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NON-LINEAR STABILITY OF ELASTIC-PLASTIC CONICAL SHELL UNDER COMBINED LOAD

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The paper presents stability analysis of an elastic-plastic sandwich open conical shell of a circular cross section under combined external load in the form of lateral pressure, longitudinal forces, and shear. The shell consists of two load-carrying faces made of an isotropic, compressible work-hardening material, and they are of different thicknesses and made of different material properties; the core material is of a soft type and it resists transversal forces only. It is also assumed that the shell can be deformed into plastic range before buckling. The flexural stiffness of the faces is taken into account, the Kirchhoff-Love hypotheses hold for the faces, and the active deformation processes are considered. The constitutive relations used in the analysis are those of the incremental Prandtl-Reuss plastic flow theory associated with the Huber-Mises yield condition. The virtual work principle is the basis to obtain the governing stability equations and the Ritz method is used to derive differential equations of the considered problem. An iterative computer algorithm was elaborated to analyse the shells both in the elastic or elastic-plastic prebuckling state of stress.

Key words: stability, yield condition, incremental theory of plasticity

1. Introduction and geometric relations

Layered sandwich shells are commonly used in civil and mechanical engineering and in aviation. They are characterised by light weight and present many other advantages as thermoisolation properties, resistance to heavy loadings, and so on. Typical sandwich structures are composed of two thin flexible carrying facings of equal or different thicknesses; between the faces a core, made of a foam plastic less rigid than the faces, is stiffly placed. The subject under consideration is a shallow open sandwich conical shell (see Fig. 1) loaded by uniformly distributed lateral pressure, longitudinal force, and shear forces applied to the edges.

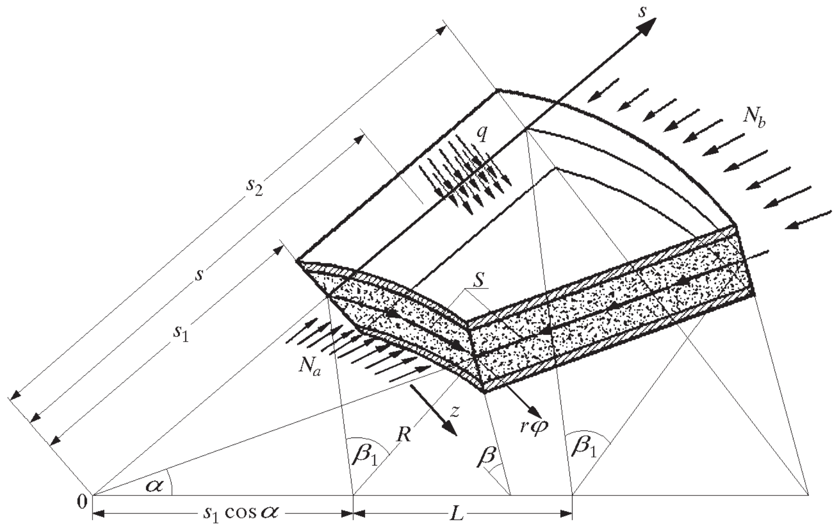

Fig. 1. Open conical sandwich shell

The shell is assumed to be simply supported at all edges. An elastic-plastic model is chosen in the stability analysis of the considered shell. This model is more suitable in the evaluation of the ability of the structure to resist external loads against stability loss.

Many different concepts in geometrically non-linear stability analysis were applied to describe specific features of the stability of elastic-plastic shells, see Croll (1984), Weichert (1984). Open and sandwich conicals shells under axial and lateral loads were also investigated to determine bifurcation loads and equilibrium paths, see Kao (1980), Zielnica (1984, 1987). The review of the most important works on the stability of elastic plastic shells is presented by Bushnell (1982); this review is rather focused on numerical methods. Problems of the linear and nonlinear stability of elastic-plastic conical and cylindrical shells; problem formulation, solution methods and numerical analyses were discussed in papers written by Maciejewski and Zielnica (1984), Zielnica (1987, 1998, 2000, 2001, 2002a,b,c), and Jaskuła and Zielnica (2001). Introduction of geometrical nonlinearity considerably complicates the governing equations. Even the introduction of simplifications following von Kármán's theory enables calculation of large elastic-plastic deflections of shells only by approximate methods.

In this work we consider the influence of shear forces on the stability loss of open sandwich shells. The incremental Prandtl-Reuss plastic flow theory is used to describe stress-strain relations for the considered shell facings. The analysis is based on the energy minimization, where the total strain ε in the shell can be expressed in terms of reference surface strains and changes in curvature, and these reference surface quantities can be then expressed in terms of displacement vector components. The Ritz method is accepted in order to derive the stability equations for the considered shell. The final form of the stability equation, being a function of a deflection function parameter, makes it possible to trace the equilibrium paths for the shell under consideration. An iterative computer algorithm was elaborated which made it possible to analyse the shells in the elastic, elastic-plastic or in totally plastic prebuckling state of stress. The numerical examples showed the influence of principal geometrical and physical parameters of the shell on the stability loss at large deflections.

The following assumptions, usually considered in theories of thin shells, are also made: (a) the displacements are small compared to the length or mean diameter of the shell, but may be of a magnitude comparable to the thickness, (b) there are no normal stresses in the radial direction and lines originally normal to the main (reference) surface remain so after load application, (c) the considered shell has perfect geometry (no imperfections), (d) we assume that there is a membrane prebuckling stress state in the shell with the following internal forces

$$\overline{N}_1 = \frac{1}{2}q\tan\alpha\Big(\frac{s_1^2}{s} - s\Big) - N_a\frac{s_1}{s} \qquad \overline{N}_2 = -qs\tan\alpha \qquad \overline{T} = S \tag{1.1}$$

The following expressions for the strains and changes in the curvature were derived for a conical shell (see Zielnica, 1984)

$$\begin{aligned}
\delta\varepsilon_s^{\pm} &= \frac{\partial u^{\pm}}{\partial s} + \Big(z \pm \frac{-c-t_{+}^{-}}{2}\Big)\frac{\partial^2 w}{\partial s^2} + \frac{1}{2}\Big(\frac{\partial w}{\partial s}\Big)^2\\
\delta\varepsilon_{\varphi}^{\pm} &= \frac{1}{s\sin\alpha}\frac{\partial v^{\pm}}{\partial\varphi} - \frac{w}{s}\cot\alpha + \Big(z \pm \frac{-c-t_{+}^{-}}{2}\Big)\Big(\frac{1}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial\varphi^2} + \frac{1}{s}\frac{\partial w}{\partial s}\Big) +\\
&\quad + \frac{1}{s^2\sin^2\alpha}\Big(\frac{\partial w}{\partial\varphi}\Big)^2\\
\delta\gamma_{s\varphi}^{\pm} &= \frac{1}{s\sin\alpha}\frac{\partial u^{\pm}}{\partial\varphi} - \frac{\partial v^{\pm}}{\partial s} + 2\Big(z \pm \frac{-c-t_{+}^{-}}{2}\Big) \times\\
&\quad \times\Big(\frac{1}{s\sin\alpha}\frac{\partial^2 w}{\partial s\partial\varphi} - \frac{1}{s^2\sin\alpha}\frac{\partial w}{\partial\varphi}\Big) + \frac{1}{s\sin\alpha}\frac{\partial w}{\partial s}\frac{\partial w}{\partial\varphi} \qquad\qquad (1.2)\\
\delta\kappa_s^{\pm} &= \frac{\partial^2 w}{\partial s^2}
\end{aligned}$$

$$\delta\kappa_\varphi^\pm = \frac{1}{s}\frac{\partial w}{\partial s} + \frac{1}{s^2\sin^2\alpha}\frac{\partial^2 w}{\partial\varphi^2} + \frac{\cos\alpha}{s^2\sin^2\alpha}\frac{\partial v^\pm}{\partial\varphi}$$

$$\delta\kappa_{s\varphi}^\pm = \frac{1}{s\sin\alpha}\frac{\partial^2 w}{\partial s\partial\varphi} - \frac{1}{s^2\sin\alpha}\frac{\partial w}{\partial\varphi} + \frac{\cos\alpha}{4s\sin\alpha}\frac{\partial v^\pm}{\partial s} - \frac{5\cos\alpha}{4s^2\sin\alpha}v^\pm + \\ + \frac{\cos\alpha}{4s^2\sin\alpha}\frac{\partial u^\pm}{\partial\varphi}$$

Here $t^- = t_1$ and $t^+ = t_2$, respectively (see Fig. 2).

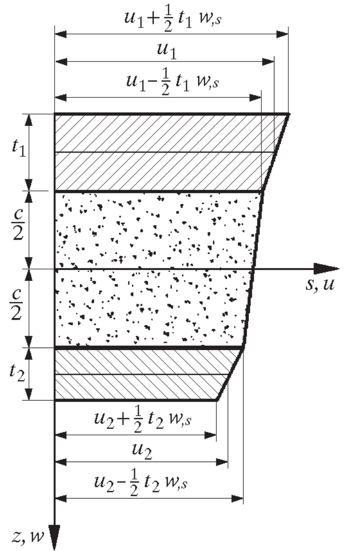

Fig. 2. Scheme of cross-section deformation

The core of the shell resists transversal shear only, thus the strains are determined according to the following expressions

$$\begin{aligned} \delta\gamma_{sz} &= \frac{2}{c}\Big(u_\beta - \frac{c+t}{2}\frac{\partial w}{\partial s}\Big) & u_\beta &= \frac{u^+ - u^-}{2} \\ \delta\gamma_{\varphi z} &= \frac{2}{c}\Big(v_\beta - \frac{c+t}{2}\frac{1}{s\sin\alpha}\frac{\partial w}{\partial\varphi}\Big) & v_\beta &= \frac{v^+ - v^-}{2} \end{aligned} \tag{1.3}$$

Here, the superscripts + and − denote the upper and the lower faces, respectively, and $t = (t_1 + t_2)/2$ is mean thickness of the faces.

The displacement vector components are as follows:

— the outer face for $-(t_1 + c/2) \leqslant z \leqslant -c/2$

$$u = u_1 - \Big(z + \frac{t_1 + c}{2}\Big)\frac{\partial w}{\partial s} \qquad v = v_1 - \Big(z + \frac{t_1 + c}{2}\Big)\frac{1}{s\sin a}\frac{\partial w}{\partial\varphi} \tag{1.4}$$

— the inner face for $c/2 \leqslant z \leqslant (t_2 + c/2)$

$$u = u_2 - \Big(z - \frac{t_2 + c}{2}\Big)\frac{\partial w}{\partial s} \qquad\qquad v = v_2 - \Big(z - \frac{t_2 + c}{2}\Big)\frac{1}{s \sin a}\frac{\partial w}{\partial \varphi} \tag{1.5}$$

— the core for $-c/2 \leqslant z \leqslant c/2$

$$\begin{aligned} u &= \frac{1}{2}\Big[u_1 + u_2 + \frac{t_2 - t_1}{2}\frac{\partial w}{\partial s} - \frac{2z}{c}\Big(u_1 - u_2 - \frac{t_1 + t_2}{2}\frac{\partial w}{\partial s}\Big)\Big] \\ v &= \frac{1}{2}\Big[v_1 + v_2 + \frac{t_2 - t_1}{2}\frac{1}{s \sin \alpha}\frac{\partial w}{\partial \varphi} - \frac{2z}{c}\Big(v_1 - v_2 - \frac{t_1 + t_2}{2}\frac{1}{s \sin \alpha}\frac{\partial w}{\partial \varphi}\Big)\Big] \end{aligned} \tag{1.6}$$

2. Stress-strain relations

In the plastic flow theory the stresses and stress increments are related with the strain increments by the constitutive flow rule and yield condition, generalized in the case of stress hardening. The basic equations of this theory of plasticity are as follows

$$D_{\dot\varepsilon} = \lambda D_\sigma + \frac{1}{2G}D_{\dot\sigma} \qquad\qquad \dot e_{ij} = \lambda s_{ij} + \frac{1}{2G}\dot s_{ij} \tag{2.1}$$

In Eqs (2.1) $D_{\dot\varepsilon}$ and $D_{\dot\sigma}$ are the deviators of the strain and stress rates, λ is a parameter of stress hardening, which can be determined from the yield condition. Here, we assume the Huber-Mises yield condition. If we neglect the yield condition and put $\lambda = 0$ in Eqs (2.1) we can describe the elastic region with these equations. When only a part of the shell undergoes plastic deformation we can obtain an equation for the elastic-plastic boundary, either from the condition $\lambda = 0$ (from a solution for the plastic region), or from the condition $\sigma_i = \sigma_Y$ (in elastic region). The increments of the plastic strains can be represented in the form

$$\begin{aligned} d\varepsilon_{ij} &= \frac{1}{2G}\Big(d\sigma_{ij} - \delta_{ij}\frac{3\nu}{1+\nu}d\sigma_m\Big) + d\lambda\Big(\sigma_{ij} - \delta_{ij}\sigma_m\Big) \\ \sigma_m &= \frac{1}{3}\sigma_{kk} \qquad\qquad d\lambda = \frac{1}{2}\frac{d\overline{\varepsilon}_i^p}{\sigma_i} \end{aligned} \tag{2.2}$$

If we assume the exponential stress-strain curve of the shell material in the general form $\sigma_i = E^{(o)}\varepsilon_i^{\xi}$, we can determine the secant E_s and tangent E_t

stress hardening moduli

$$
\begin{aligned}
E_s &= \frac{\sigma_i}{\varepsilon_i} = \begin{cases} k_1\Big(qs\sqrt{k_s(k_s-2)+4}\Big)^{\zeta} & \text{for} \quad \sigma_i \geqslant \sigma_Y = \Big(\dfrac{E^{(o)}}{E^{\xi}}\Big)^{\frac{1}{1-\xi}} \\ E & \text{for} \quad \sigma_i < \sigma_Y \end{cases} \\
E_t &= \frac{d\sigma_i}{d\varepsilon_i} = \begin{cases} \xi E_s & \text{for} \quad \sigma_i \geqslant \sigma_Y \\ E & \text{for} \quad \sigma_i < \sigma_Y \end{cases}
\end{aligned}
\tag{2.3}
$$

where k_1 and k_s are coefficients representing the prebuckling membrane stresses in the shell (see Zielnica, 2001), parameter $\zeta = (\xi-1)/\xi$, E is the elastic modulus; σ_i and σ_Y are the effective stress and yield stress, respectively.

The resultant middle surface forces and moments in the shell are defined as follows

$$
\begin{aligned}
\delta N_{\alpha\beta} &= \delta N_{\alpha\beta}^{+} + \delta N_{\alpha\beta}^{-} = \int\limits_{-\frac{c}{2}}^{-\frac{c}{2}-t_1} \delta\sigma_{\alpha\beta}\; dz + \int\limits_{\frac{c}{2}+t_2}^{\frac{c}{2}} \delta\sigma_{\alpha\beta}\; dz \\
\delta M_{\alpha\beta} &= \delta M_{\alpha\beta}^{+} + \delta M_{\alpha\beta}^{-} = \int\limits_{-\frac{c}{2}}^{-\frac{c}{2}-t_1} \delta\sigma_{\alpha\beta} z\; dz + \int\limits_{\frac{c}{2}+t_2}^{\frac{c}{2}} \delta\sigma_{\alpha\beta} z\; dz
\end{aligned}
\tag{2.4}
$$

It should be pointed out that these expressions reflect the fact that the shell thickness is small compared to the radius. If we solve Eqs (2.2) with respect to the stresses and then integratie according to (2.4), we get the following expressions for the resultant forces and moments developed by buckling in the faces

$$
\begin{aligned}
\delta N_{11} &= b_{11}\delta\varepsilon_{11} + b_{12}\delta\varepsilon_{22} - b_{13}\delta\gamma_{12} \\
\delta N_{22} &= b_{21}\delta\varepsilon_{11} + b_{22}\delta\varepsilon_{22} - b_{23}\delta\gamma_{12} \\
\delta T &= -b_{31}\delta\varepsilon_{11} - b_{32}\delta\varepsilon_{22} + b_{33}\delta\gamma_{12} \\
\delta M_1 &= -d_{11}\delta\kappa_1 - d_{12}\delta\kappa_2 + d_{13}\delta\kappa_{12} \\
\delta M_2 &= -d_{21}\delta\kappa_1 - d_{22}\delta\kappa_2 + d_{23}\delta\kappa_{12} \\
\delta H &= d_{31}\delta\kappa_1 + d_{32}\delta\kappa_2 - d_{33}\delta\kappa_{12}
\end{aligned}
\tag{2.5}
$$

where the coefficients of the local stiffness matrices b_{ij} and d_{ij} are as follows

$$
\begin{aligned}
b_{11} &= \frac{12}{t_{1,2}^2}d_{11} = \psi_0\Big\{2(1+\nu) + \psi_t\Big[\frac{1+\nu}{2}(2\overline{\sigma}_\varphi - \overline{\sigma}_s)^2 + 9\overline{\tau}_{s\varphi}^2\Big]\Big\}\\
b_{12} &= b_{21} = \frac{12}{t_{1,2}^2}d_{12} = \frac{12}{t_{1,2}^2}d_{21} =\\
&= \psi_0\Big\{2\nu(1+\nu) - \psi_t\Big[\frac{1+\nu}{2}(2\overline{\sigma}_s - \overline{\sigma}_\varphi)(2\overline{\sigma}_\varphi - \overline{\sigma}_s) + 9\overline{\tau}_{s\varphi}^2\Big]\Big\}\\
b_{13} &= b_{31} = \frac{6}{t_{1,2}^2}d_{13} = \frac{12}{t_{1,2}^2}d_{31} = 3\psi_0\psi_t\overline{\tau}_{s\varphi}\Big[(2-\nu)\overline{\sigma}_s - (1-2\nu)\overline{\sigma}_\varphi\Big]\\
b_{22} &= \frac{12}{t_{1,2}^2}d_{22} = \psi_0\Big\{2(1+\nu) + \psi_t\Big[\frac{1+\nu}{2}(2\overline{\sigma}_s - \overline{\sigma}_\varphi)^2 + 9\overline{\tau}_{s\varphi}^2\Big]\Big\}\\
b_{23} &= b_{32} = \frac{6}{t_{1,2}^2}d_{23} = \frac{12}{t_{1,2}^2}d_{32} = 3\psi_0\psi_t\overline{\tau}_{s\varphi}\Big[(2-\nu)\overline{\sigma}_\varphi - (1-2\nu)\overline{\sigma}_s\Big]\\
b_{33} &= \frac{6}{t_{1,2}^2}d_{33} = \psi_0\Big\{(1-\nu^2) + \frac{1}{4}\psi_t\Big[(5-4\nu)(\overline{\sigma}_s^2 + \overline{\sigma}_\varphi^2) - 2(4-5\nu)\overline{\sigma}_s\overline{\sigma}_\varphi\Big]\Big\}
\end{aligned}
\tag{2.6}
$$

Here

$$
\begin{aligned}
\psi_0 &= \frac{Et_{1,2}}{1+\nu}\Big\{2(1-\nu^2) + \frac{1}{2}\psi_t[(5-4\nu)(\overline{\sigma}_s^2 - \overline{\sigma}_\varphi^2) - 2(4-5\nu)\overline{\sigma}_s\overline{\sigma}_\varphi +\\
&\quad +18(1-\nu)\overline{\tau}_{s\varphi}^2]\Big\}^{-1}\\
\psi_t &= \frac{E}{E_t} - 1
\end{aligned}
$$

The over barred symbols are the relative prebuckling stresses related with the effective stress σ_i

$$
\overline{\sigma}_s = \frac{\sigma_s}{\sigma_i} \qquad \overline{\sigma}_\varphi = \frac{\sigma_\varphi}{\sigma_i} \qquad \overline{\tau}_{s\varphi} = \frac{\tau_{s\varphi}}{\sigma_i} \qquad E_t = \frac{d\sigma_i}{d\varepsilon_i} \tag{2.7}
$$

As we can see, the constitutive relations are functions of the tangent modulus E_t in the plastic flow theory. Also, the coefficients in the constitutive relations are variable, and they depend on the external loadings acting on the considered shell (see Fig. 1).

3. The potential energy and solution to the problem

The given system of the stability equations, expressed by the displacements, does not have an exact solution. Any approximate solution, found e.g.

by an orthogonalization method is complicated because appropriate calculations are time consuming. The necessity of satisfying the kinematic and static boundary conditions leads to the assumption of approximate functions in a very complicated form. In order to avoid the above mentioned difficulties the Ritz method is applied.

The conditions for the equilibrium in a classical buckling problem can be obtained from the variation of the total potential energy Π_T. In order to obtain the stability conditions from the variation relations, the principle of a stationary potential energy will be invoked, with the sandwich conical shell considered to be in a state of neutral equilibrium. Since the principle of the stationary potential energy states that the necessary condition of the equilibrium of any given state is that the variation of the total potential energy of the considered system is equal to zero, we have the following relation

$$\delta\Pi_T = \delta(W_T + L) = 0 \tag{3.1}$$

Here W_T is a change in the strain energy stored within the shell. The second term L represents the potential energy of the external loads. Equation (3.1) with its nature has a form of equilibrium equations in variational sense, and it is correct both for the pre- and postcritical deformation state. Instead of exact expressions for the displacements u_i we introduce approximate functions with coefficients A_i. These coefficients must be chosen in such a way that they fit as far as possible to real displacements. The equation

$$\delta\Pi_T = \sum_{i=1}^{k} \Pi_{T,A_i}\,\delta A_i = 0 \tag{3.2}$$

is satisfied for an arbitrary value of the variation of parameters δA_i, where $i = 1, 2, ..., k$. Thus, we have

$$\frac{\partial\Pi_T}{\partial A_i} = 0 \tag{3.3}$$

The total potential energy of the shell is obtained by summing up the particular terms related with three layers: W_T^+ (outer layer), W_T^- (inner layer) and W^C (core) with the potential energy of the external loads L, i.e.

$$\Pi_T = W_T + L = W_T^+ + W_T^- + W^C + L \tag{3.4}$$

The first terms in Eq. (3.4) related with strain energy are

$$
\begin{aligned}
W_T^{\pm} &= \frac{1}{2}\int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} (\delta N_1^{\pm}\delta\varepsilon_{11}^{\pm} + \delta N_2^{\pm}\delta\varepsilon_{22}^{\pm} + \delta T^{\pm}\delta\gamma_{12}^{\pm} + \\
&\qquad + \delta M_1^{\pm}\delta\kappa_1^{\pm} + \delta M_2^{\pm}\delta\kappa_2^{\pm} + \delta H^{\pm}\delta\kappa_{12}^{\pm}) r d\varphi\, ds \\
W^C &= \frac{1}{2}\int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} (\delta N_{sz}\delta\gamma_{sz} + \delta N_{\varphi z}\delta\gamma_{\varphi z}) r d\varphi\, ds
\end{aligned}
\tag{3.5}
$$

The term L is the potential energy of the external loads, and it is given by

$$
\begin{aligned}
L &= -\iint\limits_{A} qw\, dsr\, d\varphi - \frac{1}{2}\iint\limits_{A} N_a \frac{s_1}{s} w_{,s}^2\, ds\, r\, d\varphi + \\
&+ \int\limits_{s_1}^{s_2}\int\limits_{-(c+t_2)}^{(c+t_1)} \frac{S}{c+t_1+t_2} u(z)\, ds\, dz = -q\sin\alpha \int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} ws\, d\varphi\, ds - \\
&- \frac{1}{2}N_a s_1 \sin\alpha \int\limits_{s_1}^{s_2}\int\limits_{0}^{\beta} w_{,s}^2\, d\varphi\, ds + \int\limits_{0}^{L}\int\limits_{-(c+t_2)}^{(c+t_1)} \frac{S}{c+t_1+t_2} u(z)\, ds\, dz
\end{aligned}
\tag{3.6}
$$

Now, we substitute the local stiffness matrix coefficients b_{ij} (2.6) into Eqs (2.5), then we substitute these expressions into (3.5). Thus, using Eqs (1.1)-(1.6), we get a general form of the total potential energy Π_T for the deformed shell expressed in terms of the displacements u, v and w. Once the geometry, material constants, and load conditions are specified, we chose the displacement functions w, u and v in the following form

$$
\begin{aligned}
w(s,\varphi) &= A_1 r^2 \sin(k\psi)\sin(p\varphi + a\gamma s) \\
u_\alpha(s,\varphi) &= A_2 r^2 \cos(k\psi)\sin(p\varphi + a_1\gamma s) \\
v_\alpha(s,\varphi) &= A_4 r^2 \sin(k\psi)\cos(p\varphi + a_3\gamma s) \\
u_\beta(s,\varphi) &= A_3 r^2 \cos(k\psi)\sin(p\varphi + a_2\gamma s) \\
v_\beta(s,\varphi) &= A_5 r^2 \sin(k\psi)\cos(p\varphi + a_4\gamma s)
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{array}{lll}
k = \dfrac{m\pi}{s_2 - s_1} & \psi = s - s_1 & p = \dfrac{n\pi}{\beta} \\
r = s\sin\alpha & u_\alpha = \dfrac{1}{2}(u_1 + u_2) & u_\beta = \dfrac{1}{2}(u_1 - u_2) \\
v_\alpha = \dfrac{1}{2}(v_1 + v_2) & v_\beta = \dfrac{1}{2}(v_1 - v_2) &
\end{array}
$$

Here m, and $2n$ are parameters, equal to the number of halfwaves during buckling developed in the longitudinal and circumferential direction, respectively; a_i are multipliers that will take values 0 or 1 in numerical calculations in order to check the influence of the γ parameter on the buckling loads. Approximate functions (3.7) satisfy the kinematic boundary conditions of the simply supported shell edges

$$
\begin{aligned}
&w\Big|_{\substack{s=s_1\\ s=s_2}} = 0 \qquad w\Big|_{\substack{\varphi=0\\ \varphi=\beta}} = 0 \qquad u_\alpha\Big|_{\substack{\varphi=0\\ \varphi=\beta}} = 0\\
&v_a\Big|_{\substack{s=s_1\\ s=s_2}} = 0 \qquad u_\beta\Big|_{\substack{\varphi=0\\ \varphi=\beta}} = 0 \qquad v_\beta\Big|_{\substack{s=s_1\\ s=s_2}} = 0
\end{aligned}
\tag{3.8}
$$

We substitute approximate functions (3.7) into Eqs (1.2) and (3.5). Then, we substitute Eqs (3.5) and (3.6) into the total potential energy expression (3.4), and we obtain a complex functional of the form

$$
\begin{aligned}
\Pi_T = W_T^+ + W_T^- + W^C + L = \int\limits_{s_1}^{s_2}\int\limits_0^\beta \Lambda\Big(&u_\alpha, u_\beta, v_\alpha, v_\beta, w, \frac{\partial u_\alpha}{\partial s}, \frac{\partial u_\alpha}{\partial \varphi},\\
&\frac{\partial v_\alpha}{\partial s}, \frac{\partial v_\alpha}{\partial \varphi}, \frac{\partial u_\beta}{\partial s}, \frac{\partial u_\beta}{\partial \varphi}, \frac{\partial v_\beta}{\partial s}, \frac{\partial v_\beta}{\partial \varphi}, \frac{\partial w}{\partial s}, \frac{\partial w}{\partial \varphi}, \frac{\partial^2 w}{\partial s^2}, \frac{\partial^2 w}{\partial \varphi^2}, \frac{\partial^2 w}{\partial s \partial \varphi}\Big)\, ds\; r\, d\varphi
\end{aligned}
\tag{3.9}
$$

Then, following relations (3.3), we differentiate the total potential energy Π_T with respect to the coefficients A_i, i.e.

$$
\Pi_{T,A_i} = 0 \qquad\qquad i = 1, 2, ..., 5 \tag{3.10}
$$

Thus, we get a system of nonlinear algebraic equations written in the following general form, where the unknowns are the parameters of the displacement functions A_i

$$
\begin{aligned}
&(f_{11} + \widetilde{f}_{11})A_1 + f_{12}A_2 + f_{13}A_3 + f_{14}A_4 + f_{15}A_5 =\\
&= g_{11}A_1^2 + g_{12}A_1^3 + g_{13}A_1A_2 + g_{14}A_1A_3 + g_{15}A_4 + g_{16}A_5 + g_{17}\\
&f_{21}A_1 + f_{22}A_2 + f_{23}A_3 + f_{24}A_4 + f_{25}A_5 = g_{21}A_1^2\\
&f_{31}A_1 + f_{32}A_2 + f_{33}A_3 + f_{34}A_4 + f_{35}A_5 = g_{31}A_1^2\\
&f_{41}A_1 + f_{42}A_2 + f_{43}A_3 + f_{44}A_4 + f_{45}A_5 = g_{41}A_1^2\\
&f_{51}A_1 + f_{52}A_2 + f_{53}A_3 + f_{54}A_4 + f_{55}A_5 = g_{51}A_1^2
\end{aligned}
\tag{3.11}
$$

The coefficients f_{ij} and g_{ij} in nonlinear system of equations (3.11) are functions of the variable s (see below), and they depend on the geometrical and material parameters and also on the state of stress (elastic or elastic-plastic). The general form of these coefficients is as follows

$$
\begin{aligned}
f_{11} &= 2\sin^3\alpha\cos^2\alpha J_1\widetilde{U}_{4,4} + \frac{(t_1+t_2)^2}{24}\Big\{J_1\Big[\sin^5\alpha(4\widetilde{U}_{1,1} + 16k\widetilde{U}_{1,11} - \\
&-4k^2\widetilde{U}_{1,4} - 8k^3\widetilde{U}_{1,13} + k^4\widetilde{U}_{1,8} + 16k^2\widetilde{U}_{1,23} + 8\widetilde{U}_{2,1} + 20k\widetilde{U}_{2,11} - \\
&-4k^2\widetilde{U}_{2,4} - 2k^3\widetilde{U}_{2,13} + 4k^2\widetilde{U}_{2,23} + 4\widetilde{U}_{4,1} + 2k\widetilde{U}_{4,11} + k^2\widetilde{U}_{4,23}) + \\
&+p\sin^3\alpha(2k^2\widetilde{U}_{2,4} - 8k\widetilde{U}_{2,11} - 4\widetilde{U}_{2,1} + 2k\widetilde{U}_{4,11}) + p^4\sin\alpha\widetilde{U}_{4,1}\Big] + \\
&+4J_2p^2\sin^3\alpha(\widetilde{U}_{3,1} + 2k\widetilde{U}_{3,11} + k^2\widetilde{U}_{3,23})\Big\} + 4\frac{G_3}{c}\sin^3\alpha\Big(\frac{2c+t_1+t_2}{4}\Big)^2 \times \\
&\times\Big[J_1\sin^2\alpha(4\widetilde{U}_{5,4} + 4k\widetilde{U}_{5,13} + k^2\widetilde{U}_{5,25}) + J_2p^2\widetilde{U}_{5,4}\Big], \dots
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
g_{16} &= 2g_{51} = p\sin^3\alpha\cos\alpha\frac{2c+t_1+t_2}{4}\Big[J_3\sin^2\alpha(4V_{2,5} + 4kV_{2,14} + k^2V_{2,15}) + \\
&+p^2J_8(6V_{3,5} + kV_{3,14} - k^2V_{3,15})\Big], \dots
\end{aligned}
$$

Here k and p are the buckling mode parameters in displacement functions (3.7), J_i are the integral expresssions of the variable φ; $\widetilde{U}_{k,l}$, V_{kl} are the integral expressions of the variable s of the following general form, which will be evaluated by numerical integration

$$
\widetilde{U}_{k,l} = \widetilde{U}_{k,l}(s) = \int\limits_{s_1}^{s_2} F_k(s)C_l(s)\,ds
\tag{3.13}
$$

The other coefficients in (3.11) have a form similar to the above expressions. For the sake of brevity we introduce a function $\overline{g}$ being a combination of the coefficients f_{ij} and g_{ij}

$$
\overline{g}(\overline{x},\overline{y},r) = \sum_{i=2}^{5}\Big[\overline{x}_{1(i+r)}\Big(\sum_{j=1}^{4}(-1)^{i+j}\overline{y}_{(j+1)1}d_{ij}\Big)\Big]
\tag{3.14}
$$

where the expressions d_{ij}, $i = 2,3,4,5$, $j = 1,2,3,4$, are the minors of the determinant W_1 that is determined by the operation of deleting the ith column and jth row

$$
W_1 = \begin{vmatrix}
f_{22} & f_{23} & f_{24} & f_{25} \\
f_{32} & f_{33} & f_{34} & f_{34} \\
f_{42} & f_{43} & f_{44} & f_{45} \\
f_{52} & f_{53} & f_{54} & f_{55}
\end{vmatrix}
\tag{3.15}
$$

When we solve the set of nonlinear algebraic equations (3.11) with respect to the deflection function parameter A_1, we obtain the final stability equation in the following form

$$
\begin{aligned}
q_{(m,n,\gamma)} &= \Big\{\Big(f_{11} + W_1^{-1}\overline{g}(f,f,0)\Big)A_1 + \\
&+\Big[-g_{11} - W_1^{-1}\Big(\overline{g}(f,g,0) + \overline{g}(g,f,1)\Big)\Big]A_1^2 + \\
&+\Big(-g_{12} + W_1^{-1}\overline{g}(g,g,1)\Big)A_1^3\Big\} \times \\
&\times\Big[s_1^2 J_1 \sin^5\alpha(4\widetilde{U}_{5,2} + 4k\widetilde{U}_{5,12} + k^2\widetilde{U}_{5,24})\kappa A_1 + \kappa_1 A_1 + \sin^3\alpha J_4\widetilde{U}_{5,3}\Big]^{-1}
\end{aligned}
\tag{3.16}
$$

The final form of stability equation (3.16), being a function of the deflection function parameter, makes it possible to trace the equilibrium paths for the elastic-plastic open sandwich conical shell under consideration, and to test different forms of the stability loss.

4. Numerical calculations and concluding remarks

Because the analysed problem considers two types of nonlinearities, both physical and geometrical, and large amount of calculations is to be made in order to determine the equilibrium paths for the elastic-plastic problem, a special computer algorithm has been developed. Numerical calculations were carried out to analyse the postcritical equilibrium paths for arbitrary combinations of the shear force and lateral-to-longitudinal load. In the analysis of stability equation (3.16), which is a transcendental one (local stiffness matrix coefficients (2.6) depend on the external load acting on the shell), aiming at the determination of the "upper" $(q_+^*; N_{a+}^*; S_+^*)$ and "lower" $(q_-^*; N_{a-}^*; S_-^*)$ critical load, we procede according to the following steps:

(i) we assume geometrical and material data for the shell and fixed ratios of the lateral pressure and shear force to the longitudinal loads $\kappa = qs_1/N_a$, $\kappa_1 = S/N_a$

(ii) we adopt a series of values for the parameters m, n and γ

(iii) for a series of increasing values of A_1 we calculate the respective maximum deflection w and loads q, (S, N_a)

(iv) in the system of coordinates (w, q) or (w, N_a) we draw a two-parameter family of the curves $q(w; m, n)$ or $N_a(w; m, n)$

(v) from the family of curves we choose the points of less values of q (or N_a) with specified values of the variable w, and we obtain a curve which constitutes the solution

(vi) the local maximum and minimum of the curve determine the "upper" $(q_+^*; N_{a+}^*)$ and the "lower" $(q_-^*; N_{a-}^*)$ critical loads, respectively.

The starting point in the analysis of the shell being partially or totally in the plastic region a certain value of the initial load $q_i(QI)$ is assumed to be on the equilibrium path, basing on the value from the previous step q_{i-1}. Thus, the local stiffness matrix coefficients b_{ij}, see (2.6), can be determined, and stability equation (3.16) is now an equation with known coefficients where the deflection $w^* = w/H$ $(H = t_1 + t_2 + 2c)$ is the parameter. Finally, such a value is accepted for QI, which satisfies the condition $|Q - QI| < \varepsilon_i$, where ε_i is the parameter of the assumed calculation accuracy. A linear interpolation rule (regula falsi) has been assumed to find the initial load. To determine the integrals in the stability equation, the Simpson rule of numerical integration has been adopted.

It has been assumed that the shell material for the faces was an aluminium alloy with the following material constants: Young's modulus $E = 7.1 \cdot 10^4\,\mathrm{MPa}$, tangent modulus $E_t = 0.95 \cdot 10^4\,\mathrm{MPa}$, yield stress $\sigma_Y = 380\,\mathrm{MPa}$, $\nu = 0.3$. The core material was an industrial foam plastic "Moltopren" with $E_s = 53\,\mathrm{MPa}$, $G_s = 27\,\mathrm{MPa}$, $\nu_s = 0$. The other basic geometrical parameters were as follows: shell thickness $H = 12\,\mathrm{mm}$, face thickness $t_1 = t_2 = 0.001\,\mathrm{m}$, $2c = 0.01\,\mathrm{mm}$, distance from the apex to the upper base $s_1 = 10\,\mathrm{m}$, shell length along the generatrix $l = 0.7\,\mathrm{m}$, apex angle $\alpha = 10°$, shell angle $\beta = 16°$.

The other variable parameters are shown in the diagrams.

Figures 3-5 show the nonlinear equilibrium paths for the analysed shell. The diagram shown in Fig. 3 presents the numerical results in the form of curves representing the longitudinal load N_a as a function of the deflection w^* for different ratios of the shear force S to N_a. It was assumed here that the lateral pressure q equals 0. It can be seen from the diagram that both upper and lower critical loads decrease when the ratio S/N_a increases. Moreover, the difference between the upper and lower critical loads drops if the ratio S/N_a increases. Fig. 4 is a diagram enabling the evaluation of the influence of different ratios of the shear force S to the lateral pressure q on the equilibrium paths and the values of critical loads. It was accepted there that the longitudinal force N_a equals 0. The same tendency in the relation between the upper and lower critical loads was observed in the diagram as in the previous case. Finally, Fig. 5 presents the equilibrium paths $N_a - w^*$ for different face

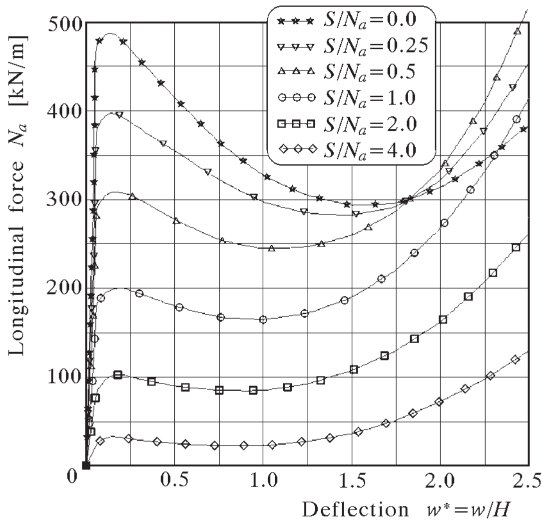

Fig. 3. Equilibrium paths for different shear/longitudinal force ratios

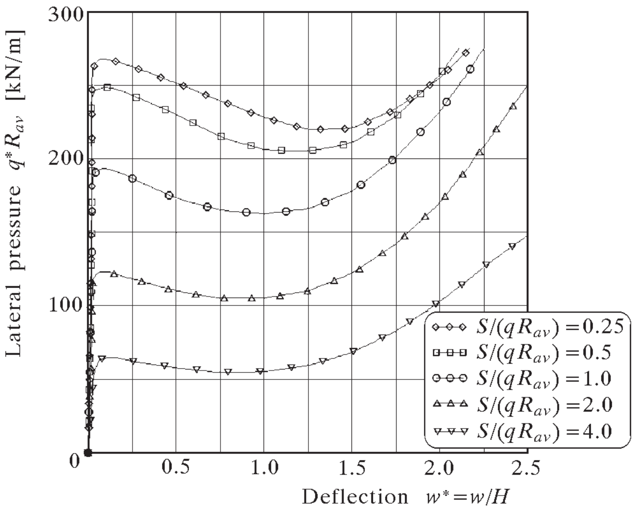

Fig. 4. Equilibrium paths for different shear/lateral pressure ratios

thicknesses t to total shell thickness H. As it could be expected, greater shell face thicknesses, enlarge the upper and lower critical loads. Generally, it can be stated that the number of waves during buckling was $m = 1$ and $n = 1$ in most of the analysed cases.

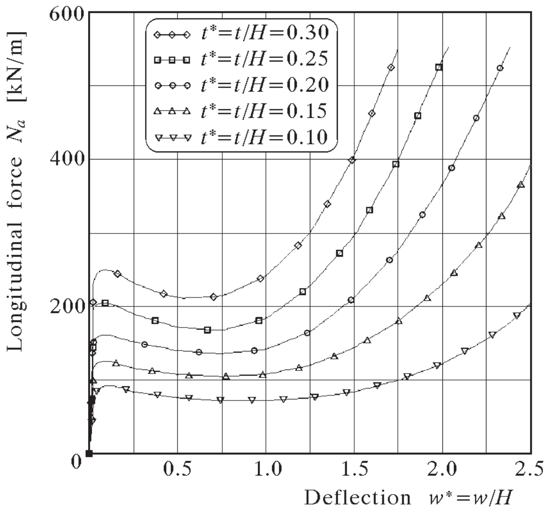

Fig. 5. Equilibrium paths for different shell thicknesses of the shell

The results presented in the paper can be valuable for engineering practice. The analysis and numerical calaculations point out the ways for the determination of values of the basic shell parameters preventing the system from the instabilities observed in open elastic-plastic sandwich conical shells.

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Nieliniowa stateczność sprężysto-plastycznej powłoki stożkowej przy złożonym obciążeniu

Streszczenie

W pracy przedstawiono analizę stateczności sprężysto-plastycznej otwartej powłoki stożkowej pod wpływem złożonego obciążenia w postaci siły podłużnej, ciśnienia poprzecznego i sił tnących. Powłoka składa się z dwóch warstw nośnych wykonanych z różnych materiałów wykazujących umocnienie i mających różną grubość. Warstwa

wypełniąca jest typu lekkiego i zakłada się, że przenosi wyłącznie siły ścinające. Zakłada się również, że pod wpływem sił zewnętrznych powłoka może przejść częściowo lub całkowicie w stan plastyczny. Uwzględnia się sztywność zginania warstw nośnych, ważność hipotez Kirchhoffa-Lova i przyjmuje się koncepcję wzrastającego obciążenia Shanleya. Analizę oparto na teorii plastycznego płynięcia Prandtla-Reussa stowarzyszonej z warunkiem uplastycznienia Hubera-Misesa. Podstawowe równania stateczności wyprowadzono z zasady prac wirtualnych, a do ich rozwiązania wykorzystano metodę Ritza. Dla analizy i obliczeń numerycznych opracowano specjalny algorytm iteracyjny, który umożliwia obliczenia stateczności dla powłok częściowo lub całkowicie uplastycznionych, a nawet dla powłok sprężystych.

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