \tag{4.2}$$

$$\begin{aligned}T^N(x,y,t) &= -\frac{2}{bl}\sum_{k=1}^{\infty}\frac{(-1)^k q_0\cos\alpha_k y}{\alpha_k^2}\mathrm{e}^{-\alpha_k^2 t} - \frac{2}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}}{\lambda_n^2}\mathrm{e}^{-\lambda_n^2 t}\cos\lambda_n x + \\ &-\frac{4}{bl}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k q_{n1}\cos\alpha_k y}{\lambda_n^2+\alpha_k^2}\mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}\cos\lambda_n x\end{aligned} \tag{4.3}$$

$$\begin{aligned}T^S(x-vt,y) &= \frac{2}{bl}\sum_{k=1}^{\infty}\frac{(-1)^k q_0\cos\alpha_k y}{\alpha_k^2} + \frac{2}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}}{\lambda_n^2}\cos\lambda_n(x-vt) + \\ &+\frac{4}{bl}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k q_{n1}\cos\alpha_k y}{\lambda_n^2+\alpha_k^2}\cos\lambda_n(x-vt)\end{aligned} \tag{4.4}$$

$$\begin{aligned}T^B(x,y,t) &= -\frac{2}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}\sin\gamma_{n0}}{\lambda_n^2}\Big\{\sin[\lambda_n(x-vt)+\gamma_{n0}] + \\ &-\mathrm{e}^{-\lambda_n^2 t}\sin[\lambda_n x+\gamma_{n0}]\Big\} - \frac{4}{bl}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k q_{n1}\sin\gamma_{nk}\cos\alpha_k y}{\lambda_n^2+\alpha_k^2}\cdot \\ &\cdot\Big\{\sin[\lambda_n(x-vt)+\gamma_{nk}] - \mathrm{e}^{-(\lambda_n^2+\alpha_k^2)t}\sin(\lambda_n x+\gamma_{nk})\Big\}\end{aligned} \tag{4.5}$$

and

$$\sin\gamma_{nk} = \frac{v\lambda_n}{\sqrt{(v\lambda_n^2)^2 + (\lambda_n^2+\alpha_k^2)^2}}$$

By applying formula (5.14) in Grysa (1977b) with $x = b$, equation (4.4) takes the form

$$\begin{aligned} T^S(x-vt,y) &= \frac{2}{bl}\Big\{q_0\Big[\frac{\eta(y-b)}{2}(b-y)+\frac{2+3y^2-6b+3b^2}{12}\Big]+ \\ &+\sum_{n=1}^{\infty}\frac{q_{n1}}{\lambda_n^2}\cos\lambda_n(x-vt)\Big\}+\frac{4}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}\cos\lambda_n(x-vt)}{2\lambda_n\sinh\lambda_n}\cdot \\ &\cdot\Big\{\eta(y-b)\sinh\lambda_n\sinh[\lambda_n(b-y)]+\cosh\lambda_n y\cosh[\lambda_n(1-b)]\Big\} \end{aligned} \tag{4.6}$$

where $\eta(z)$ denotes the Heaviside function.

In formula (4.1),its individual terms have the following sense:

- $T^K(x,y,t)$ is a linear term representing the heat accumulation within the area under analysis
- $T^N(x,y,t)$ describes temperature changes at individual points of the area, which result from the heating, and which are not affected by motion of the source
- $T^B(x,y,t)$ describes the thermal inertia caused by the heating
- $T^S(x-vt,y)$ is actually a function of two variables: y and the difference $x-vt$.

The change in time t by Δt leads to a similar change in the value of the function T^S as the change in x by $\Delta x = -v\Delta t$.

The temperature distribution expressed by formula (4.3) may be interpreted as a temperature field generated by a stationary source after a lapse of very long time described in the system x,y. A characteristic feature of the term is the fact that it does not depend on physical properties of the body, its dimensions and initial conditions. In a purely nonstationary heating process, none of the expressions present in relations (4.1), (4.2), (4.3) and (4.5) can be excluded. In a regular heating process, the function assumes on a much simpler form. All expressions containing $\exp[-(\lambda_n^2+\alpha_k^2)t]$, except for $\exp(-\alpha_1^2 t)$, can be excluded. The expressions highlighted in relation (4.1) take the forms

$$T^N(x,y,t) = \frac{2}{bl}\frac{q_0\cos\alpha_1 y}{\alpha_1^2}(\mathrm{e}^{-\alpha_1^2 t}) \tag{4.7}$$

$$\begin{aligned} T^B(x,y,t) = T^B(x,y,t) &= -\frac{2}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}\sin\gamma_{n0}}{\lambda_n^2}\sin[\lambda_n(x-vt)+\gamma_{n0}]+ \\ -\frac{4}{bl}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}&\frac{(-1)^k q_{n1}\sin\gamma_{nk}\cos\alpha_1 y}{\lambda_n^2+\alpha_1^2}\sin[\lambda_n(x-vt)+\gamma_{nk}] \end{aligned} \tag{4.8}$$

Expressions (4.2), (4.6) do not change their forms. Therefore, for $t > 0.5$, relation (4.1) can be presented as follows

$$T(x,y,t) = T^K(x,y,t) + T^N(y,t) + T^B(x-vt,y) + T^S(x-vt,y) \tag{4.9}$$

With time running, the second of the highlighted expressions quickly approaches zero, the third and the fourth are actually functions of two variables; they describe a quasi-steady state.

$T(x,y,t) \to \infty$, $t \to \infty$ because $T^K(x,y,t) \to \infty$, $t \to \infty$ and $T^N(x,y,t)$, $T^B(x,y,t)$, $T^S(x,y,t)$ are limited. The component $T^K(x,y,t)$ describes the accumulation of heat collected inside the body. The formulated model and the calculated temperature function are applied to determine the temperature distribution in the brake drum over a short time of the vehicle braking.

The velocity v significantly affects the thermal inertia, because the value of the function $T^B(x-vt,y)$ depends on $\sin\gamma_{nk}$. For velocities satisfying the condition $v \ll 1$, we have $\sin\gamma_{nk} \ll 1$ and $T^B(x-vt,y) \approx 0$. The temperature field will be then expressed by the relation

$$T(x,y,t) \approx T^K(x,y,t) + T^N(y,t) + T^S(x-vt,y) \tag{4.10}$$

which indicates that the movement of the source does not really affect the way in which heat penetrates the inside of the area. Figures 9a,b present the temperature distribution for the dimensionless velocity $v = 0.001$. When the velocity is sufficiently high, then $\cos\gamma_{nk} \approx 0$ and the terms with $\cos\gamma_{nk}$ disappear in expression (3.9), and the temperature may be approximated with the formula

$$T(y,t) = \frac{q_0 t}{bl} + \frac{2}{bl}\sum_{k=1}^{\infty} \frac{(-1)^k q_0 \cos\alpha_k y}{\alpha_k^2}(1 - \mathrm{e}^{-\alpha_k^2 t}) \tag{4.11}$$

It is apparent from formula (4.11) that the temperature depends only on one spatial variable, so the temperature distribution is one-dimensional and heat flow is one-directional. The same solution would be achieved if it was assumed that the source with the consistent intensiveness equal to q_0/l affected the entire surface of the body, i.e. there was an insulation on the opposite surface and the heat conduction equation had the form

$$\frac{\partial^2 T}{\partial y^2} - \frac{\partial T}{\partial t} = 0 \qquad \text{for } y \in (0,b), \quad t > 0$$

In formula (4.11), q_0 is formulated as follows

$$q_0 = \int_0^l f(x,t)\,dx \tag{4.12}$$

where $f(x,t)$ is expressed by relation $(2.2)_3$. Figures 5a,b present the temperature distribution for the dimensionless velocity $v = 2.1 \cdot 10^4$. For $t > 1.5$, the expression $T^N(y,t)$ introduces a correction of the order $10^{-8} q_0$ into formula (4.9), which can be neglected. Formula (4.9) has the form

$$T(x,y,t) \approx T^K(x,y,t) + T^B(x - vt, y) + T^S(x - vt, y) \tag{4.13}$$

Upon introduction of a notion of the characteristic time $\tau_0 = 1/\alpha_1^2$ (it is the time that characterises the heating rate of the brake drum in the regular heating process), one can notice that the condition $t > 0.5$ means $t > (\pi/b)^2 \tau_0/2$, and $t > 1.5$ for which the expression $T^N(y,t)$ can be excluded in formula (4.9), means $t > 3(\pi/b)^2 \tau_0/2$. When the source is not moving $(v = 0)$, relation (4.1) is transformed into the form

$$T(x,y,t) = T^K(x,y,t) + T^N(x,y,t) + T^S(x,y) \tag{4.14}$$

where

$$\begin{aligned} T^S(x,t) &= \frac{2}{bl}\Big\{q_0\Big[\frac{\eta(y-b)}{2}(b-y) + \frac{2+3y^2-6b+3b^2}{12}\Big] + \sum_{n=1}^{\infty}\frac{q_{n1}}{\lambda_n^2}\cos\lambda_n x\Big\} + \\ &+ \frac{4}{bl}\sum_{n=1}^{\infty}\frac{q_{n1}\cos\lambda_n x}{2\lambda_n \sinh\lambda_n}\Big\{\eta(y-b)\sinh\lambda_n \sinh[\lambda_n(b-y)] + \\ &+ \cosh\lambda_n y \cosh[\lambda_n(1-b)]\Big\} \end{aligned} \tag{4.15}$$

T^K, T^N do not change their forms.

5. Numerical example

Let us consider a thermal field in a cast-iron brake drum of a lorry riding down a road inclined at an angle $\alpha = 10°$ with a constant velocity. We assume the following numerical data: $\lambda = 50$ W/mK, $\kappa = 0.125 \cdot 10^{-4}$ m^2/s, $\overline{l} = 1.3$ m, $\overline{b} = 0.013$ m, $\overline{a} = 0.3$ m, $\varTheta_0 = 293$ K, $\overline{q}_n = (\sigma G \delta \overline{v} \sin\alpha)/(2S)$, $\delta = 0.4$ – ratio of braking force acting on front wheels to braking force, $\sigma = 0.95$ – heat distribution between brake lining and drum, $\alpha = 10°$ – road inclination angle, $\overline{v}$ – vehicle velocity [m/s^2], $S = 0.093$ m^2 – area of contact between two brake linings and brake drum, $G = 108 \cdot 10^3$ N – vehicle weight.

The data and nomenclature were taken from the literature on the subject, see Łukomski *et al.* (1976), Wrzesiński (1978).

The temperature field was calculated for a source moving with velocities: $\overline{v} = 72$ km/h, $\overline{v} = 10.8$ km/h and $\overline{v} \approx 0$ km/h. The $\overline{q}_n = 7.7 \cdot 10^5$ W/m^2 was adopted for a source moving with a velocity $\overline{v} = 72$ km/h. Figure 3 presents the temperature distribution through the entire drum thickness at the moment $\overline{t} = 0.02$ s. Figures 4a,b present the temperature distribution in a fragment of the drum close to the heat-affected surface at the same moment.

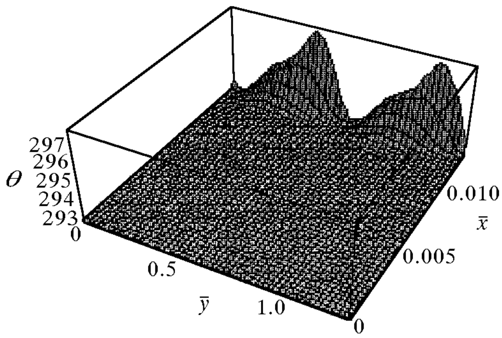

Fig. 3. The temperature distribution at the moment $\overline{t} = 0.02$ s for the velocity $\overline{v} = 72$ km/h

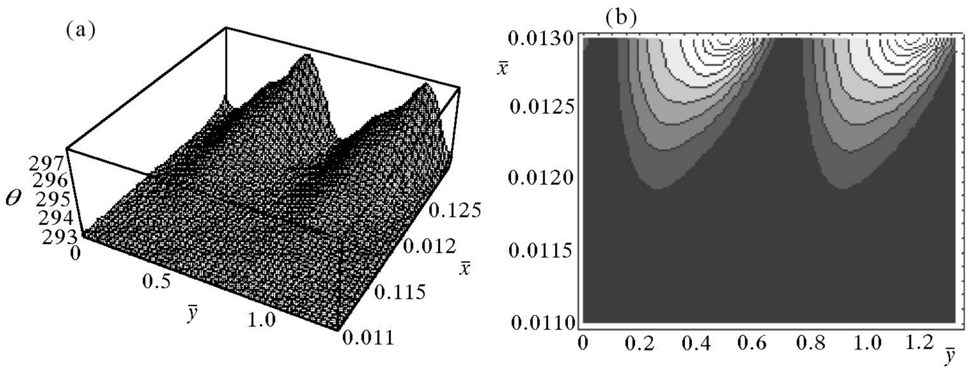

Fig. 4. (a) The temperature distribution in a fragment of the drum close to the heat-affected surface at the moment $\overline{t} = 0.02$ s for the velocity $\overline{v} = 72$ km/h; (b) a contour diagram

Figures 5a,b present the temperature distribution through the entire drum thickness at the moment $\overline{t} = 30$ s. Figures 6a,b present the temperature distribution in a fragment of the drum close to the heat-affected surface at the same moment.

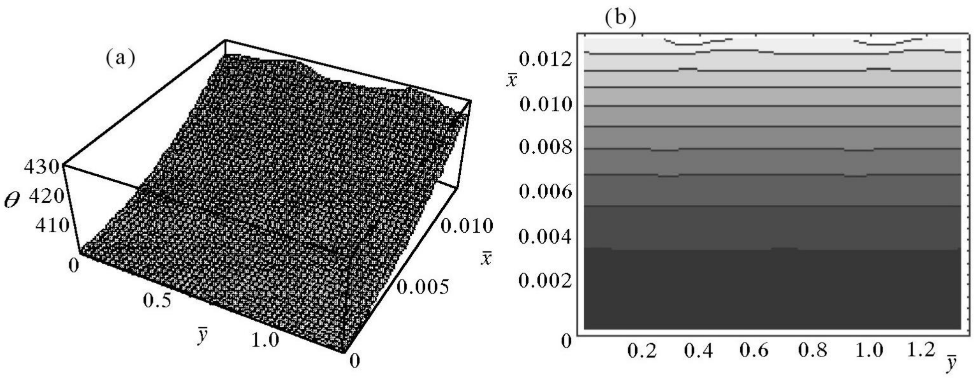

Fig. 5. (a) The temperature distribution at the moment $\bar{t} = 30$ s for the velocity $\bar{v} = 72$ km/h; (b) a contour diagram

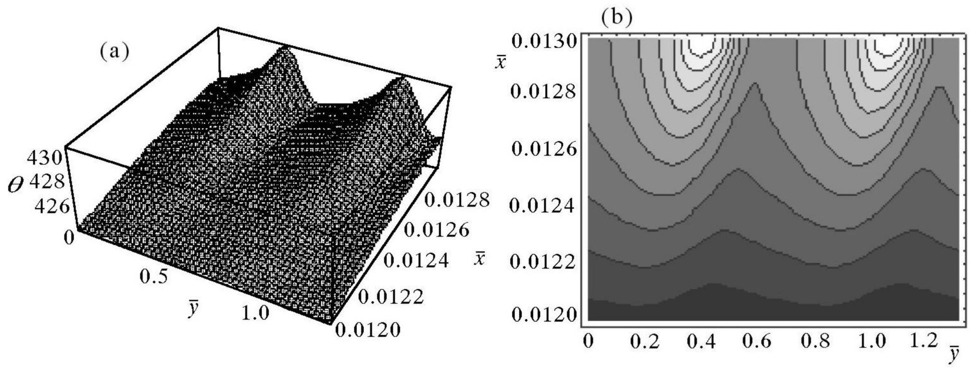

Fig. 6. (a) The temperature distribution in a fragment of the drum close to the heat-affected surface at the moment $\bar{t} = 30$ s for the velocity $\bar{v} = 72$ km/h; (b) a contour diagram

The value of $\bar{q}_n = 1.2 \cdot 10^5$ W/m^2 is adopted for a source moving with the velocity $\bar{v} = 10$ km/h. Figures 7a,b present the temperature distribution through the entire drum thickness at the moment $\bar{t} = 30$ s. Figures 8a,b present the temperature distribution in a fragment of the drum close to the heat-affected surface at the same moment.

The $\bar{q}_n = 0.04$ W/m^2 is adopted for a source moving with the velocity $\bar{v} \approx 0$ km/h ($\bar{v} = 10^{-7}$ km/h). Figures 9a,b present the temperature distribution at the moment $\bar{t} = 20$ s.

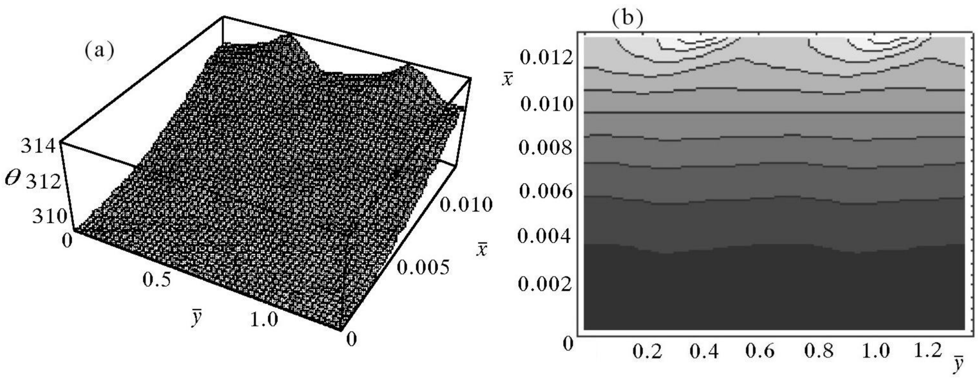

Fig. 7. (a) The temperature distribution at the moment $\overline{t} = 30$ s for the velocity $\overline{v} = 10$ km/h; (b) a contour diagram

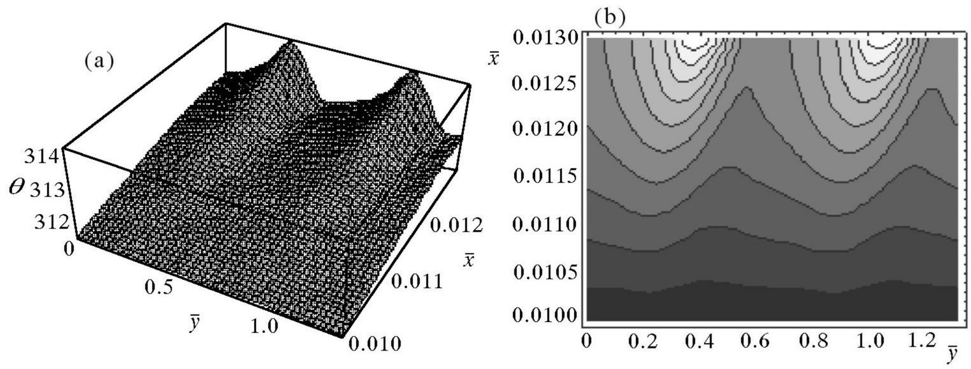

Fig. 8. (a) The temperature distribution in a fragment of the drum close to the heat-affected surface at the moment $\overline{t} = 30$ s for the velocity $\overline{v} = 10$ km/h; (b) a contour diagram

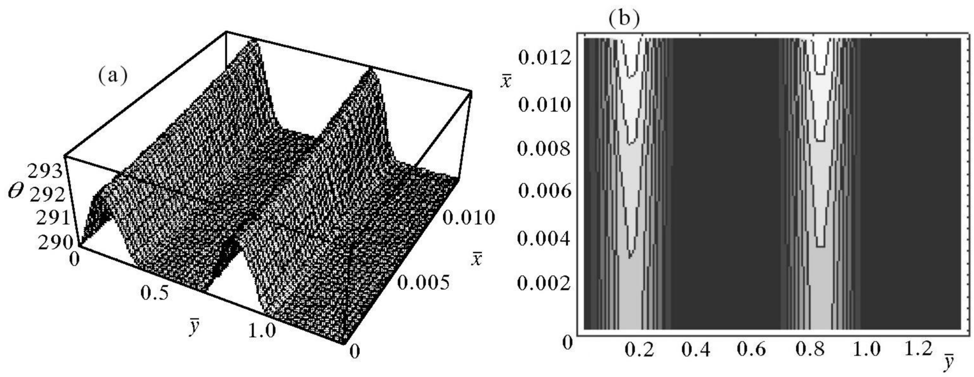

Fig. 9. (a) The temperature distribution at the moment $\overline{t} = 20$ s for the velocity $\overline{v} \approx 0$ km/h; (b) a contour diagram

6. Conclusions

It is apparent from Fig. 1 to fig. 9 that the postulates adopted in the second section of the paper have been satisfied, i.e. at the beginning and at the end of the unrolled drum the temperatures and heat fluxes are equal (the tangents to isotherms are parallel). The perpendicularity of isotherms to the surface $\overline{y} = 0.013$ beyond the area affected by the sources indicates the satisfaction of insulation that was assumed to be present there. For a source moving with the velocity $\overline{v} = 10$ km/h, the predicted effect of inertia caused by movement of the sources can be observed, and with the velocity $\overline{v} = 72$ km/h, for the purpose of better visualisation of that effect, some fragments of the figures had to be enlarged. As it was also expected, that effect is not present at a velocity of the sources close to zero, which can be observed in Fig. 9. The heat penetration depth inside the drum depends mostly on velocity and the number of cycles. The slower the velocity is, the deeper the heat penetrates the inside. The opposite holds true for the number of consecutive passes of the sources – the greater it is, the deeper the heat penetrates the inside and is accumulated there, which is represented by the expression T^K expressed by formula (4.2).

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Nieustalone pole temperatury w obszarze prostokąta z ruchomymi źródłami ciepła na jego brzegu

Streszczenie

W pracy rozwiązane zostało, w sposób ścisły, niestacjonarne dwuwymiarowe zagadnienie przepływu ciepła z poruszającymi się źródłami ciepła wzdłuż brzegu obszaru. W celu znalezienia rozwiązania zastosowana została skończona transformata Fouriera. Rozwiązanie podane zostało w postaci sumy czterech składników. Zastosowane zostało do wyznaczenia rozkładu temperatury w bębnie hamulcowym podczas utrzymywania stałej prędkości samochodu zjeżdżającego z pochyłości. Okładziny hamulcowe trąc o bęben hamulca w trakcie hamowania stanowią poruszające się źródła ciepła. Ze względu na charakter badanego procesu można przyjąć, że wymiana ciepła jest dwuwymiarowa. Wymiary bębna hamulcowego (stosunek promienia wewnętrznego do zewętrznego wynosi około 0.95) i poczynione uproszczenia pozwalają modelować go obszarem o kształcie prostokąta.

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