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OPTIMAL DESIGN OF MEMBRANE SHELLS. HOMOGENIZATION-BASED RELAXATION OF THE TWO-PHASE LAYOUT PROBLEM

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The aim of using shell structures instead of plates is to avoid bending, hence the vital role of the membrane theory. Within this theory the classical optimum design problem is formulated: lay out two isotropic materials such that the shell becomes the stiffest possible. The amount of both the materials is fixed. The aim of the present paper is to reformulate this problem in a form assuring its well-posedness. The membrane approximation can be introduced from the very beginning or be imposed upon the relaxation. In the present paper it is shown that the latter modelling leads to a better formulation. It does not lose its stability even if one material degenerates to a void, thus leading to a well-posed shape design problem.

Key words: membrane shells, homogenization, minimum compliance problem, relaxation by homogenization, Michell's sphere

1. Introduction

A prerequisite for solving the classical shape design problem is understanding the following more general problem: lay out two non-degenerated isotropic materials in a feasible domain such that the two-phase body obtained becomes the stiffest among all possible bodies transmitting a given surface loading to a given support. The volume of one (or both) material must be fixed to make the problem solvable. It turns out that this problem requires a relaxation.

If applied to 2D or 3D elasticity problems, the relaxation by homogenization is now well-understood and considered as a standard method for finding stable optimal layouts, see Tartar (2000), Cherkaev (2000), Allaire (2002). Two-phase compliance minimization problem of thin plates was solved by Gibiansky and Cherkaev (1984) and further cleared up by Lipton (1994) and Lewiński and Telega (2000, Sec. 26). New layouts have been recently reported by Czarnecki and Lewiński (2001) and Kolanek and Lewiński (1999, 2003). This latter work reports new results for the old problem of designing of circular and annular plates, for which the first relaxed results were found by Cheng and Olhoff (1981).

Optimization of shells is less developed, although the first relaxed numerical solutions were already announced by Suzuki and Kikuchi (1991) and Tenek and Hagiwara (1994).

The relaxation requires the homogenization formulae for shells. Within the bending theory of thin shells such formulae were derived in Lewiński and Telega (1988) and Telega and Lewiński (1998), see Lewiński and Telega (2000, Sec. 17). These formulae are fairly complicated, since they couple the membrane and flexural effects. The simplifying assumption of shallowness cancels this coupling, hence making the final formulae similar to those for plates. Just these formulae are usually used to relax the optimum design problem of shells.

The aim of the present paper is to consider a specific case when the bending effects can be neglected. This simplification can be introduced at various stages of the optimization process. In the present paper two possible methods of modelling are discussed: neglecting the bending effects before relaxation and then after relaxation. It is shown that only the latter method can lead to a correct formulation of shape design of membrane shells.

2. Two-phase layout problem for a membrane shell formed on a given middle surface

2.1. Equilibrium problem

Let Ω be a given plane domain whose image $S \subset \mathbb{R}^3$ is a middle surface of the shell, the transformation of Ω into S being denoted by Φ, i.e.

$$\begin{aligned} &\Phi : \Omega \to S \subset \mathbb{R}^3 \\ &\boldsymbol{\xi} = (\xi^1, \xi^2) \in \Omega \to \Phi(\boldsymbol{\xi}) \in S \end{aligned} \tag{2.1}$$

The shell considered is formed around its middle surface S such that its thickness h is kept constant. Assume that the shell is supported along ∂S_u, loaded by tractions $\boldsymbol{T}(s)$ along ∂S_T and subjected to the surface loading $\boldsymbol{q}(\boldsymbol{\xi})$, $\boldsymbol{q} = (q^1, q^2, q^3)$. Here $\partial S = \partial S_u \cup \partial S_T$, $\overline{\partial S_u} \cap \overline{\partial S_T} = \emptyset$, $\partial S_u = \varPhi(\varGamma_u)$, $\partial S_T = \varPhi(\varGamma_T)$, $\varGamma = \varGamma_u \cup \varGamma_T$ and $\overline{\varGamma}_u \cap \overline{\varGamma}_T = \emptyset$.

The deformation of the shell is determined by the displacement field (u_1, u_2, w) with $\boldsymbol{u} = (u_1, u_2)$ representing the tangent displacement and w being the displacement normal to S.

To introduce the set of admissible displacements and then to formulate the boundary-value problem, we define first

$$V = \Big\{(\boldsymbol{v}, v) \,|\, v_\alpha \in H^1(\varOmega), \quad v \in H^2(\varOmega), \quad v_\alpha = 0 \text{ on } \varGamma_u\Big\} \tag{2.2}$$

and endow this space with the norm

$$\|(\boldsymbol{v}, v)\|_V^2 = \sum_{\alpha=1}^{2} \|v_\alpha\|_{H^1(\varOmega)}^2 + \|v\|_{H^2(\varOmega)}^2 \tag{2.3}$$

From now onward the small Greek indices: α, β, λ, μ will take values 1 or 2.

Let us recall the formulae for membrane strains

$$\epsilon_{\alpha\beta}(\boldsymbol{u}, w) = \frac{1}{2}(u_{\alpha\|\beta} + u_{\beta\|\alpha}) - b_{\alpha\beta} w \tag{2.4}$$

where $(\cdot)_{\|\alpha}$ represents the covariant derivative in the tangent plane and $(b_{\alpha\beta})$ is a curvature tensor, see Bernadou (1996) and Lewiński and Telega (2000, Sec. 16). Assume that $\mathbf{A}(\boldsymbol{\xi}) = \big(A^{\alpha\beta\lambda\mu}(\boldsymbol{\xi})\big)$ represent a membrane stiffness tensor which satisfies the usual symmetry and positive definiteness properties. We define the bilinear form

$$a_0(\boldsymbol{u}, w; \boldsymbol{v}, v) = \int\limits_S A^{\alpha\beta\lambda\mu}(\boldsymbol{\xi}) \epsilon_{\lambda\mu}(\boldsymbol{u}, w) \epsilon_{\alpha\beta}(\boldsymbol{v}, v) \, dS \tag{2.5}$$

and the norm

$$\|(\boldsymbol{u}, w)\|_0 = [a_0(\boldsymbol{u}, w; \boldsymbol{u}, w)]^{\frac{1}{2}} \tag{2.6}$$

Now let V_0 be the completion of V in the norm $\|\cdot\|_0$. We observe that in general

$$V^0 \subset \Big\{(\boldsymbol{v}, v) \,|\, v_\alpha \in H^1(\varOmega), \quad v \in L^2(\varOmega), \quad v_\alpha = 0 \text{ on } \varGamma_u\Big\}$$

and the inclusion is strong, see Sanchez-Hubert and Sanchez-Palencia (1997). The space V_0 plays the role of the space of kinematically admissible displacements, within the membrane shell theory. Note that no boundary condition can be imposed on w and v since in the norm of the space V^0 the derivative of these fields does not intervene. Consequently the value (trace) of w and v on $\partial\Gamma$ cannot be determined.

The equilibrium problem of the membrane shell considered has the form

$$
(P_1) \quad \left| \begin{array}{l} \text{find } (\boldsymbol{u}, w) \in V^0 \text{ such that} \\ \quad a_0(\boldsymbol{u}, w; \boldsymbol{v}, v) = f(\boldsymbol{v}, v) \quad \forall\, (\boldsymbol{v}, v) \in V^0 \\ \text{with} \\ \quad f(\boldsymbol{v}, v) = \displaystyle\int_S (q^\alpha v_\alpha + qv)\, dS + \int_{\partial S_T} T^\alpha v_\alpha\, ds \end{array} \right. \tag{2.7}
$$

and s parametrizes ∂S.

The membrane stress resultants are given by

$$
N^{\alpha\beta} = A^{\alpha\beta\lambda\mu} \epsilon_{\lambda\mu}(\boldsymbol{u}, w) \tag{2.8}
$$

They satisfy the following local equations of equilibrium

$$
N^{\beta\alpha}_{\|\beta} + q^\alpha = 0 \qquad\qquad b_{\alpha\beta} N^{\alpha\beta} + q = 0 \tag{2.9}
$$

in Ω.

The local equations (2.9) completed with boundary conditions on Γ_T are equivalent to the variational equation

$$
\int_S N^{\alpha\beta} \epsilon_{\alpha\beta}(\boldsymbol{v}, v)\, dS = f(\boldsymbol{v}, v) \qquad \forall\, (\boldsymbol{v}, v) \in V^0 \tag{2.10}
$$

Remark 1. There exist problems for which the fields $(N^{\alpha\beta})$ can be found by solving (2.10). Such problems are called statically determinate. For these problems the *set of statically admissible membrane forces*

$$
\mathbb{S}(\Omega) = \Big\{ \mathbf{N} \in L^2(\Omega, \mathbb{E}^s_2) \mid N^{\alpha\beta}_{\|\beta} \in L^2(\Omega) \quad \text{and (2.10) is satisfied} \Big\} \tag{2.11}
$$

is a one-element set. In general, $\mathbb{S}(\Omega)$ is an affine set. □

2.2. The layout problem

Assume that the membrane shell is composed of two isotropic materials, the filling being transversely homogeneous. Thus the membrane stiffness tensor has the following representation

$$\mathbf{A}(\boldsymbol{\xi}) = \chi_1(\boldsymbol{\xi})\mathbf{A}_1 + \chi_2(\boldsymbol{\xi})\mathbf{A}_2 \tag{2.12}$$

where

$$\mathbf{A}_\alpha = 2k_\alpha \mathbf{I}_1 + 2\mu_\alpha \mathbf{I}_2 \tag{2.13}$$

Here $k_2 > k_1$, $\mu_2 > \mu_1$ are stiffnesses due to in-plane uniform stress and shear stresses, respectively. Tensors $\mathbf{I}_1$, $\mathbf{I}_2$ are expressed in terms of the metric tensor $\mathbf{g} = (g_{\alpha\beta})$:

$$\begin{aligned} I_1^{\alpha\beta\lambda\mu} &= \frac{1}{2} g^{\alpha\beta} g^{\lambda\mu} \\ I_2^{\alpha\beta\lambda\mu} &= \frac{1}{2}(g^{\alpha\lambda}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\lambda} - g^{\alpha\beta}g^{\lambda\mu}) \end{aligned} \tag{2.14}$$

they have properties of projection operators. The function $\chi_\alpha(\boldsymbol{\xi})$ is a characteristic function of the domain Ω_α corresponding to the domain S_α around which the α-th material is located. Thus $\Omega_1 \cup \Omega_2 = \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ and

$$\chi_\alpha(\boldsymbol{\xi}) = \begin{cases} 1 & \text{if } \boldsymbol{\xi} \in \Omega_\alpha \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

The tensor of compliances

$$\mathfrak{a} = \mathbf{A}^{-1} \tag{2.16}$$

is given by

$$\mathfrak{a}(\boldsymbol{\xi}) = \chi_1(\boldsymbol{\xi})\mathfrak{a}_1 + \chi_2(\boldsymbol{\xi})\mathfrak{a}_2 \tag{2.17}$$

with

$$\begin{aligned} \mathfrak{a}_\alpha &= 2K_\alpha \mathbf{I}_1 + 2L_\alpha \mathbf{I}_2 \\ K_\alpha &= (k_\alpha)^{-1} \qquad\qquad L_\alpha = (\mu_\alpha)^{-1} \end{aligned} \tag{2.18}$$

The compliance C of the shell is defined in a standard manner

$$C = f(\boldsymbol{u}, w) \tag{2.19}$$

The following equality

$$C = \inf_{\mathbf{N}\in\mathbb{S}(\Omega)} \int_S \mathbf{N} : (\mathfrak{a}\mathbf{N})\, dS \tag{2.20}$$

expresses the Castigliano theorem for membrane shells. It can be proved by using the duality theory expounded in Ekeland and Temam (1976) or by a hybrid approach developed by the second author, see Telega (2003, Sec. 3.2). This problem will be treated in a separate paper.

Note that if a shell problem is statically determinate, then inf in (2.20) is redundant, since then $\mathbb{S}(\Omega)$ is a one element set.

Now we are ready to formulate the layout problem

$$\inf\Big\{C = f(\boldsymbol{u}, w)\,|\,\chi_2 \in L^\infty(\Omega;\{0,1\}), \quad \int_S \chi_2\, dS = \mathcal{A}\Big\} \tag{2.21}$$

where $(\boldsymbol{u}, w)$ depend on χ_2 and $\mathcal{A}$ is a given area occupied by the material 2. To simplify further notation we put $\chi = \chi_2$.

By Castigliano's theorem the layout problem (2.21) can be put in the form

$$(P) \quad \left|\begin{array}{l} \displaystyle\inf_{\chi\in L^\infty(\Omega;\{0,1\})} \inf_{\mathbf{N}\in\mathbb{S}(\Omega)} \int_S \mathbf{N} : (\mathfrak{a}\mathbf{N})\, dS \\ \displaystyle\int_S \chi\, dS = \mathcal{A} \end{array}\right. \tag{2.22}$$

Now we introduce the Lagrangian multiplier λ associated with the isoperimetric condition and change the position of supremum over λ before both infima. This was done in elasticity, see Kohn and Strang (1986), and is equally justified in the case of membrane shells. For fixed λ we find

$$(P_\lambda) \quad \left| \quad \inf_{\chi\in L^\infty(\Omega;\{0,1\})} \inf_{\mathbf{N}\in\mathbb{S}(\Omega)} \int_S [\mathbf{N} : (\mathfrak{a}\mathbf{N}) + \lambda\chi]\, dS \right. \tag{2.23}$$

The problems (P) and (P_λ) require relaxation. Indeed, imagine that the direct method of the calculus of variations is applied to solve these problems. In this case such an approach will involve a sequence of characteristic functions, say $\{\chi_n\}_{n\in\mathbb{N}}$. Its *weak*-$*$ limit in $L^\infty(\Omega)$ is not a characteristic function but a function $0 \leqslant m(\boldsymbol{\xi}) \leqslant 1$, $\boldsymbol{\xi} \in \Omega$.

Construction of the relaxed problem for (P_λ) will be the subject of the next section.

3. Relaxation of the layout problem (P_λ) for membrane shells

As we already know, relaxation means admitting weak limits of sequences $\{\chi_n\}_{n\in\mathbb{N}}$ of two-phase designs. These *weak*-$*$ limits in $L^\infty(\Omega;\{0,1\})$ are understood by

$$\lim_{n\to\infty}\int\limits_{\Omega}\varphi\chi_n\,d\boldsymbol{\xi}\to\int\limits_{\Omega}\varphi m\,d\boldsymbol{\xi}\qquad\forall\,\varphi\in L^1(\Omega)\tag{3.1}$$

The limit m belongs to $L^\infty(\Omega;[0,1])$ and $(L^1(\Omega))^*=L^\infty(\Omega)$ in the sense of duality between L^1 and L^∞, cf. Ekeland and Temam (1976).

According to Remark 1, there exist statically determinate problems of membrane shells in which $\mathbb{S}(\Omega)$ is one-element set, say $\{\widehat{\mathbf{N}}\}=\{(\widehat{N}^{\alpha\beta})\}$. This element is independent of χ_n. The construction of the relaxation problem for (P_λ) depends heavily on whether the equilibrium problem is statically determinate or not.

3.1. Case of the equilibrium problem being statically determined

Let $\mathbb{S}(\Omega)=\{\widehat{\mathbf{N}}\}$. Then (P_λ) assumes the form

$$\inf_{\chi\in L^\infty(\Omega;\{0,1\})}\int\limits_{S}[\widehat{\mathbf{N}}:(\mathfrak{a}\widehat{\mathbf{N}})+\lambda\chi]\,dS\tag{3.2}$$

Note that $\widehat{\mathbf{N}}$ does not depend on χ. We recall (2.17) and (2.18) and compute

$$\widehat{\mathbf{N}}:(\mathfrak{a}\widehat{\mathbf{N}})=[1-\chi(\boldsymbol{\xi})]\widehat{\mathbf{N}}:(\mathfrak{a}_1\widehat{\mathbf{N}})+\chi(\boldsymbol{\xi})\widehat{\mathbf{N}}:(\mathfrak{a}_2\widehat{\mathbf{N}})\tag{3.3}$$

Consider now a sequence $\{\chi_n\}$ such that $\chi_n\overset{*}{\rightharpoonup}m\in L^\infty(\Omega;[0,1])$. Hence $\mathfrak{a}$ is replaced with $\mathfrak{a}_n$ and

$$\begin{aligned}\widehat{\mathbf{N}}:(\mathfrak{a}_n\widehat{\mathbf{N}})\to 2W^*(\widehat{\mathbf{N}},m)&=[1-m(\boldsymbol{\xi})]\widehat{\mathbf{N}}:(\mathfrak{a}_1\widehat{\mathbf{N}})+\\ +m(\boldsymbol{\xi})\widehat{\mathbf{N}}:(\mathfrak{a}_2\widehat{\mathbf{N}})&\qquad\textit{weak-}*\text{ in }L^\infty\end{aligned}\tag{3.4}$$

Thus the problem (P_λ) is replaced by

$$(\overline{P}{}^1_\lambda)\quad\Big|\quad\min_{m\in L^\infty(\Omega;[0,1])}\int\limits_{S}[2W^*(\widehat{\mathbf{N}},m)+\lambda m]\,dS\tag{3.5}$$

The result above holds *irrespective* of the isotropy assumption. If both the phases are isotropic, the potential W^* can be easily expressed in terms of

$(\operatorname{tr}\widehat{\mathbf{N}})^2$ and $\operatorname{tr}\widehat{\mathbf{N}}^2$. Note yet that both the phases must be non-degenerated. The relaxation does not pave the way for a formulation of the shape design problem in which the shell is made of one material.

3.2. The statically indeterminate case

In the case considered the layout of both the materials influences the distribution of the stress resultants $(N^{\alpha\beta})$. Therefore, the limit behaviour of the sequence of functionals

$$\inf_{\mathbf{N}\in\mathbb{S}(\Omega)} \int_S [\mathbf{N} : (\mathfrak{a}(\chi_n)\mathbf{N}) + \lambda\chi_n]\, dS \tag{3.6}$$

where $\mathfrak{a}(\chi_n)$ is given by (2.17), should be considered within the framework of the Γ-convergence theory.

The above convergence problem has already been solved in a broader context of the Koiter shell theory, see Telega and Lewiński (1998) and Lewiński and Telega (2000, Sec. 17). We use here these results and conclude that (3.6) must be replaced by

$$\inf_{\mathbf{N}\in\mathbb{S}(\Omega)} \int_S [2W^*(\mathbf{N}, m) + \lambda m]\, dS \tag{3.7}$$

where the potential W^* is defined as follows. First we assign a basic cell $Y = (0, l_1) \times (0, l_2)$ to each point $\boldsymbol{\xi}$ of Ω. This cell is composed of both the materials, their distribution being given by the characteristic functions

$$\chi^Y = \chi_2^Y \qquad \chi_1^Y = 1 - \chi^Y$$
$$\chi^Y = \chi^Y(\boldsymbol{y}) \quad \boldsymbol{y} = (y_1, y_2) \in Y$$

Averaging over Y is denoted by $\langle\cdot\rangle$ and defined by

$$\langle f\rangle = \frac{1}{|Y|}\int_Y f\, d\boldsymbol{y} \qquad |Y| = l_1 l_2 \tag{3.8}$$

Distribution of the flexibilities $\mathfrak{a} = (\mathfrak{a}_{\alpha\beta\lambda\mu})$ within Y is expressed as follows

$$\mathfrak{a} = [1 - \chi^Y(\boldsymbol{y})]\mathfrak{a}_1 + \chi^Y(\boldsymbol{y})\mathfrak{a}_2 \tag{3.9}$$

Let us define the set

$$\begin{aligned}\mathbb{S}^{per}(Y) = \Big\{\widetilde{\mathbf{N}} \in L^2(Y, \mathbb{E}_2^s) \;\Big|\; &\frac{\partial \widetilde{N}^{\alpha\beta}}{\partial y_\beta} = 0 \text{ in } Y,\\ &\widetilde{N}^{\alpha\beta}\nu_\beta \text{ take opposite values at opposite sides of } Y\Big\}\end{aligned} \tag{3.10}$$

Here $\boldsymbol{\nu} = (\nu_\alpha)$ represents the unit vector normal to ∂Y.

Further, we introduce the set G_m^{per} of tensors $\mathfrak{a}_h$ such that

$$\mathbf{N} : (\mathfrak{a}_h \mathbf{N}) = \min\Big\{ \langle \widetilde{\mathbf{N}} : (\mathfrak{a}\widetilde{\mathbf{N}}) \rangle \,|\, \widetilde{\mathbf{N}} \in \mathbb{S}^{per}(Y), \ \ \langle \chi^Y \rangle = m, \ \ \langle \widetilde{\mathbf{N}} \rangle = \mathbf{N} \Big\} \tag{3.11}$$

where $\mathfrak{a}$ is given by (3.9). Its closure is denoted by G_m, i.e.

$$G_m = \overline{G_m^{per}} \tag{3.12}$$

where the completion $\overline{(\cdot)}$ is understood as admitting hierarchical microstructures within Y, like e.g. laminates of higher rank, see e.g. Cherkaev (2000).

Now we are ready to define the potential W^*

$$W^*(\mathbf{N}, m) = \min\Big\{ \frac{1}{2} \mathbf{N} : (\mathfrak{a}\mathbf{N}) \,|\, \mathfrak{a} \in G_m \Big\} \tag{3.13}$$

A detailed analysis of the passage from (3.6) to (3.7), (3.13) will be given elsewhere.

To put it briefly, the homogenization process retains the highest derivatives of (2.9) at the microstructural level; hence the simplified form of the equilibrium equations in the definition of the set $\mathbb{S}(Y)$.

Let us note now that the definition (3.11) coincides with that of the plane elasticity problem, see Lewiński and Telega (2000, Sec. 28.4). In the case of the phases being isotropic, an explicit form of W^* is known. It was found by Gibiansky and Cherkaev (1987). Prior to recalling this expression let us introduce some auxiliary notation. The invariants of $\mathbf{N}$ are chosen as

$$I(\mathbf{N}) = \frac{1}{\sqrt{2}} \operatorname{tr} \mathbf{N} \qquad\qquad II(\mathbf{N}) = \frac{1}{\sqrt{2}} [(\operatorname{tr} \mathbf{N})^2 - 4 \det \mathbf{N}]^{\frac{1}{2}} \tag{3.14}$$

Next we set

$$\zeta_{\mathbf{N}} = \frac{II(\mathbf{N})}{|I(\mathbf{N})|} \tag{3.15}$$

If f takes two values f_1 and f_2, we define

$$\begin{aligned} &\langle f \rangle_m = (1-m) f_1 + m f_2 \qquad\qquad \Delta f = |f_2 - f_1| \\ &[f]_m = (1-m) f_2 + m f_1 \end{aligned} \tag{3.16}$$

We assume here isotropy of both phases, as in Eqs (2.17) and (2.18). Let us define the auxiliary quantities

$$\begin{aligned} &\check{K} = \frac{K_1K_2 + L_2\langle K\rangle_m}{L_2 + [K]_m} && \check{L} = \frac{L_1L_2 + K_2\langle L\rangle_m}{K_2 + [L]_m} \\ &\zeta_1 = \frac{K_2 + [L]_m}{m\Delta L} && \zeta_2 = \frac{m\Delta K}{[K]_m + L_2} \\ &a_L = \check{K} \qquad a_R = K_2 && c_L = L_2 \qquad c_R = \check{L} \\ &A_L = \frac{m\Delta L(L_2 + [K]_m)}{[K+L]_m} && A_R = \frac{m(1-m)(\Delta L)^2[K]_m}{[L]_m[K+L]_m} \end{aligned} \tag{3.17}$$

Further, we introduce the function

$$H(\zeta) = \begin{cases} H_L(\zeta) & \text{if} \quad \zeta \in [0,\zeta_2] \\ H_i(\zeta) & \text{if} \quad \zeta \in [\zeta_2,\zeta_1] \\ H_R(\zeta) & \text{if} \quad \zeta \geqslant \zeta_1 \end{cases} \tag{3.18}$$

where

$$H_L(\zeta) = a_L + c_L\zeta^2 \qquad\qquad H_R(\zeta) = a_R + c_R\zeta^2 \tag{3.19}$$

and

$$H_i(\zeta) = H_L(\zeta) + A_L(\zeta - \zeta_2)^2 \tag{3.20}$$

or

$$H_i(\zeta) = H_R(\zeta) + A_R(\zeta - \zeta_1)^2 \tag{3.21}$$

The potential W^* assumes the following form, cf. Lewiński and Telega (2000, Sec. 28.4)

$$2W^*(\mathbf{N},m) = \begin{cases} \frac{1}{2}I^2(\mathbf{N})H(\zeta_{\mathbf{N}}) & \text{if} \quad I(\mathbf{N}) \neq 0 \\ \dfrac{1}{2}\check{L}\,II^2(\mathbf{N}) & \text{if} \quad I(\mathbf{N}) = 0 \end{cases} \tag{3.22}$$

Let us note that $W^*(\cdot,m)$ is smooth along the interfaces $\zeta_{\mathbf{N}} = \zeta_2$ and $\zeta_{\mathbf{N}} = \zeta_1$ of three regimes occurring in (3.18). Thus $W^*(\cdot,m)$ is smooth for all $\mathbf{N}$.

We conclude that the relaxation of (P_λ) gives

$$(\overline{P}^2_\lambda) \quad \left| \begin{array}{l} \displaystyle\min_{m\in L^\infty(\Omega;[0,1])} \min_{\mathbf{N}\in\mathbb{S}(\Omega)} \int\limits_S [2W^*(\mathbf{N},m) + \lambda m]\, dS \\ \text{and} \\ \displaystyle\int\limits_S m\, dS = \mathcal{A} \end{array} \right. \tag{3.23}$$

This formulation looks like a similar problem for two-dimensional elasticity. The shell characteristics are concealed in $\mathbb{S}(\Omega)$, where the differential equilibrium equations involve the metric tensor and the curvature tensor of the shell middle surface.

The sub-problem

$$\min_{\mathbf{N}\in\mathbb{S}(\Omega)} \int\limits_S W^*(\mathbf{N}, m)\, dS \tag{3.24}$$

is a non-linear equilibrium problem of a hypothetic physically nonlinear membrane shell.

4. Shape design problem

Shape design means forming a shell from one given material, thus admitting some voids in S such that the isoperimetric condition $(3.23)_2$ holds. In the shape design problem we usually assume that the surface loading $\boldsymbol{q}$ is absent, to prevent from cutting out a loaded part of the shell.

Shape design formulation should emerge as a result of passing to zero: $k_1 \to 0$, $\mu_1 \to 0$ or $K_1 \to +\infty$, $L_1 \to +\infty$.

In the case of the shell problem being statically determined, the above passage to the limit is not allowable. The definition (3.4) of W^* cannot be used if k_1 or μ_1 tend to zero.

In the statically indeterminate problems the passage to the limit $k_1 \to 0$, $\mu_1 \to 0$ is admissible, although the potential $W^*(\mathbf{N}, m)$ loses its smoothness. Then it assumes the form

$$W^*(\mathbf{N}, m) = W_0^*(\mathbf{N}) + \frac{1-m}{m} G(\mathbf{N}) \tag{4.1}$$

where

$$\begin{aligned} W_0^*(\mathbf{N}) &= \frac{1}{4} K\, I^2(\mathbf{N}) + \frac{1}{4} L\, II^2(\mathbf{N}) \\ G(\mathbf{N}) &= \frac{1}{4}(K+L)(|N_I|^2 + |N_{II}|^2) \end{aligned} \tag{4.2}$$

where $K = K_2$, $L = L_2$ and N_I, N_{II} represent the principal values of the tensor $\mathbf{N}$. The potential W_0^* refers to the one-phase material of moduli $k = k_2$, $\mu = \mu_2$; $K = 1/k$, $L = 1/\mu$. The expression (4.1) is the same as in the plane elasticity case, see Allaire and Kohn (1993).

Having found (4.1) one can consider the degenerated case when $\mathcal{A}$ is a very small number. This corresponds to the case of the Lagrangian multiplier λ being a large number. Following the arguments of Allaire and Kohn (1993), we conclude that the optimization problem reduces to

$$\min_{\mathbf{N}\in\mathbb{S}(\Omega)} \int\limits_S (|N_I| + |N_{II}|)\, dS \tag{4.3}$$

see also Lewiński and Telega (2001). The problem above is free of any material characteristics. It resembles the Michell formulation, see Strang and Kohn (1983). Using the same duality arguments we can pass from (4.3) to a dual formulation

$$\max_{\epsilon(u,w)\in B} \int\limits_{\partial S_T} \boldsymbol{T}\cdot\boldsymbol{u}\, ds \tag{4.4}$$

with

$$B = \Big\{\boldsymbol{\epsilon} \in \mathbb{E}_2^s \,|\, |\epsilon_I| \leqslant 1, \quad |\epsilon_{II}| \leqslant 1\Big\} \tag{4.5}$$

Problem (4.4) can be viewed as a locking problem, while B can be treated as a locking locus, see Telega and Jemioło (1998).

The literature on Michell's structures, see e.g. Hemp (1973), concerns plane problems with one exception: the problem of forming the stiffest spatial gridwork subjected to two opposite torques. Michell (1904) claims that the stiffest network should be formed on a sphere, yet the proof of this property has never been published. This famous Michell's sphere problem can also be put in the form: find the optimal layout of fibres forming the lightest spherical gridwork capable of resisting two opposite concentrated torques applied at two given points. These points are taken as poles of the optimal spherical gridwork, see Michell (1904) and Hemp (1973). This Michell problem is formulated by (4.4). Its solution can be found in Hemp (1973). At first one should find $(\boldsymbol{u}, w)$ such that $\epsilon_I = 1$ and $\epsilon_{II} = -1$, uniformly on the sphere. The work done by the tractions $\boldsymbol{T}$ (which replace the torques) determines the optimal weight. This solution is conditioned by the assumption of the spherical shape of the shell.

5. Final remarks

In the problem considered the shell middle surface S is taken as known. More challenging problem is to admit certain variations of S and find its best shape. As indicated above, an open problem is whether just a sphere is the stiffest among all shells of revolution subjected to two opposite torques, cf. Michell (1904) and Hemp (1973). A general treatment of the problems with varying middle surfaces seems to be a difficult task – let us remind mathematical difficulties appearing in the classical problem of minimal surfaces, see Nitsche (1975), Dierkes et al. (1992) and Pilz (1997).

Our considerations did not take into account prestressing of a membrane to enforce a free of folding membrane behaviour. The role of prestressing is described in Barnes (1988). Including prestressing in the formulation presented here is a challenge for future work.

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Optymalne projektowanie powłok w zakresie pracy bezmomentowej. Relaksacja zadania optymalizacji rozkładu dwu materiałów z wykorzystaniem metod homogenizacji

Streszczenie

Odpowiednie kształtowanie konstrukcji powłokowych zezwala na minimalizację efektów zginania. Konstrukcje zaprojektowane idealnie powinny pracować bezmomentowo, co podkreśla szczególną rolę teorii powłok błonowych, czyli powłok nie podlegających zginaniu. W pracy rozpatrujemy klasyczne zadanie optymalizacji rozmieszczenia dwu materiałów izotropowych w powłoce pracującej bezmomentowo w celu maksymalizacji jej sztywności. Ilość obu materiałów jest z góry ustalona. Celem pracy jest przeformułowanie tego zagadnienia do postaci dobrze postawionej. Założenie bezmomentowej pracy powłoki może być narzucone od początku lub przyjęte już po procesie relaksacji (w sensie rachunku wariacyjnego). W tej pracy wykazujemy, że ta

ostatnia metoda modelowania jest bardziej korzystna. Otrzymuje się sformułowanie, które zachowuje się stabilnie nawet wtedy, gdy jeden z materiałów degeneruje się do pustek, co zezwala na otrzymanie dobrze sformułowanego zadania optymalizacji kształtu.

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