JOURNAL OF THEORETICAL
AND APPLIED MECHANICS
41, 3, pp. 487-508, Warsaw 2003

ASYMPTOTIC ANALYSIS OF NONLINEARLY ELASTIC SHELLS WITH VARIABLE THICKNESS

Liliana Gratie

Liu Bie Ju Center for Mathematical Sciences, City University of Hong Kong
e-mail: mcgratie@cityu.edu.hk

P.G. Ciarlet recently proposed, and justified with A. Roquefort through the method of formal asymptotic expansions, a nonlinear shell model for shells with constant thickness. This model is analogous in its form to the model formerly proposed by W.T. Koiter, but is more amenable to numerical computations. In the same spirit, we propose and we justify here, again by the method of formal asymptotic expansions, a more general nonlinear model, which is valid for shells with variable thickness.

Key words: asymptotic analysis, nonlinearly elastic shells, Koiter's model, variable thickness, energy functional, variational problems

1. Introduction and technical preliminaries

In this paper, we propose and, using the method of formal asymptotic expansions, we justify a shell model "of Koiter's type" for nonlinearly elastic shells with variable thickness, which extends that proposed by Ciarlet (2000b) for shells with constant thickness. In doing so, we show that nonlinearly elastic shells with variable thickness have two essentially distinct limit behaviors as their thickness approaches zero, either that of a nonlinearly elastic membrane shell or that of a nonlinearly elastic flexural shell with variable thickness. Complete proofs and further details will be found in Gratie (2003).

We emphasize here that "membrane and flexural shells" represents a general terminology about shells that is commonly used in the Western literature, as in e.g., Ciarlet (2000a). Other terminologies are often favored. In this direction, the author is grateful to the referee, who pointed out that "membrane shells" and "flexural shells" could be equally well labeled as "geometrically rigid shells" and "geometrically bendable shells". This latter terminology is adopted in the present article.

Note that, while there is a huge literature about shells with constant thickness (see e.g., the extensive list of references provided in Ciarlet (2000a)), comparatively much less attention has been paid to the analysis of shells with *variable thickness*. This problem however, was addressed in the pioneering contributions of Ladevèze (1976) and Busse (1997) for *linearly elastic shells*.

The derivation of variational equations of our model is based on the method of asymptotic expansions. We use here the well-established "*variational approach of Ciarlet*" (to paraphrase Gilbert and Vashakmadze (2000)) and, in particular, we use the same notations as in Ciarlet (2000a). As is customary in the mathematical elasticity theory, Greek indices or exponents: α, β, μ, etc. take their values in the set $\{1,2\}$, while Latin indices or exponents: i, j, k, etc. take their values in the set $\{1,2,3\}$, and we use the summation convention with respect to repeated indices and exponents.

Let ω be a domain in $\mathbb{R}^2$, i.e., an open, bounded, connected subset with a Lipschitz- continuous boundary $\gamma = \partial\omega$, such that the set ω is locally on one side of γ, and let $y = (y_\alpha)$ denote a generic point in the closed set $\overline{\omega}$. The area element in ω is dy and the partial derivatives with respect to the variable y are denoted $\partial_\alpha = \partial/\partial y_\alpha$ and $\partial_{\alpha\beta} = \partial^2/(\partial y_\alpha \partial y_\beta)$. The length element along the boundary γ is denoted $d\gamma$, the unit outer normal vector and the unit tangent vector along γ are respectively denoted (ν_α) and (τ_α), where $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$. We denote by $\partial_\nu f = \nu_\alpha \partial_\alpha f$ the outer normal derivative of a function f along the boundary γ, and similarly, its tangential derivative by $\partial_\tau f = \tau_\alpha \partial_\alpha f$. Sometimes, the "horizontal" curvilinear coordinates x_α will be also denoted y_α.

Let $\boldsymbol{\theta} : \overline{\omega} \subset \mathbb{R}^2 \to \mathbb{R}^3$ be an injective and smooth enough mapping, such that the two vectors $\boldsymbol{a}_\alpha(y) := \partial_\alpha \boldsymbol{\theta}(y)$ are linearly independent at all points $y = (x_1, x_2) \in \overline{\omega}$. They form the covariant basis of the tangent plane to the surface $\mathcal{S} := \boldsymbol{\theta}(\overline{\omega})$ at the point $\boldsymbol{\theta}(y)$. On the other hand, the two vectors $\boldsymbol{a}^\alpha(y)$, defined by the relations $\boldsymbol{a}^\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \delta^\alpha_\beta$, form the contravariant basis of the same tangent plane.

We consider a third vector, normal to $\mathcal{S}$ at the point $\boldsymbol{\theta}(y)$, with Euclidean norm one, defined by

$$\boldsymbol{a}_3(y) = \boldsymbol{a}^3(y) = \frac{\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)}{|\boldsymbol{a}_1(y) \times \boldsymbol{a}_2(y)|}$$

The triple $(\boldsymbol{a}^1(y), \boldsymbol{a}^2(y), \boldsymbol{a}^3(y))$ is the contravariant basis at $\boldsymbol{\theta}(y)$, and similarly, $(\boldsymbol{a}_1(y), \boldsymbol{a}_2(y), \boldsymbol{a}_3(y))$ is the covariant basis at the same point.

A general shell structure can be fully represented by a middle surface geometry and the thickness at each point of its middle surface.

We intend to model a family of nonlinearly elastic thin shells having in common the middle surface $\mathcal{S} := \boldsymbol{\theta}(\overline{\omega})$, and such that for each "small" parameter $\varepsilon > 0$, the variable thickness of each shell is defined by $h(y) := 2\varepsilon e(y)$ for all $y \in \overline{\omega}$, where

$$e : \overline{\omega} \to \mathbb{R}$$

is a given function, which does not depend on ε. We shall assume for definiteness that $e \in W^{2,\infty}(\omega)$. We also assume that the "thickness function" e does not vanish in ω. Thus, there exist two positive constants e_0 and e_1 such that

$$0 < e_0 \leqslant e(y) \leqslant e_1 \qquad \forall\, y \in \overline{\omega}$$

Furthermore, we consider that the shells are *symmetric* with respect to their middle surface $\mathcal{S}$.

Thereby, we focus our study on elastic bodies whose reference configurations consist of all points within a distance less than $\varepsilon e(y)$ from the middle surface $\mathcal{S}$. The reference configuration of the shell is the three-dimensional set $\boldsymbol{\Theta}^e(\overline{\Omega}^\varepsilon)$, where $\Omega^\varepsilon = \omega \times]-\varepsilon, \varepsilon] \subset \mathbb{R}^3$, and the mapping $\boldsymbol{\Theta}^e : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ is defined by

$$\boldsymbol{\Theta}^e(y, x_3^\varepsilon) = \boldsymbol{\theta}(y) + e(y) x_3^\varepsilon \boldsymbol{a}_3(y)$$

for all $x^\varepsilon = (x_1, x_2, x_3^\varepsilon) = (y, x_3^\varepsilon) \in \overline{\Omega}^\varepsilon$. The curvilinear coordinate $x_3^\varepsilon \in [-\varepsilon, \varepsilon]$ is called the transverse variable.

For a generic point $x^\varepsilon = (x_i^\varepsilon) \in \overline{\Omega}^\varepsilon$, we let $\partial_i^\varepsilon = \partial/\partial x_i^\varepsilon$. For $\varepsilon > 0$ small enough, the mapping $\boldsymbol{\Theta}^e : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ is injective (see Ciarlet, 2000a) and the three vectors $\boldsymbol{g}_i^{e,\varepsilon}(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}^e(x^\varepsilon)$ are linearly independent. This shows that the physical problem is well posed since the set $\boldsymbol{\Theta}^e(\overline{\Omega}^\varepsilon)$ does not interpenetrate itself.

The three vectors $\boldsymbol{g}_i^{e,\varepsilon}(x^\varepsilon)$ form the covariant basis (of the tangent space, here $\mathbb{R}^3$, to the manifold $\boldsymbol{\Theta}^e(\overline{\Omega}^\varepsilon)$) at the point $\boldsymbol{\Theta}^e(x^\varepsilon)$, and the three vectors $\boldsymbol{g}^{i,e,\varepsilon}(x^\varepsilon)$ defined by the relations $\boldsymbol{g}^{i,e,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_j^{e,\varepsilon}(x^\varepsilon) = \delta_j^i$ form the contravariant basis at $\boldsymbol{\Theta}^e(x^\varepsilon)$.

Each shell is subjected to:

- a boundary condition of place along the portion $\boldsymbol{\Theta}^e(\gamma_0 \times [-\varepsilon, \varepsilon])$ of its lateral face $\boldsymbol{\Theta}^e(\gamma \times [-\varepsilon, \varepsilon])$, where $\gamma_0 \subset \gamma$ with $length(\gamma_0) > 0$; this means that the displacement vanishes on $\boldsymbol{\Theta}^e(\gamma_0 \times [-\varepsilon, \varepsilon])$

- applied body forces $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ in its interior $\boldsymbol{\Theta}^e(\Omega^\varepsilon)$

- applied surface forces $h^{i,\varepsilon} \in L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$ on its upper and lower faces $\boldsymbol{\Theta}^e(\Gamma_+^\varepsilon)$ and $\boldsymbol{\Theta}^e(\Gamma_-^\varepsilon)$, where $\Gamma_+^\varepsilon := \omega \times \{\varepsilon\}$ and $\Gamma_-^\varepsilon := \omega \times \{-\varepsilon\}$.

We recall now some elementary notions from differential geometry in $\mathbb{R}^3$. The area element along the surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$ is $\sqrt{a}\,dy$ where $a = \det\{a_{\alpha\beta}(y)\}$, and

$$a_{\alpha\beta}(y) = \boldsymbol{a}_\alpha(y) \cdot \boldsymbol{a}_\beta(y) = \partial_\alpha \boldsymbol{\theta}(y) \cdot \partial_\beta \boldsymbol{\theta}(y)$$

are the *covariant components of the metric tensor* of the surface $\mathcal{S}$ (also named the first fundamental form of $\mathcal{S}$). Similarly, the contravariant components of the metric tensor of $\mathcal{S}$ are defined by $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$.

Note that the matrix $\{a_{\alpha\beta}(y)\}$ is positive definite since the vectors $\boldsymbol{a}_\alpha(y)$ are assumed to be linearly independent. In particular, there exists a positive constant a_0 such that $0 < a_0 \leqslant a(y)$, for all $y \in \overline{\omega}$.

Having given a surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$ and a displacement field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ of $\mathcal{S}$ with smooth enough covariant components $\eta_i : \overline{\omega} \to \mathbb{R}$, we let

$$\boldsymbol{\eta}^e := \eta_\alpha \boldsymbol{a}^\alpha + \frac{1}{e}\eta_3 \boldsymbol{a}^3 \qquad \boldsymbol{a}_\alpha^e(\boldsymbol{\eta}) := \partial_\alpha(\boldsymbol{\theta} + \boldsymbol{\eta}^e)$$

$$a_{\alpha\beta}^e(\boldsymbol{\eta}) := \boldsymbol{a}_\alpha^e(\boldsymbol{\eta}) \cdot \boldsymbol{a}_\beta^e(\boldsymbol{\eta}) \quad G_{\alpha\beta}^e(\boldsymbol{\eta}) := \frac{1}{2}(a_{\alpha\beta}^e(\boldsymbol{\eta}) - a_{\alpha\beta})$$

The displacement field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ of the middle surface $\mathcal{S}$ is said to be *admissible* if it vanishes along the curve $\boldsymbol{\theta}(\gamma_0)$, where $\gamma_0 \subset \gamma = \partial\omega$ has *length*$(\gamma_0) > 0$, which means that $\boldsymbol{\eta} = \mathbf{0}$ on γ_0.

Extending the definition given in Miara (1998) and Ciarlet (2000a, Chapter 9), we say that a shell is a *nonlinearly elastic, geometrically rigid shell with variable thickness* if

$$\{\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}^{2,p}(\omega); \quad \boldsymbol{\eta} = \mathbf{0} \;\text{ on }\; \gamma_0, \quad a_{\alpha\beta}^e(\boldsymbol{\eta}) - a_{\alpha\beta} = 0 \;\text{ in }\; \omega\} = \{\mathbf{0}\}$$

Note that the various regularities mentioned above or subsequently are simply chosen so that the energies (to be introduced later) are differentiable.

The covariant and mixed components of the *curvature tensor* of $\mathcal{S}$ (also named the second fundamental form of the surface) are respectively defined by

$$b_{\alpha\beta} = \boldsymbol{a}^3 \cdot \partial_\beta \boldsymbol{a}_\alpha \qquad \text{and} \qquad b_\alpha^\beta = a^{\beta\sigma} b_{\sigma\alpha}$$

If the two vectors $a_\alpha^e(\boldsymbol{\eta})$ are linearly independent in ω, we let

$$R_{\alpha\beta}^e(\boldsymbol{\eta}) := b_{\alpha\beta}^e(\boldsymbol{\eta}) - b_{\alpha\beta}$$

where

$$b_{\alpha\beta}^e(\boldsymbol{\eta}) := \partial_{\alpha\beta}(\boldsymbol{\theta} + \boldsymbol{\eta}^e) \cdot \frac{\boldsymbol{a}_1^e(\boldsymbol{\eta}) \times \boldsymbol{a}_2^e(\boldsymbol{\eta})}{|\boldsymbol{a}_1^e(\boldsymbol{\eta}) \times \boldsymbol{a}_2^e(\boldsymbol{\eta})|}$$

Next, we define functions

$$\eta^e_{\beta//\alpha} := \partial_\alpha \eta_\beta - \varGamma^\sigma_{\alpha\beta}\eta_\sigma - \frac{1}{e} b_{\alpha\beta}\eta_3$$
$$\eta^e_{3//\alpha} := b^\sigma_\alpha \eta_\sigma + \partial_\alpha\Big(\frac{1}{e}\eta_3\Big)$$

where the *Christoffel symbols* of the surface $\mathcal{S}$ are given by $\varGamma^\sigma_{\alpha\beta} = \boldsymbol{a}^\sigma \cdot \partial_\beta \boldsymbol{a}_\alpha$.

Accordingly, we can rewrite the components $G^e_{\alpha\beta}(\boldsymbol{\eta})$ as

$$G^e_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(a^e_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta}) = \frac{1}{2}(\eta^e_{\alpha//\beta} + \eta^e_{\beta//\alpha} + a^{mn}\eta^e_{m//\alpha}\eta^e_{n//\beta})$$

where $a^{i3} = a^{3i} := \delta^{i3}$.

The Gâteaux derivatives of each function $G^e_{\alpha\beta} : \mathbb{W}^{1,4}(\omega) \to L^2(\omega)$ are given by

$$(G^e_{\alpha\beta})'(\boldsymbol{\zeta})\boldsymbol{\eta} := \frac{1}{2}[\eta^e_{\alpha//\beta} + \eta^e_{\beta//\alpha} + a^{mn}(\zeta^e_{m//\alpha}\eta^e_{n//\beta} + \zeta^e_{n//\beta}\eta^e_{m//\alpha})]$$

We assume for simplicity that the shells are made of an homogeneous isotropic material of Saint Venant-Kirchhoff's type. This implies in particular that the reference configuration $\boldsymbol{\varTheta}^e(\overline{\varOmega}^\varepsilon)$ is a natural state, i.e. stress-free. Hence, the material is characterized by its two Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$, and the contravariant components $a^{\alpha\beta\sigma\tau,\varepsilon}$ of its two-dimensional elasticity tensor are given by

$$a^{\alpha\beta\sigma\tau,\varepsilon} := \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon + 2\mu^\varepsilon} a^{\alpha\beta}a^{\sigma\tau} + 2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

Extending the definition given in Lods and Miara (1998) and Ciarlet (2000a, Chapter 10), we say that a nonlinearly elastic shell with the middle surface $\mathcal{S}$, subjected to a boundary condition of place along the portion of its lateral face with $\boldsymbol{\theta}(\gamma_0)$ as its middle curve, where $\gamma_0 \subset \gamma$ and $length(\gamma_0) > 0$, is a *nonlinearly elastic, geometrically bendable shell with variable thickness*, if the manifold

$$\mathbb{W}^e_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}^{2,4}(\omega);\ \ \boldsymbol{\eta} = \partial_\nu\boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0,\ \ a^e_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta} = 0 \text{ in } \omega\}$$

and its tangent space

$$T_\zeta\mathbb{W}^e_F(\omega) := \{\boldsymbol{\eta} \in \mathbb{W}^{2,4}(\omega);\ \ \boldsymbol{\eta} = \partial_\nu\boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0,\ \ (G^e_{\alpha\beta})'(\boldsymbol{\zeta})\boldsymbol{\eta} = 0 \text{ in } \omega\}$$

contains nonzero functions, i.e., $\mathbb{W}^e_F(\omega) \neq \{\boldsymbol{0}\}$ and $T_\zeta\mathbb{W}^e_F(\omega) \neq \{\boldsymbol{0}\}$ at each $\boldsymbol{\zeta} \in \mathbb{W}^e_F(\omega)$.

Note that the admissible displacement field must satisfy in this case the "two-dimensional boundary conditions of strong clamping" along the curve $\boldsymbol{\theta}(\gamma_0)$, i.e. not only the points and the tangents spaces (as for the weaker boundary conditions of clamping $\eta_i = \partial_\nu \eta_3 = 0$ on γ_0), but also the vectors tangent to the coordinate lines of the deformed and undeformed middle surfaces coincide along the curve $\boldsymbol{\theta}(\gamma_0)$. This remark emphasizes the essential role played by the set $\boldsymbol{\theta}(\gamma_0)$ for determining the type of a shell.

2. Two-dimensional variational scaled problems for geometrically rigid and bendable, nonlinearly elastic shells with variable thickness

In this section, we convert into "the displacement approach" the two-dimensional equations of nonlinearly elastic, geometrically rigid and geometrically bendable shells with variable thickness, as they were identified by Roquefort (2001, Chapter 4), through "the deformation approach".

Our aim is to study the behavior of the displacement field $u_i^\varepsilon \boldsymbol{g}^{i,e,\varepsilon} : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$ that the shell undergoes the influence of the applied forces as $\varepsilon \to 0$, by means of the method of formal asymptotic expansions. The unknown in the three-dimensional formulation is the vector field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon) : \overline{\Omega}^\varepsilon \to \mathbb{R}^3$, where the functions $u_i^\varepsilon : \overline{\Omega}^\varepsilon \to \mathbb{R}$ represent the covariant components of the displacement field of the shell.

This method relies in particular on two essential guiding rules: no restriction should be put on the applied forces and the linearization of any nonlinear equation found in this process should provide an equation from the linear theory ("linearization requirement").

The first task in the asymptotic analysis consists in transforming the three-dimensional problems $P^e(\Omega^\varepsilon)$ (for a geometrically rigid or geometrically bendable shell) into "asymptotically equivalent" problems posed over a domain independent of ε.

More specifically, we let

$$\Omega := \omega \times]-1,1[\quad \Gamma_0 := \gamma_0 \times [-1,1]$$
$$\Gamma_+ := \omega \times \{1\} \qquad \Gamma_- := \omega \times \{-1\}$$

where $x = (x_1, x_2, x_3)$ denotes a generic point in the closure $\overline{\Omega}$ of the fixed domain Ω, and $\partial_i := \partial/\partial x_i$. We make then the change of the variable from the fixed domain Ω to the domain Ω^ε through the bijection

$$\begin{aligned}\pi^\varepsilon : x = (x_1, x_2, x_3) \in \overline{\Omega} \to \pi^\varepsilon(x_1, x_2, x_3) = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = \\ = (x_1, x_2, \varepsilon x_3) = x^\varepsilon \in \overline{\Omega}^\varepsilon\end{aligned}$$

where the coordinate $x_3 \in [-1, 1]$ is the scaled transverse variable. The relations between the first derivatives with respect to the variable $x^\varepsilon \in \overline{\Omega}^\varepsilon$ and the derivatives of the same order with respect to the scaled variable belonging to the fixed domain $x \in \overline{\Omega}$ are

$$\partial_\alpha^\varepsilon = \partial_\alpha \qquad \text{and} \qquad \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$$

The scaled unknown $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbb{R}^3$ satisfies the scaled three-dimensional nonlinear variational problem $P^e(\varepsilon; \Omega)$ of a shell with variable thickness (in Section 4 of Roquefort (2001), it is derived by means of the deformations of the middle surface). To begin the asymptotic analysis, we first write the scaled unknown as a formal expansion in terms of powers of the thickness (considered as usually as a "small" parameter)

$$\boldsymbol{u}(\varepsilon) = \varepsilon^{-k}\boldsymbol{u}^{-k} + ... + \varepsilon^{-2}\boldsymbol{u}^{-2} + \varepsilon^{-1}\boldsymbol{u}^{-1} + \boldsymbol{u}^0 + \varepsilon\boldsymbol{u}^1 + \varepsilon^2\boldsymbol{u}^2 + ...$$

for some integer $k \geqslant 0$.

Given a function $\boldsymbol{v} : \omega\times]-1,1[\to \mathbb{R}^3$, let $\overline{\boldsymbol{v}} : \overline{\omega} \to \mathbb{R}^3$ represent its average defined by the integral

$$\overline{\boldsymbol{v}}(y) := \frac{1}{2}\int_{-1}^{1} \boldsymbol{v}(y, x_3)\, dx_3$$

Then we have (note that it can be proved as in Miara (1998) that there are no negative powers, i.e. the first nonzero term of the formal series is indeed $\boldsymbol{u}^0$):

Theorem 2.1. Consider a family of nonlinearly elastic, geometrically rigid shells with nonvanishing variable thickness $h(y) = 2\varepsilon e(y)$, $e \in W^{2,\infty}(\omega)$, with the same middle surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$ and with each subjected to a boundary condition of place along a portion of their lateral face having the same curve $\boldsymbol{\theta}(\gamma_0)$ as their middle line. Assume that the scaled unknown $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ satisfying the scaled three-dimensional variational problem $P^e(\varepsilon; \Omega)$ admits a formal asymptotic expansion of the form

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 + \varepsilon\boldsymbol{u}^1 + \varepsilon^2\boldsymbol{u}^2 + ...$$

Then, to free the applied forces from any restriction and to satisfy the linearization requirement, the Lamé constants and contravariant components of the applied loading must be of the form

$$\begin{aligned} &\lambda^\varepsilon = \lambda && \mu^\varepsilon = \mu \\ &f^{i,\varepsilon}(x^\varepsilon) = f^{i,0}(x) && \text{for} \quad x^\varepsilon = \pi^\varepsilon(x) \in \Omega^\varepsilon \\ &h^{i,\varepsilon}(x^\varepsilon) = \varepsilon h^{i,1}(x) && \text{for} \quad x^\varepsilon = \pi^\varepsilon(x) \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \end{aligned}$$

where the constants $\lambda > 0$, $\mu > 0$ and the scaled functions $f^{i,0}(x) \in L^2(\Omega)$, $h^{i,1}(x) \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of ε.

Under these hypotheses, the leading term $\boldsymbol{u}^0$ is independent of the transverse variable x_3 and its average

$$\boldsymbol{\zeta}^0 := (\zeta_i^0) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}^0 \, dx_3 = \overline{\boldsymbol{u}}^0$$

satisfies the scaled two-dimensional variational problem $P_M^e(\omega)$ of a nonlinearly elastic, geometrically rigid shell with variable thickness:

Find

$$\boldsymbol{\zeta}^0 \in \mathbb{W}_M(\omega) := \{\boldsymbol{\eta} \in \mathbb{W}^{1,4}(\omega); \quad \boldsymbol{\eta} = \mathbf{0} \ \text{ on } \gamma_0\}$$

such that

$$\int_\omega a^{\alpha\beta\sigma\tau} G_{\sigma\tau}^e(\boldsymbol{\zeta}^0)[(G_{\alpha\beta}^e)'(\boldsymbol{\zeta}^0)\boldsymbol{\eta}] e\sqrt{a} \, dy = \int_\omega p^{i,0} \eta_i e\sqrt{a} \, dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}_M(\omega)$, where, for any $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{W}^{1,4}(\omega)$

$$G_{\alpha\beta}^e(\boldsymbol{\eta}) := \frac{1}{2}[a_{\alpha\beta}^e(\boldsymbol{\eta}) - a_{\alpha\beta}] = \frac{1}{2}(\eta_{\alpha//\beta}^e + \eta_{\beta//\alpha}^e + a^{mn}\eta_{m//\alpha}^e \eta_{n//\beta}^e)$$

$$\eta_{\beta//\alpha}^e := \partial_\alpha \eta_\beta - \Gamma_{\alpha\beta}^\sigma \eta_\sigma - \frac{1}{e} b_{\alpha\beta}\eta_3$$

$$\eta_{3//\alpha}^e := b_\alpha^\sigma \eta_\sigma + \partial_\alpha\Big(\frac{1}{e}\eta_3\Big)$$

$$(G_{\alpha\beta}^e)'(\boldsymbol{\zeta})\boldsymbol{\eta} := \frac{1}{2}[\eta_{\alpha//\beta}^e + \eta_{\beta//\alpha}^e + a^{mn}(\zeta_{m//\alpha}^e \eta_{n//\beta}^e + \zeta_{n//\alpha}^e \eta_{m//\alpha}^e]$$

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma})$$

$$p^{i,0} := \int_{-1}^{1} f^{i,0} \, dx_3 + h_+^{i,1} + h_-^{i,1}$$

$$h_+^{i,1} = h^{i,1}(\cdot, +1) \qquad h_-^{i,1} = h^{i,1}(\cdot, -1)$$

■

Let the scaled two-dimensional energy $j_M^e : \mathbb{W}_M(\omega) \to \mathbb{R}$ of a nonlinearly elastic, geometrically rigid shell with variable thickness be defined by

$$j_M^e(\boldsymbol{\eta}) = \frac{1}{8}\int\limits_{\omega} a^{\alpha\beta\sigma\tau}[a_{\sigma\tau}^e(\boldsymbol{\eta}) - a_{\sigma\tau}][a_{\alpha\beta}^e(\boldsymbol{\eta}) - a_{\alpha\beta}]e\sqrt{a}\,dy - \int\limits_{\omega} p^{i,0}\eta_i e\sqrt{a}\,dy =$$

$$= \frac{1}{2}\int\limits_{\omega} a^{\alpha\beta\sigma\tau}G_{\sigma\tau}^e(\boldsymbol{\eta})G_{\alpha\beta}^e(\boldsymbol{\eta})e\sqrt{a}\,dy - \int\limits_{\omega} p^{i,0}\eta_i e\sqrt{a}\,dy$$

The functional j_M^e is differentiable over the Sobolev space $\mathbb{W}^{1,4}(\omega)$, hence also over its subspace $\mathbb{W}_M(\omega)$, and $\boldsymbol{\zeta}^0 \in \mathbb{W}_M(\omega)$ is a solution to the variational problem $P_M^e(\omega)$ of Theorem 2.1 if and only if it is a stationary point of the functional j_M^e over the space $\mathbb{W}_M(\omega)$, which means that $(j_M^e)'(\boldsymbol{\zeta}^0) = 0$. Hence, particular solutions to the problem $P_M^e(\omega)$ can be obtained by solving the minimization problem:

Find $\boldsymbol{\zeta} \in \mathbb{W}_M(\omega)$ such that

$$j_M^e(\boldsymbol{\zeta}) = \inf_{\eta \in \mathbb{W}_M(\omega)} j_M^e(\boldsymbol{\eta})$$

where the scaled unknown is the two-dimensional displacement vector field $\boldsymbol{\zeta} = (\zeta_i)$ and ζ_i are the covariant components of the displacement $\zeta_i \boldsymbol{a}^i : \overline{\omega} \to \mathbb{R}^3$ of the points of the middle surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$. More precisely, $\zeta_i(y)\boldsymbol{a}^i(y)$ is the displacement of the point $\boldsymbol{\theta}(y) \in \mathcal{S}$.

Thus we emphasize that, as expected for shells with nonconstant thickness, the specific computation leads to the fact that the "thickness function" $e : \overline{\omega} \to \mathbb{R}$ appears in the energy functional.

Consider next the case of geometrically bendable shells.

Theorem 2.2. Assume that the manifold $\mathbb{W}_F^e(\omega)$ defined in Section 1 contains nonzero elements and possesses nonzero tangent vectors at each of its points. Consider a family of nonlinearly elastic, geometrically bendable shells, with the nonvanishing variable thickness $h(y) = 2\varepsilon e(y)$, where $e \in W^{2,\infty}(\omega)$. Assume that they all have the same middle surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$ and that they are subjected to a boundary condition of place along a portion of their lateral face having the same curve $\boldsymbol{\theta}(\gamma_0)$ as their middle line. Finally, assume that the scaled unknown $\boldsymbol{u}(\varepsilon) = (u_i(\varepsilon))$ appearing in the scaled three-dimensional variational problem $P^e(\varepsilon; \Omega)$ admits a formal asymptotic expansion of the form

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 + \varepsilon\boldsymbol{u}^1 + \varepsilon^2\boldsymbol{u}^2 + ...$$

Let the Lamé constants be independent of ε, i.e. $\lambda^\varepsilon = \lambda$ and $\mu^\varepsilon = \mu$. Then, assuming that no restriction can be put on the applied forces involved into the equations verified by the leading term $\boldsymbol{u}^0$, and that the linearization requirement must be satisfied, their components must be scaled as follows

$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^{i,2}(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon(x) \in \Omega^\varepsilon$$
$$h^{i,\varepsilon}(x^\varepsilon) = \varepsilon^3 h^{i,3}(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon(x) \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$$

where the functions $f^{i,2}(x) \in L^2(\Omega)$ and $h^{i,3}(x) \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of ε.

This being the case, the leading term $\boldsymbol{u}^0 : \overline{\Omega} \to \mathbb{R}^3$ is independent of the transverse variable x_3 and its average

$$\boldsymbol{\zeta}^0 := (\zeta_i^0) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{u}^0 \, dx_3$$

satisfies the following scaled two-dimensional variational problem $P_F^e(\omega)$ of a nonlinearly elastic, geometrically bendable shell with variable thickness:

Find

$$\boldsymbol{\zeta}^0 \in \mathbb{W}_F^e(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,4}(\omega); \quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0, \quad G_{\alpha\beta}^e(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

such that

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau} R_{\sigma\tau}^e(\boldsymbol{\zeta}^0)[(R_{\alpha\beta}^e)'(\boldsymbol{\zeta}^0)\boldsymbol{\eta}]e^3\sqrt{a}\, dy = \int_\omega p^{i,2}\eta_i e\sqrt{a}\, dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{T}_{\zeta^0}\mathbb{W}_F^e(\omega)$, with

$$T_{\zeta^0}\mathbb{W}_F^e(\omega) := \{\boldsymbol{\eta} \in \mathbb{W}^{2,4}(\omega); \quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0, \quad (G_{\alpha\beta}^e)'(\boldsymbol{\zeta}^0)\boldsymbol{\eta} = 0 \text{ in } \omega\}$$

where

$$R_{\alpha\beta}^e(\boldsymbol{\eta}) := b_{\alpha\beta}^e(\boldsymbol{\eta}) - b_{\alpha\beta}$$
$$b_{\alpha\beta}^e(\boldsymbol{\eta}) := \partial_{\alpha\beta}(\boldsymbol{\theta} + \boldsymbol{\eta}^e) \cdot \frac{\boldsymbol{a}_1^e(\boldsymbol{\eta}) \times \boldsymbol{a}_2^e(\boldsymbol{\eta})}{|\boldsymbol{a}_1^e(\boldsymbol{\eta}) \times \boldsymbol{a}_2^e(\boldsymbol{\eta})|}$$

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma})$$

$$p^{i,2} = \int_{-1}^{1} f^{i,2}\, dx_3 + h_+^{i,3} + h_-^{i,3}$$

$$h_+^{i,3} = h^{i,3}(\cdot, +1) \qquad h_-^{i,3} = h^{i,3}(\cdot, -1)$$

■

For the sequel, we need to recast the two-dimensional variational problem $P_F^e(\omega)$ as a minimization problem. To this end, let the scaled two-dimensional energy of a nonlinearly elastic, geometrically bendable shell with variable thickness $j_F^e : \mathbb{W}_F^e(\omega) \to \mathbb{R}$ be defined by

$$j_F^e(\boldsymbol{\eta}) = \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}[b_{\sigma\tau}^e(\boldsymbol{\eta}) - b_{\sigma\tau}][b_{\alpha\beta}^e(\boldsymbol{\eta}) - b_{\alpha\beta}]e^3\sqrt{a}\, dy - \int_\omega p^{i,2}\eta_i e\sqrt{a}\, dy =$$

$$= \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau} R_{\sigma\tau}^e(\boldsymbol{\eta}) R_{\alpha\beta}^e(\boldsymbol{\eta}) e^3\sqrt{a}\, dy - \int_\omega p^{i,2}\eta_i e\sqrt{a}\, dy$$

The functional j_F^e is differentiable over the space $\mathbb{W}_F^e$, and $\boldsymbol{\zeta}^0$ is a solution to the variational problem $P_F^e(\omega)$ of Theorem 2.2 if and only if it is a stationary point of functional j_F^e over the space $\mathbb{W}_F^e$, which means that $(j_F^e)'(\boldsymbol{\zeta}^0) = 0$. Hence, particular solutions to problem $P_F^e(\omega)$ can be obtained by solving the minimization problem:

Find $\boldsymbol{\zeta} \in \mathbb{W}_F^e(\omega)$ such that

$$j_F^e(\boldsymbol{\zeta}) = \inf_{\eta \in \mathbb{W}_F^e(\omega)} j_F^e(\boldsymbol{\eta})$$

3. A two-dimensional nonlinear shell model of Koiter's type with variable thickness

Koiter's approach to nonlinear, constant thickness, shell theory is based upon two *a priori* assumptions (see Koiter, 1966):

- The first one is of a geometrical nature and it asserts that the normals to the middle surface stay normal to the deformed middle surface and the distance of any point on these normals to the middle surface remains constant (the Kirchhoff-Love assumption).

- The second one is of a mechanical nature and it consists in assuming that the state of stress inside the shell is planar and the stresses parallel to the middle surface vary linearly across the thickness. This assumption was justified in the fundamental work of John (1965).

Using these assumptions, Koiter showed that the displacement field across the thickness of the shell can be completely expressed in terms of the displacement field of the middle surface, and he determined a two-dimensional problem for finding this field.

By analogy, the strain energy for our *shell model of Koiter's type with variable thickness* could thus be simply the sum of the strain energy of a nonlinearly elastic, geometrically rigid shell and that of a nonlinearly elastic, geometrically bendable shell, both with variable thickness $h(y) = 2\varepsilon e(y)$, where $e \in W^{2,\infty}(\omega)$.

The unknown vector field $\boldsymbol{\zeta}^\varepsilon = (\zeta_i^\varepsilon) : \overline{\omega} \to \mathbb{R}^3$, where the functions $\zeta_i^\varepsilon : \overline{\omega} \to \mathbb{R}$ are the covariant components of the displacement field $\zeta_i^\varepsilon \boldsymbol{a}^i$ of the middle surface should thus solve the following two-dimensional variational problem $P_K^{\varepsilon,e}(\omega)$ for an *ad hoc* $p > 2$:

Find

$$\boldsymbol{\zeta}^\varepsilon \in \mathbb{W}_K(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,p}(\omega); \quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\}$$

such that

$$\varepsilon \int\limits_\omega a^{\alpha\beta\sigma\tau} G^e_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)[(G^e_{\alpha\beta})'(\boldsymbol{\zeta}^\varepsilon)\boldsymbol{\eta}] e\sqrt{a}\, dy +$$

$$+\frac{\varepsilon^3}{3} \int\limits_\omega a^{\alpha\beta\sigma\tau} R^e_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)[(R^e_{\alpha\beta})'(\boldsymbol{\zeta}^\varepsilon)\boldsymbol{\eta}] e^3\sqrt{a}\, dy = \int\limits_\omega p^{i,\varepsilon}\eta_i e\sqrt{a}\, dy$$

where

$$p^{i,\varepsilon} := \int\limits_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}\, dx_3^\varepsilon + h_+^{i,\varepsilon} + h_-^{i,\varepsilon}$$

$$h_+^{i,\varepsilon} = h^{i,\varepsilon}(\cdot, +1) \qquad\qquad h_-^{i,\varepsilon} = h^{i,\varepsilon}(\cdot, -1)$$

or, equivalently, the covariant components of the displacement field of the surface $\mathcal{S}$, should be a stationary point of the energy functional defined by

$$j_K^{\varepsilon,e}(\boldsymbol{\eta}) = \frac{\varepsilon}{2} \int\limits_\omega a^{\alpha\beta\sigma\tau} G^e_{\sigma\tau}(\boldsymbol{\eta}) G^e_{\alpha\beta}(\boldsymbol{\eta}) e\sqrt{a}\, dy +$$

$$+\frac{\varepsilon^3}{6} \int\limits_\omega a^{\alpha\beta\sigma\tau} R^e_{\sigma\tau}(\boldsymbol{\eta}) R^e_{\alpha\beta}(\boldsymbol{\eta}) e^3\sqrt{a}\, dy - \int\limits_\omega p^i \eta_i e\sqrt{a}\, dy$$

Unfortunately, the functions $b^e_{\alpha\beta}(\boldsymbol{\eta})$ are not defined at those points of $\overline{\omega}$ where the two vectors

$$a^e_\alpha(\boldsymbol{\eta}) = \partial_\alpha(\boldsymbol{\theta} + \eta_\alpha \boldsymbol{a}^\alpha + \frac{1}{e}\eta_3 \boldsymbol{a}^3)$$

are collinear. Hence, it appears the difficulty of choosing the right manifold for minimizing the energy. To avoid this ambiguity, we replace in the strain energy the functions $R^e_{\alpha\beta}(\boldsymbol{\eta}) := b^e_{\alpha\beta}(\boldsymbol{\eta}) - b_{\alpha\beta}$ by the new functions (by analogy with Thm. 10.3-2, Ciarlet (2000a); see also Ciarlet (2000b))

$$R^{\#,e}_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{\sqrt{a}}\partial_{\alpha\beta}(\boldsymbol{\theta} + \boldsymbol{\eta}^e) \cdot \{\boldsymbol{a}^e_1(\boldsymbol{\eta}) \times \boldsymbol{a}^e_2(\boldsymbol{\eta})\} - b_{\alpha\beta}$$

which have the advantage of being *well defined for all smooth enough fields* $\boldsymbol{\eta}^e$*, irrespective of whether or not the two vectors* $a^e_\alpha(\boldsymbol{\eta})$ *are collinear in a subset of* $\overline{\omega}$. Obviously, $R^{\#,e}_{\alpha\beta} \equiv R^e_{\alpha\beta}$, when $\boldsymbol{\eta} = (\eta_i)$ is such that $a^e_{\alpha\beta}(\boldsymbol{\eta}) - a_{\alpha\beta} = 0$ in ω.

Consequently, the energy functional now takes the form

$$j^{e,\varepsilon}_K(\boldsymbol{\eta}) = \frac{\varepsilon}{2}\int_\omega a^{\alpha\beta\sigma\tau} G^e_{\sigma\tau}(\boldsymbol{\eta}) G^e_{\alpha\beta}(\boldsymbol{\eta}) e\sqrt{a}\, dy +$$

$$+\frac{\varepsilon^3}{6}\int_\omega a^{\alpha\beta\sigma\tau} R^{\#,e}_{\sigma\tau}(\boldsymbol{\eta}) R^{\#,e}_{\alpha\beta}(\boldsymbol{\eta}) e^3\sqrt{a}\, dy - \int_\omega p^i \eta_i e\sqrt{a}\, dy$$

The minimization problem will be:

Find

$$\boldsymbol{\zeta}^\varepsilon \in \mathbb{W}_K(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,p}(\omega),\ p > 2;\quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \mathbf{0} \ \text{ on } \ \gamma_0\}$$

such that

$$\varepsilon\int_\omega a^{\alpha\beta\sigma\tau} G^e_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)[(G^e_{\alpha\beta})'(\boldsymbol{\zeta}^\varepsilon)\boldsymbol{\eta}] e\sqrt{a}\, dy +$$

$$+\frac{\varepsilon^3}{3}\int_\omega a^{\alpha\beta\sigma\tau} R^{\#,e}_{\sigma\tau}(\boldsymbol{\zeta}^\varepsilon)[(R^{\#,e}_{\alpha\beta})'(\boldsymbol{\zeta}^\varepsilon)\boldsymbol{\eta}] e^3\sqrt{a}\, dy = \int_\omega p^i \eta_i e\sqrt{a}\, dy$$

where

$$(R^{\#,e}_{\alpha\beta})'(\boldsymbol{\zeta})\boldsymbol{\eta} := \frac{1}{\sqrt{a}}\partial_{\alpha\beta}(\boldsymbol{\theta} + \boldsymbol{\zeta}^e) \cdot \{\boldsymbol{a}^e_1(\boldsymbol{\zeta}) \times \partial_2(\boldsymbol{\eta}^e) + \partial_1(\boldsymbol{\eta}^e) \times \boldsymbol{a}^e_2(\boldsymbol{\zeta})\} +$$

$$+\frac{1}{\sqrt{a}}\partial_{\alpha\beta}(\boldsymbol{\eta}^e) \cdot \{\boldsymbol{a}^e_1(\boldsymbol{\zeta}) \times \boldsymbol{a}^e_2(\boldsymbol{\zeta})\}$$

with

$$\boldsymbol{a}^e_\alpha(\boldsymbol{\zeta}) := \partial_\alpha\Big(\boldsymbol{\theta} + \zeta_\alpha \boldsymbol{a}^\alpha + \frac{1}{e}\zeta_3 \boldsymbol{a}^3\Big)$$

Once this variational problem for a shell with variable thickness is written *in extenso* (see *supra*), its specific form suggests that we once again use the ansatz of the formal asymptotic method in order to justify it.

4. Asymptotic analysis of Koiter's model of shells with variable thickness

This section shows that the leading term $\boldsymbol{\zeta}^0$ of the formal asymptotic expansion of the two-dimensional scaled unknown $\boldsymbol{\zeta}(\varepsilon)$ satisfies *ad hoc* limit two-dimensional nonlinear equations that are exactly either the geometrically rigid or the geometrically bendable equations found in Section 2, according to which family of shells we would consider.

To this end, we will identify in Theorem 4.4 and in Theorem 4.7 two classes of variational problems that the leading term $\boldsymbol{\zeta}^0$ should verify, according to specific assumptions on the geometry of the middle surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$ of the shell, specific boundary conditions, and to specific powers of ε that affect the components of the applied forces.

We now carry out the formal asymptotic analysis for the model of Koiter's type for shells with variable thickness, introduced in the previous section. More specifically, our objective is to study the behavior as $\varepsilon \to 0$ of a two-dimensional displacement field $\boldsymbol{\zeta}^\varepsilon$ that satisfies the problems $P^{\varepsilon,e}_K(\omega)$. Our first task consists in "scaling" the problems $P^{\varepsilon,e}_K(\omega)$; accordingly, we let

$$\Omega = \omega \times]-1,1[\qquad \Gamma_+ = \omega \times \{+1\} \qquad \Gamma_- = \omega \times \{-1\}$$

and with each point $x \in \overline{\Omega}$, we associate the point $x^\varepsilon \in \overline{\Omega}^\varepsilon$ through the bijection

$$\pi^\varepsilon : x = (x_1, x_2, x_3) \in \overline{\Omega} \to x^\varepsilon = (x^\varepsilon_i) = (x_1, x_2, \varepsilon x_3) \in \overline{\Omega}^\varepsilon$$

Theorem 4.1. On the assumptions that there exist functions $f^i(\varepsilon) \in L^2(\Omega)$, $h^i(\varepsilon) \in L^2(\Gamma_+ \cup \Gamma_-)$ and $p^i(\varepsilon) \in L^2(\omega)$, independent of ε, such that

$$f^i(\varepsilon)(x) := f^{i,\varepsilon}(x^\varepsilon) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$

$$h^i(\varepsilon)(x) := h^{i,\varepsilon}(x^\varepsilon) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Gamma^\varepsilon_+ \cup \Gamma^\varepsilon_-$$

$$\varepsilon p^i(\varepsilon)(y) := p^{i,\varepsilon}(y) \quad \text{for all} \quad y \in \omega$$

the scaled unknown $\boldsymbol{\zeta}(\varepsilon) := \boldsymbol{\zeta}^{\varepsilon}$ satisfies the following two-dimensional scaled variational problem $P_K^e(\varepsilon;\omega)$, for the shell model with variable thickness:

Find

$$\boldsymbol{\zeta}(\varepsilon) = (\zeta_i(\varepsilon)) \in \mathbb{W}_K(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,p}(\omega),\ \ p > 2; \quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \mathbf{0} \ \text{ on } \ \gamma_0\}$$

such that

$$\int_\omega a^{\alpha\beta\sigma\tau} G^e_{\sigma\tau}(\boldsymbol{\zeta}(\varepsilon))[(G^e_{\alpha\beta})'(\boldsymbol{\zeta}(\varepsilon))\boldsymbol{\eta}]e\sqrt{a}\,dy +$$
$$+\frac{\varepsilon^2}{3}\int_\omega a^{\alpha\beta\sigma\tau} R^{\#;e}_{\sigma\tau}(\boldsymbol{\zeta}(\varepsilon))[(R^{\#;e}_{\alpha\beta})'(\boldsymbol{\zeta}(\varepsilon))\boldsymbol{\eta}]e^3\sqrt{a}\,dy = \int_\omega p^i(\varepsilon)\cdot\eta_i e\sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}_K(\omega)$, where

$$p^i(\varepsilon) := \int_{-1}^{1} f^i(\varepsilon)\,dx_3 + \frac{1}{\varepsilon}h^i_+(\varepsilon) + \frac{1}{\varepsilon}h^i_-(\varepsilon)$$
$$h^i_+(\varepsilon) = h^i(\varepsilon)(\cdot,+1) \qquad\qquad h^i_-(\varepsilon) = h^i(\varepsilon)(\cdot,-1)$$

■

According to the procedure set up by Miara (1998), our asymptotic analysis will be guided by two requirements:

- we do not wish to retain limit equations where restrictions must be imposed on the applied force densities in order that these equations possess solutions
- by linearization with respect to the unknown, we should find the problem solved by the leading term of the linear theory ("linearization requirement"); in other words, taking formal limits as $\varepsilon \to 0$ and linearizing should commute.

Remark. As Roquefort (2001, Chapter 4) noticed, the order of the forces is the same for shells with constant thickness as for shells with variable thickness.

To begin with, we have the following analog of Theorem 5.1 from Ciarlet and Roquefort (2001).

Theorem 4.2. Assume that for some integer N, the scaled solution $\boldsymbol{\zeta}(\varepsilon)$ of the above problem $P_K^e(\varepsilon;\omega)$ admits a formal asymptotic expansion in the form of polynomial ansatz

$$\boldsymbol{\zeta}(\varepsilon) = \varepsilon^{-N}\boldsymbol{\zeta}^{-N} + ... + \varepsilon^{-1}\boldsymbol{\zeta}^{-1} + \varepsilon^0\boldsymbol{\zeta}^0 + \varepsilon^1\boldsymbol{\zeta}^1 + ...$$

such that $\boldsymbol{\zeta}^{-N} = (\zeta_i^{-N}) \in \mathbb{W}_K(\omega)$ and $\boldsymbol{\zeta}^{-N} \neq \mathbf{0}$.

Then $N \leqslant 0$, which means that the first nonzero term has to be $\varepsilon^0\boldsymbol{\zeta}^0 = \boldsymbol{\zeta}^0$. ■

We focus now on the variational problems solved by the leading term $\boldsymbol{\zeta}^0$. The formal asymptotic expansion of the scaled unknown $\boldsymbol{\zeta}(\varepsilon) = \boldsymbol{\zeta}^0 + \varepsilon\boldsymbol{\zeta}^1 + ...$ induces the following expansions (the leading terms $G_{\sigma\tau}^{0,e}$ and $H_{\alpha\beta}^{0,e}(\boldsymbol{\eta})$ are given in the statement of the next theorem)

$$G_{\sigma\tau}^e(\boldsymbol{\zeta}(\varepsilon)) = G_{\sigma\tau}^{0,e} + ... \qquad (G_{\alpha\beta}^e)'(\boldsymbol{\zeta}(\varepsilon))\boldsymbol{\eta} = H_{\alpha\beta}^{0,e}(\boldsymbol{\eta}) + ...$$

The smallest power of ε found in the left-hand side of the variational equations in the problem $P_K^e(\varepsilon;\omega)$ is then ε^0; accordingly, we have to choose $p^i(\varepsilon) = \varepsilon^0 p^{i,0} = p^{i,0}$, where the new scaled functions $p^{i,0} \in L^2(\omega)$ are independent of ε. Cancelling the factor of ε^0 in the new formulation of the problem $P_K^e(\varepsilon;\omega)$ in terms of "leading terms", immediately gives the equations that should hold for all $\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}_M(\omega)$

$$\int_\omega a^{\alpha\beta\sigma\tau} G_{\sigma\tau}^{0,e} H_{\alpha\beta}^{0,e}(\boldsymbol{\eta}) e\sqrt{a}\, dy = \int_\omega p^{i,0}\eta_i e\sqrt{a}\, dy$$

hence we obtain the following result.

Theorem 4.3. Assume that the scaled displacement can be written as

$$\boldsymbol{\zeta}(\varepsilon) = \boldsymbol{\zeta}^0 + \varepsilon\boldsymbol{\zeta}^1 + \varepsilon^2\boldsymbol{\zeta}^2 + ...$$

and that the leading term of this formal asymptotic expansion satisfies $\boldsymbol{\zeta}^0 \in \mathbb{W}_K(\omega)$.

Then, in order that the leading term $\boldsymbol{\zeta}^0$ may be computed without any restriction on the applied forces and in order that the linearization requirement is satisfied, we must have

$$f^{i,\varepsilon}(x^\varepsilon) = f^{i,0}(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$

$$h^{i,\varepsilon}(x^\varepsilon) = \varepsilon h^{i,1}(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$$

with functions $f^{i,0} \in L^2(\Omega)$, $h^{i,1} \in L^2(\Gamma_+ \cup \Gamma_-)$, independent of ε; more specifically, the functions involved in the right-hand side of the problem must be of the final form

$$p^i(\varepsilon) = p^{i,0} \qquad \text{with} \qquad p^{i,0} \in L^2(\omega)$$

Moreover, the leading term $\boldsymbol{\zeta}^0$ solves the variational equation:

Find

$$\boldsymbol{\zeta}^0 \in \mathbb{W}_M(\omega) := \{\boldsymbol{\eta} \in \mathbb{W}^{1,4}(\omega); \quad \boldsymbol{\eta} = \mathbf{0} \ \text{ on } \ \gamma_0\}$$

such that

$$\int\limits_\omega a^{\alpha\beta\sigma\tau} G^{0,e}_{\sigma\tau} H^{0,e}_{\alpha\beta}(\boldsymbol{\eta}) e\sqrt{a}\, dy = \int\limits_\omega p^{i,0}\eta_i e\sqrt{a}\, dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}_M(\omega)$, where

$$\begin{aligned}
G^{0,e}_{\alpha\beta} &:= \frac{1}{2}(\zeta^{0,e}_{\alpha//\beta} + \zeta^{0,e}_{\beta//\alpha} + a^{mn}\zeta^{0,e}_{m//\alpha}\zeta^{0,e}_{n//\beta})\\
H^{0,e}_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}[\eta^e_{\alpha//\beta} + \eta^e_{\beta//\alpha} + a^{mn}(\zeta^0_{m//\alpha}\eta^e_{n//\beta} + \zeta^0_{n//\beta}\eta^e_{m//\alpha})]\\
\eta^e_{\beta//\alpha} &:= \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - \frac{1}{e} b_{\alpha\beta}\eta_3\\
\eta^e_{3//\alpha} &:= b^\sigma_\alpha \eta_\sigma + \partial_\alpha\Big(\frac{1}{e}\eta_3\Big)\\
p^{i,0} &:= \int\limits_{-1}^{1} f^{i,0}\, dx_3 + h^{i,1}_+ + h^{i,1}_-\\
h^{i,1}_+ &= h^{i,1}(\cdot,+1) \qquad\qquad h^{i,1}_- = h^{i,1}(\cdot,-1)
\end{aligned}$$

■

We now reformulate Theorem 4.3 in a form more close to the variational problem found in Theorem 2.1.

Theorem 4.4. Consider a family of nonlinearly elastic, geometrically rigid shells with variable thickness $h(y) = 2\varepsilon e(y)$, and with the same middle surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$. Assume that they satisfy a boundary condition of place along a portion of their lateral face with the same middle curve $\boldsymbol{\theta}(\gamma_0)$, and that they are subjected to the same applied forces as in Theorem 4.3.

Then the leading term $\boldsymbol{\zeta}^0 : \overline{\omega} \to \mathbb{R}^3$ of the formal asymptotic expansion of the scaled displacement $\boldsymbol{\zeta}(\varepsilon) = \boldsymbol{\zeta}^0 + \varepsilon\boldsymbol{\zeta}^1 + \varepsilon^2\boldsymbol{\zeta}^2 + ...$ satisfies the following scaled two-dimensional variational equations of a nonlinearly elastic, geometrically rigid shell:

Find

$$\boldsymbol{\zeta}^0 \in \mathbb{W}_M(\omega) := \{\boldsymbol{\eta} \in \mathbb{W}^{1,4}(\omega); \quad \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0\}$$

such that

$$\int_\omega a^{\alpha\beta\sigma\tau} G^e_{\sigma\tau}(\boldsymbol{\zeta}^0)[(G^e_{\alpha\beta})'(\boldsymbol{\zeta}^0)\boldsymbol{\eta}]e\sqrt{a}\, dy = \int_\omega p^{i,0}\eta_i e\sqrt{a}\, dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \mathbb{W}_(\omega)$. ■

Let us now consider the other case. Combining the linearization requirement and the presentation from Ciarlet (2000a; Sections 3.4 and 10.1), we get the following result.

Theorem 4.5. Assume that the scaled solution $\boldsymbol{\zeta}(\varepsilon)$ of the problem $P^e_K(\varepsilon;\omega)$ admits a formal asymptotic expansion of the form: $\boldsymbol{\zeta}(\varepsilon) = \boldsymbol{\zeta}^0 + \varepsilon^1\boldsymbol{\zeta}^1 + ...$, with the leading term satisfying $\boldsymbol{\zeta}^0 \in \mathbb{W}_K(\omega)$. In addition, assume that the manifold $\boldsymbol{M}^e_0(\omega)$ has the properties

$$\boldsymbol{M}^e_0(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,p}(\omega),\ \ p > 2; \quad \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0, \quad G^e_{\alpha\beta}(\boldsymbol{\eta}) = 0 \text{ in } \omega\} \neq \{\mathbf{0}\}$$

$$\boldsymbol{T}_\zeta\boldsymbol{M}^e_0(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,p}(\omega),\ \ p > 2; \quad \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0, \quad (G^e_{\alpha\beta})'(\boldsymbol{\zeta})\boldsymbol{\eta} = 0 \text{ in } \omega\} \neq \{\mathbf{0}\}$$

at each $\boldsymbol{\zeta} \in \boldsymbol{M}^e_0(\omega)$. Then $p^{i,0} = 0$ (the functions $p^{i,0} \in L^2(\omega)$ are defined in Theorem 4.3), and $G^e_{\alpha\beta}(\boldsymbol{\zeta}^0)$ vanish in ω, hence $\boldsymbol{\zeta}^0 \in \boldsymbol{M}^e_0(\omega)$. ■

The next result is the final step in the asymptotic analysis of our model of Koiter's type for nonlinearly elastic, geometrically bendable shells with variable thickness.

Theorem 4.6. Assume that $\boldsymbol{M}^e_0(\omega) \neq \{\mathbf{0}\}$, $\boldsymbol{T}_\zeta\boldsymbol{M}^e_0(\omega) \neq \{\mathbf{0}\}$ for all $\boldsymbol{\zeta} \in \boldsymbol{M}^e_0\omega)$, and that the scaled unknown $\boldsymbol{\zeta}(\varepsilon)$ of the problem $P^e_K(\varepsilon;\omega)$ admits the formal asymptotic expansion

$$\boldsymbol{\zeta}(\varepsilon) = \boldsymbol{\zeta}^0 + \varepsilon\boldsymbol{\zeta}^1 + \varepsilon^2\boldsymbol{\zeta}^2 + ...$$

with $\boldsymbol{\zeta}^0, \boldsymbol{\zeta}^1 \in \mathbb{W}_K(\omega)$ and $\boldsymbol{\zeta}^2 \in \mathbb{W}^{2,p}(\omega)$.

Then, in order that the leading term $\boldsymbol{\zeta}^0$ may be computed without any restriction on the applied forces and in order that the linearization requirement be satisfied, we must have

$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^{i,2}(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon$$
$$h^{i,\varepsilon}(x^\varepsilon) = \varepsilon^3 h^{i,3}(x) \quad \text{for all} \quad x^\varepsilon = \pi^\varepsilon x \in \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon$$

with functions $f^{i,2} \in L^2(\Omega)$, $h^{i,3} \in L^2(\Gamma_+ \cup \Gamma_-)$ independent of ε; more specifically, the functions involved in the right-hand side of the variational equations must be of the form

$$p^i(\varepsilon) = \varepsilon^2 p^{i,2} \qquad \text{with} \qquad p^{i,2} \in L^2(\omega)$$

Then, the leading term $\boldsymbol{\zeta}^0$ satisfies the following variational problem: Find

$$\boldsymbol{\zeta}^0 \in \mathbb{W}_F^e(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,4}(\omega); \quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \mathbf{0} \ \text{ on } \ \gamma_0, \quad G^e_{\alpha\beta}(\boldsymbol{\eta}) = 0 \ \text{ in } \ \omega\}$$

such that

$$\frac{1}{3}\int\limits_\omega a^{\alpha\beta\sigma\tau} R^{0,e}_{\sigma\tau} \mathcal{S}^{0,e}_{\alpha\beta}(\boldsymbol{\eta}) e^3 \sqrt{a}\, dy = \int\limits_\omega p^{i,2}\eta_i e\sqrt{a}\, dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{T}_{\zeta^0}\mathbb{W}_F^e(\omega)$, where

$$T_{\zeta^0}\mathbb{W}_F^e(\omega) := \{\boldsymbol{\eta} \in \mathbb{W}^{2,4}(\omega); \quad \boldsymbol{\eta} = \partial_\nu\boldsymbol{\eta} = \mathbf{0} \ \text{ on } \ \gamma_0,$$
$$(G^e_{\alpha\beta})'(\boldsymbol{\zeta}^0)\boldsymbol{\eta} = 0 \ \text{ in } \ \omega\}$$

denotes the tangent space to the manifold $\mathbb{W}_F^e(\omega)$ at $\boldsymbol{\zeta}^0$

$$p^{i,2} := \int\limits_{-1}^{1} f^{i,2}\, dx_3 + h_+^{i,3} + h_-^{i,3}$$
$$h_+^{i,3} = h^{i,3}(\cdot, +1) \quad h_-^{i,3} = h^{i,3}(\cdot, -1)$$
$$R^{0,e}_{\sigma\tau} = R^{\#,e}_{\sigma\tau}(\boldsymbol{\zeta}^0) \quad \mathcal{S}^{0,e}_{\alpha\beta}(\boldsymbol{\eta}) = (R^{\#,e}_{\alpha\beta})'(\boldsymbol{\zeta}^0)\boldsymbol{\eta}$$

■

Let us now recast the above result in a form more reminiscent of that of Theorem 2.2.

Theorem 4.7. Consider a family of nonlinearly elastic, geometrically bendable shells with variable thickness $h(y) = 2\varepsilon e(y)$. Assume that they have the same middle surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$, they satisfy a boundary condition of place along a portion of their lateral face with the same middle curve $\boldsymbol{\theta}(\gamma_0)$, and they are subjected to the same applied forces as in Theorem 4.6.

Then, the leading term $\boldsymbol{\zeta}^0 : \overline{\omega} \to \mathbb{R}^3$ of the asymptotic series associated with the scaled displacement field $\boldsymbol{\zeta}(\varepsilon)$ solves the following scaled two-dimensional variational problem $P_F^e(\omega)$ of a nonlinearly elastic, geometrically bendable shell:

Find

$$\boldsymbol{\zeta}^0 \in \mathbb{W}_F^e(\omega) = \{\boldsymbol{\eta} \in \mathbb{W}^{2,4}(\omega); \quad \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta} = \mathbf{0} \text{ on } \gamma_0, \quad G_{\alpha\beta}^e(\boldsymbol{\eta}) = 0 \text{ in } \omega\}$$

such that

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau} R_{\sigma\tau}^{\#,e}(\boldsymbol{\zeta}^0)[(R_{\alpha\beta}^{\#,e})'(\boldsymbol{\zeta}^0)\boldsymbol{\eta}]e^3\sqrt{a}\,dy = \int_\omega p^{i,2}\eta_i e\sqrt{a}\,dy$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{T}_{\zeta^0}\mathbb{W}_F^e(\omega)$. ■

5. Concluding remarks

- The variational problems found in Theorems 4.4 and 4.7 can be equivalently expressed in terms of energy functionals and, moreover, to be physically meaningful, these variational problems can be "de-scaled".
- If $e(y) \equiv 1$ for all $y \in \overline{\omega}$, then we recover the equations for shells with constant thickness 2ε, proposed by Ciarlet (2000b).
- The main conclusion is that this new model of Koiter's type has two advantages: firstly, the strain energy has no longer a possibly vanishing denominator, and secondly one does not have to know in advance if the shell is a geometrically rigid shell or a geometrically bendable shell, since it will automatically adjust itself to the appropriate model, for small enough values of the parameter ε.

Acknowledgement

This work was partially supported by Research Project No. 9380037 from City University of Hong Kong, whose contribution is gratefully acknowledged.

References

1. Busse S., 1997, Sur quelques questions en théorie de coques en coordonnées curvilignes, Doctoral Dissertation, Université Pierre et Marie Curie, Paris
2. Ciarlet P.G., 2000a, *Mathematical Elasticity*, Vol. III: *Theory of Shells*, North-Holland, Amsterdam
3. Ciarlet P.G., 2000b, Un modéle bi-dimensionnel non linéaire de coque analogue á celui de W.T. Koiter, *C. R. Acad. Sci. Paris Sér. I*, **331**, 405-410
4. Ciarlet P.G., Roquefort A., 2001, Justification of a two-dimensional nonlinear shell model of Koiter's type, *Chinese Annals of Mathematics*, **22B**, 129-144
5. Gilbert R.P., Vashakmadze T.S., 2000, A two-dimensional nonlinear theory of anisotropic plates, *Mathematical and Computer Modelling*, **32**, 855-875
6. Gratie L., 2003, Two-dimensional nonlinear shell model of Koiter's type with variable thickness, *Math. Mech. Solids* (to appear)
7. John F., 1965, Estimates for the derivatives of the stresses in a thin shell and interior shell equations, *Comm. Pure Appl. Math.*, **18**, 235-267
8. Koiter W.T., 1966, On the nonlinear theory of thin elastic shells, *Proc. Konik. Ned. Akad. Wetensch.*, **B 69**, 1-54
9. Ladevèze P., 1976, Justification de la théorie linéaire des coques élastiques, *J. Mécanique*, **15**, 813-856
10. Lods V., Miara B., 1998, Nonlinearly elastic shell models II, The flexural model, *Arch. Rational Mech. Anal.*, **142**, 355-374
11. Miara B., 1998, Nonlinearly elastic shell models I, The membrane model, *Arch. Rational Mech. Anal.*, **142**, 331-353
12. Roquefort A., 2001, Sur quelques questions liées aux modéles non linéaires de coques minces, Ph.D. Thesis, Université Pierre et Marie Curie, Paris

Asymptotyczna analiza nieliniowo sprężystych powłok o zmiennej grubości

Streszczenie

W pracy odniesiono się do nieliniowego modelu powłoki o stałej grubości, który uprzednio został zweryfikowany za pomocą metody formalnych rozwinięć asymptotycznych. Jego konstrukcja ma właściwości analogiczne do innych modeli spotykanych w literaturze, ale jest bardziej dogodna przy zastosowaniu symulacji numerycznej. W tym samym duchu zaprezentowano w pracy metodę formalnych rozwinięć asymptotycznych do zbudowania bardziej ogólnego modelu o podobnym charakterze, ale dotyczącego powłoki o zmiennej grubości.

Manuscript received December 3, 2002; accepted for print April 14, 2003