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INTERACTION BETWEEN A STRATIFIED ELASTIC HALF-SPACE AND AN IRREGULAR BASE ALLOWING FOR THE INTERCONTACT GAS

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The paper is devoted to the investigation of contact interaction of a laminated half-space and a rigid body with a smooth cylindrical depression under conditions of plane deformation allowing for an intercontact ideal gas. To describe the homogenized model of the laminated body with microlocal parameters and to describe the behavior of the gas – equations of ideal gas state are used. Applying the method of complex potentials the problem is reduced to the singular integral equation for the height of intercontact gap and its solution is obtained in a closed form. To find the length of the gap, its volume and the gas pressure a system of three equations is derived. With the aid of this system the dependence of the external loading and amount of the gas in the gap on the contact pressure and geometrical parameters of the gap is analyzed.

Key words: contact problem, laminated half-space, gas, complex potentials, singular integral equations

1. Introduction

For the time being, the theory of contact problems, which deals with the Hertz contact (e.g. Johnson, 1985) when conjugated bodies touch at point or along the line before loading, is developed sufficiently well. Demands of modern technical engineering require also the development of much less investigated models to study a contact of bodies with conformable surfaces that initially coincide everywhere excluding zones that are much less than the contact area. Such models take into account imperfections of surfaces related to their small deviations from a flat onto local parts (Martynyak, 1985). They are physically grounded since real surfaces of conjugated bodies frequently have unevennesses caused by their manufacturing, defects of various kinds and technological imperfections. Despite the fact that these surface factors are small, considerable perturbations of stress and strain fields appear in their vicinity (Monastyrskyy, 1999; Shvets *et al.*, 1996), therefore the considering of surface irregularities is necessary from the point of view of contact durability and fracture. As a result of the unevennesses of mated surfaces intercontact gaps between the bodies appear. Very often they are filled by a certain substance. Depending on the type of the compound and conditions of its exploitation it could be, for example, a gas or liquid. The filler of the intercontact gaps takes a considerable part in the contact interaction as far as along the zones of contact it transmits traction between the surfaces, and this in one turn changes the distribution of contact pressure and geometrical parameters of the gaps.

The contact interaction of bodies allowing for an intercontact substance was considered in literature earlier. Kuznetsov (1985) considered contact of rough bodies in the presence of a fluid lubricant. Martynyak (1998) studied the influence of the ideal gas within the gap on the contact of half-spaces. An interaction of bodies taking into account "gas-liquid" phase transition in the filler of the gap was analyzed by Martynyak and Machyshyn (2000) in the case of surface depression possessing corner points at its ends providing constancy of the gap length under compression of the bodies.

Fast development of building and mechanical engineering along with the improvement of technology demands creation of new modern materials which must satisfy *a priori* given mechanical characteristics. One of the ways to solve this problem is to produce materials which are compounds of elements with known characteristics, namely composites. To describe composites of periodically repeated structures on the macro level different methods of averaging of their properties (e.g. Bahvalov and Panasenko, 1984; Christensen, 1979) are used.

This paper is devoted to the investigation of contact interaction between a layered composite half-space and an uneven rigid base in the presence of the ideal gas in the intercontact gap. As a tool of the averaging of characteristics of a stratified body the homogenization model with microlocal parameters established by Woźniak (1986, 1987) and developed in numerous publications (e.g. Kaczyński and Matysiak, 1998; Matysiak and Nagórko, 1989; Nagórko, 1989) is proposed. In contrast to other methods of homogenization it allows one to derive mean values of stresses and strains as well as their local values, that is stresses and strains in each component of the stratified body.

2. Model of an elastic stratified body

2.1. Governing equations for a periodic n-layered half-space

A stratified half-space, in which every periodically repeated lamina of the thickness δ consists of n elastic isotropic layers of the thickness $\delta_1, \delta_2, ..., \delta_n$ ($\delta = \sum_{i=1}^{n} \delta_i$), is considered (Fig. 1). Perfect bonding between the layers and laminas takes place. The Cartesian system of coordinates $Ox_1x_2x_3$ is introduced so that the axis Ox_3 is parallel to the planes of the layers and the axis Ox_1 is situated on the boundary of the half-space. We do not define the loading of the system so far, but we demand that it ensures the conditions of plane deformation. This makes it possible to consider an arbitrary half-plane perpendicular to the planes of the layers instead of the three-dimensional body. As such we choose the half-plane x_1Ox_2. According to the model of homogenization with microlocal parameters (Woźniak, 1986, 1987) the displacement in the considered half-plane can be written in the form

$$
\begin{aligned}
U_1(x_1, x_2) &= u_1(x_1, x_2) + h^A(x_2) V_1^A(x_1, x_2) \\
U_2(x_1, x_2) &= u_2(x_1, x_2) + h^A(x_2) V_2^A(x_1, x_2)
\end{aligned}
\tag{2.1}
$$

where the summation convention for the index $A = 1, \dots, n-1$ is accepted; $u_1(x_1, x_2)$, $u_2(x_1, x_2)$ are components of the macrodisplacement vector; $V_i^A(x_1, x_2)$ – unknown functions of microlocal parameters, $i = 1, 2$; $h^A(x_2)$ – *a priori* given δ-periodic piece-wise linear functions (so called shape functions), which will be determined afterwards.

To find $2n$ unknown functions, namely $u_1(\cdot)$, $u_2(\cdot)$, $V_1^A(\cdot)$, $V_2^A(\cdot)$ we have a system of $2n$ equations (Woźniak, 1987)

$$
\begin{aligned}
&\frac{1}{2}\langle B_{\alpha\beta\gamma\delta} h^A_{,\beta}\rangle (u_{\gamma,\delta} + u_{\delta,\gamma}) + \langle B_{\alpha\beta\gamma\delta} h^A_{,\delta} h^B_{,\gamma}\rangle V^B_\beta = 0 \\
&\frac{1}{2}\langle B_{\alpha\beta\gamma\delta}\rangle (u_{\gamma,\delta\beta} + u_{\delta,\gamma\beta}) + \langle B_{\alpha\beta\gamma\delta} h^A_{,\delta}\rangle V^A_{\gamma,\beta} = 0
\end{aligned}
\tag{2.2}
$$

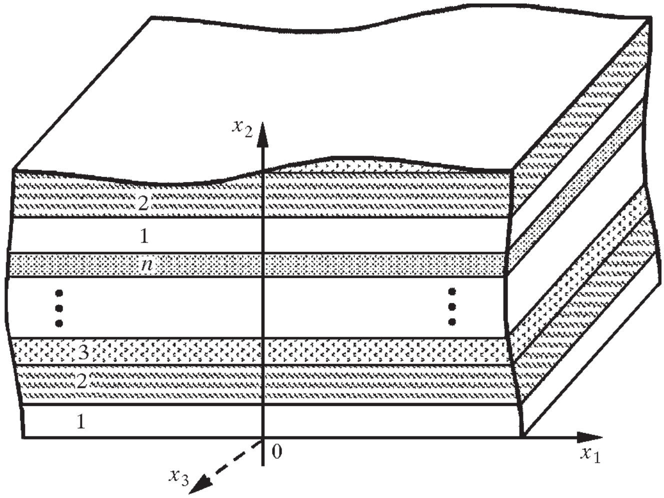

Fig. 1. Scheme of n-layered periodic half-space

where the arguments of the functions are omitted in order to simplify the presentation of the expressions; comma denotes partial differentiation; $\alpha,\beta,\gamma,\delta = 1,2$; $A,B = 1,\dots,n-1$; $h^A_{,1} = 0$; the summation convention for all repeated indices are accepted, $\langle B_{\alpha\beta\gamma\delta}\rangle$ are the components of the tensor of elastic constants, and the symbol $\langle\varphi\rangle$ denotes the averaging of the function φ along the thickness of the lamina

$$\langle\varphi\rangle = \frac{1}{\delta}\int\limits_0^\delta \varphi(x_2)\,dx_2 \tag{2.3}$$

After eliminating $2n-2$ unknown functions V_i^A from system (2.2) we receive equations of equilibrium for the displacement components $u_1(\cdot)$ and $u_2(\cdot)$. The obtaining of coefficients of these equations in an explicit form and in the general case of n layers is related with very complicated analytical transformations. In the case of two layers the coefficients of Lamé equations were calculated (Kaczyński and Matysiak, 1987) and the next subsection of this paper is devoted to the case of three layers.

2.2. The equilibrium equation on displacements for a periodic three-layered body

Let us consider described in 2.1 stratified half-space in the case of $n = 3$, that is periodically repeated lamina of the thickness δ comprises three layers of the thickness $\delta_1,\delta_2,\delta_3$ (Fig. 2). The Lamé coefficients λ,μ of such a body are

δ-periodic, piece-wise constant functions along the thickness of the half-space (in the direction of the Ox_2 axis), and can be written in the form

$$\lambda(x_2) = \begin{cases} \lambda_1 & \text{for} \quad 0 \leqslant x_2 \leqslant \delta_1 \\ \lambda_2 & \text{for} \quad \delta_1 \leqslant x_2 \leqslant \delta_1 + \delta_2 \\ \lambda_3 & \text{for} \quad \delta_1 + \delta_2 \leqslant x_2 \leqslant \delta \end{cases}$$
$$\mu(x_2) = \begin{cases} \mu_1 & \text{for} \quad 0 \leqslant x_2 \leqslant \delta_1 \\ \mu_2 & \text{for} \quad \delta_1 \leqslant x_2 \leqslant \delta_1 + \delta_2 \\ \mu_3 & \text{for} \quad \delta_1 + \delta_2 \leqslant x_2 \leqslant \delta \end{cases} \tag{2.4}$$

where λ_i, μ_i for $i = 1, 2, 3$ denote the Lamé coefficients of the ith layer.

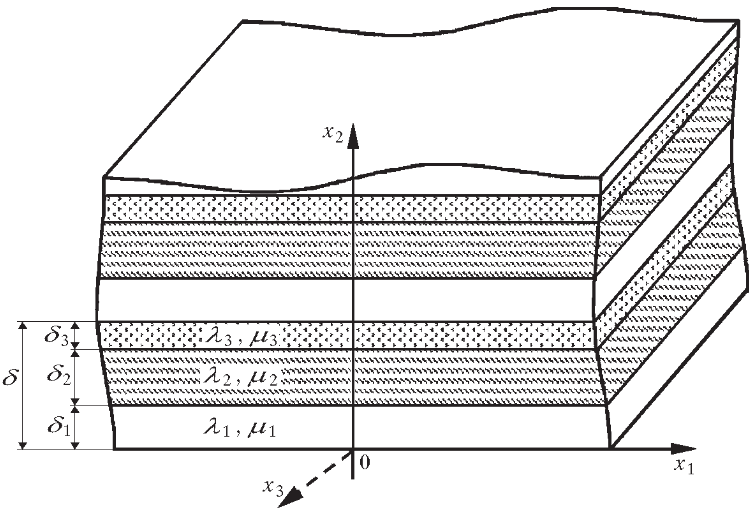

Fig. 2. Scheme of three-layered periodic half-space

By virtue of isotropy of the layers, the components of the tensor of the elastic constants $B_{\alpha\beta\gamma\xi}$ can be expressed via Lamé coefficients

$$B_{\alpha\beta\gamma\xi} = \lambda(x_2)\delta_{\alpha\beta}\delta_{\gamma\xi} + \mu(x_2)(\delta_{\alpha\gamma}\delta_{\beta\xi} + \delta_{\alpha\xi}\delta_{\beta\gamma}) \tag{2.5}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol.

The functions $h^1(x_2)$, $h^2(x_2)$, given in the form

$$h^1(x_2) = \begin{cases} x_2 - \dfrac{1}{2}\delta_1 & \text{for} \quad 0 \leqslant x_2 \leqslant \delta_1 \\ -\dfrac{\delta_1}{\delta_2+\delta_3}x_2 - \dfrac{1}{2}\delta_1 + \dfrac{\delta_1\delta}{\delta_2+\delta_3} & \text{for} \quad \delta_1 < x_2 \leqslant \delta \end{cases}$$

$$h^2(x_2) = \begin{cases} x_2 - \dfrac{1}{2}(\delta_1+\delta_2) & \text{for} \quad 0 \leqslant x_2 \leqslant \delta_1+\delta_2 \\ -\dfrac{\delta_1+\delta_2}{\delta_3}x_2 - \dfrac{1}{2}(\delta_1+\delta_2) + \dfrac{(\delta_1+\delta_2)\delta}{\delta_3} & \text{for} \quad \delta_1+\delta_2 < x_2 \leqslant \delta \end{cases} \tag{2.6}$$

and system (2.2) taking into account relations (2.5), can be written in the form

$$\begin{aligned} &\langle\lambda+\mu\rangle u_{\alpha,\alpha\beta} + \langle\mu\rangle u_{\alpha,\beta\beta} + \langle\lambda h^A_{,\beta}\rangle V^A_{\beta,\alpha} + \langle\mu h^A_{,\beta}\rangle V^A_{\alpha,\beta} + \langle\mu h^A_{,\alpha}\rangle V^A_{\beta,\beta} = 0 \\ &\langle\lambda h^A_{,\alpha}\rangle u_{\beta,\beta} + \langle\mu h^A_{,\beta}\rangle u_{\alpha,\beta} + \langle\mu h^A_{,\beta}\rangle u_{\beta,\alpha} + \langle\lambda h^A_{,\alpha} h^B_{,\beta}\rangle V^B_\beta + \\ &+\langle\mu h^A_{,\beta} h^B_{,\beta}\rangle V^B_\alpha + \langle\mu h^A_{,\beta} h^B_{,\alpha}\rangle V^B_\beta = 0 \end{aligned} \tag{2.7}$$

With the use of relations (2.3), (2.6) we can average the functions

$$\begin{aligned} \langle\varphi\rangle &= \frac{1}{\delta}[\delta_1\varphi_1 + \delta_2\varphi_2 + \delta_3\varphi_3] \\ \langle\varphi h^1_{,2}\rangle &= \frac{\delta_1}{\delta}\Big[\varphi_1 - \frac{\delta_2\varphi_2 + \delta_3\varphi_3}{\delta_2+\delta_3}\Big] \\ \langle\varphi h^2_{,2}\rangle &= \frac{1}{\delta}[\delta_1\varphi_1 + \delta_2\varphi_2 - (\delta_1+\delta_2)\varphi_3] \\ \langle\varphi (h^1_{,2})^2\rangle &= \frac{1}{\delta}\Big[\delta_1\varphi_1 + \frac{\delta_1^2}{(\delta_2+\delta_3)^2}(\delta_2\varphi_2 + \delta_3\varphi_3)\Big] \\ \langle\varphi (h^2_{,2})^2\rangle &= \frac{1}{\delta}\Big[\delta_1\varphi_1 + \delta_2\varphi_2 + \frac{(\delta_1+\delta_2)^2}{\delta_3}\varphi_3\Big] \\ \langle\varphi h^1_{,2} h^2_{,2}\rangle &= \frac{\delta_1}{\delta}\Big[\varphi_1 - \frac{\delta_2\varphi_2 - (\delta_1+\delta_2)\varphi_3}{\delta_2+\delta_3}\Big] \end{aligned} \tag{2.8}$$

where φ is arbitrary, δ – periodic function of x_2, which equals a constant value φ_i, $i = 1, 2, 3$ within layer i.

After eliminating the microlocal parameters from relations (2.7) we receive equations on the macrodisplacements

$$\begin{aligned} A_2 u_{1,11} + (B_* + C)u_{2,21} + Cu_{1,22} &= 0 \\ A_1 u_{2,22} + (B_{**} + C)u_{1,12} + Cu_{2,11} &= 0 \end{aligned} \tag{2.9}$$

where

$$
\begin{aligned}
A_2 &= \langle \lambda + 2\mu \rangle + \frac{G}{H}\langle \lambda h^1_{,2}\rangle + \frac{J}{H}\langle \lambda h^2_{,2}\rangle \\
A_1 &= \langle \lambda + 2\mu \rangle + \frac{F}{H}\langle (\lambda + 2\mu) h^1_{,2}\rangle + \frac{I}{H}\langle (\lambda + 2\mu) h^2_{,2}\rangle \\
C &= \langle \mu \rangle + \frac{L}{M}\langle \mu h^1_{,2}\rangle + \frac{N}{M}\langle \mu h^2_{,2}\rangle \\
B_* &= \langle \lambda \rangle + \frac{F}{H}\langle \lambda h^1_{,2}\rangle + \frac{I}{H}\langle \lambda h^2_{,2}\rangle \\
B_{**} &= \langle \lambda \rangle + \frac{G}{H}\langle (\lambda + 2\mu) h^1_{,2}\rangle + \frac{J}{H}\langle (\lambda + 2\mu) h^2_{,2}\rangle
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
F &= \langle (\lambda + 2\mu) h^1_{,2}\rangle \langle (\lambda + 2\mu)(h^2_{,2})^2\rangle - \langle (\lambda + 2\mu) h^1_{,2} h^2_{,2}\rangle \langle (\lambda + 2\mu) h^2_{,2}\rangle \\
G &= \langle \lambda h^1_{,2}\rangle \langle (\lambda + 2\mu)(h^2_{,2})^2\rangle - \langle \lambda h^2_{,2}\rangle \langle (\lambda + 2\mu) h^1_{,2} h^2_{,2}\rangle \\
H &= \langle (\lambda + 2\mu) h^1_{,2} h^2_{,2}\rangle^2 - \langle (\lambda + 2\mu)(h^1_{,2})^2\rangle \langle (\lambda + 2\mu)(h^2_{,2})^2\rangle \\
I &= \langle (\lambda + 2\mu) h^2_{,2}\rangle \langle (\lambda + 2\mu)(h^1_{,2})^2\rangle - \langle (\lambda + 2\mu) h^1_{,2}\rangle \langle (\lambda + 2\mu) h^1_{,2} h^2_{,2}\rangle \\
J &= \langle \lambda h^2_{,2}\rangle \langle (\lambda + 2\mu)(h^1_{,2})^2\rangle - \langle \lambda h^1_{,2}\rangle \langle (\lambda + 2\mu) h^1_{,2} h^2_{,2}\rangle \\
L &= \langle \mu h^1_{,2} h^2_{,2}\rangle \langle \mu h^2_{,2}\rangle - \langle \mu h^1_{,2}\rangle \langle \mu (h^2_{,2})^2\rangle \\
M &= \langle \mu (h^1_{,2})^2\rangle \langle \mu (h^2_{,2})^2\rangle - \langle \mu h^1_{,2} h^2_{,2}\rangle^2 \\
N &= \langle \mu h^1_{,2} h^2_{,2}\rangle \langle \mu h^1_{,2}\rangle - \langle \mu (h^1_{,2})^2\rangle \langle \mu h^2_{,2}\rangle
\end{aligned}
\tag{2.11}
$$

As the obtaining of coefficients (2.10) requires complicated calculations in order to solve this problem, we have involved a powerful instrument of analytical transformations, that is Mathematica 4.0 package. Here is the find result

$$
\begin{aligned}
A_1 &= \frac{L_1 L_2 L_3}{\eta_3 L_1 L_2 + \eta_2 L_1 L_3 + \eta_1 L_2 L_3} \\
A_2 &= A_1 + \frac{4(\mu_2 - \mu_3)(\lambda_2 - \lambda_3 + \mu_2 - \mu_3)(\mu_2\mu_3 L_1 + \mu_1\mu_3 L_2 + \mu_1\mu_2 L_3)}{\eta_3 L_1 L_2 + \eta_2 L_1 L_3 + \eta_1 L_2 L_3}
\end{aligned}
\tag{2.12}
$$

$$B_* = B_{**} \equiv B = \frac{\eta_1\lambda_1 L_2 L_3 + \eta_2\lambda_2 L_1 L_3 + \eta_3\lambda_3 L_1 L_2}{\eta_3 L_1 L_2 + \eta_2 L_1 L_3 + \eta_1 L_2 L_3}$$
$$C = \frac{\mu_1\mu_2\mu_3}{\eta_1\mu_2\mu_3 + \eta_2\mu_1\mu_3 + \eta_3\mu_1\mu_2}$$

where

$$\eta_i = \frac{\delta_i}{\delta} \qquad L_i = \lambda_i + 2\mu_i \qquad i = 1,2,3$$

We can see from the last relations that $B_* = B_{**} \equiv B$. This makes it possible to overwrite equation (2.9) in the form

$$\begin{aligned} A_2 u_{1,11} + (B+C)u_{2,21} + Cu_{1,22} &= 0 \\ A_1 u_{2,22} + (B+C)u_{1,12} + Cu_{2,11} &= 0 \end{aligned} \tag{2.13}$$

whose coefficients are defined by relations (2.12).

Note, that allowing for two arbitrary neighbor layers within the lamina to be made of the same material (then we have a two-layered body) coefficients (2.12) equal the corresponding coefficients obtained by Kaczyński and Matysiak (1987) for periodically two-layered body.

3. Model of the contact problem

The problem of the contact interaction between the stratified half-plane described in 2.2 and a rigid base with a cylindrical depression taking into account the intercontact gas is formulated in this Section. The interaction is considered under conditions of plane deformation that allows us to replace the bodies by half-planes D_2 and D_1, which are cross-sections of the bodies perpendicular to the generators of the depression on the surface of the rigid body, Fig. 3. The Cartesian system of coordinates Ox_1x_2 is introduced so that the axis Ox_1 coincide with the undeformed boundary D_0 of the half-plane D_2.

The upper half-plane D_2 is elastic stratified, and is composed of periodic three-layered laminas. The lower half-plane D_1 is rigid, and possesses a boundary depression within the interval $(-b, b)$, described by a function $r(x_1)$, which satisfies the conditions

$$|r(x_1)| \ll b \qquad |r'(x_1)| \ll 1 \qquad r(\pm b) = 0 \qquad r'(\pm b) = 0 \tag{3.1}$$

It means that the depression is supposed to be shallow and smooth.

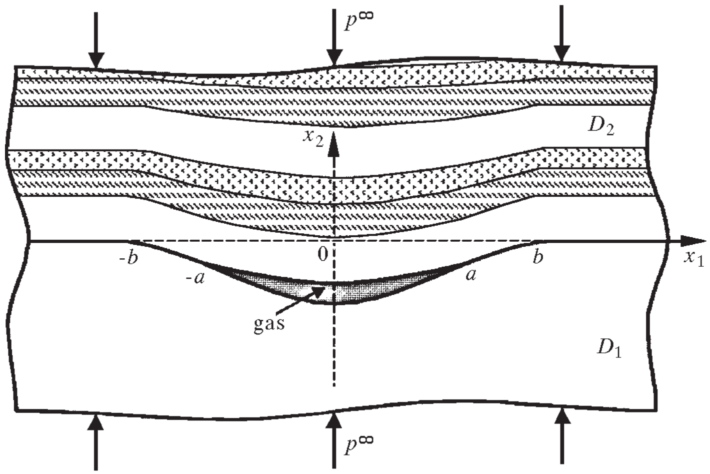

Fig. 3. Scheme of the contact

The half-planes are compressed by the uniformly distributed pressure p^∞ at infinity. The intercontact gap, located within the depression on the symmetric region of the unknown length $2a$ is assumed to be filled by the ideal gas, whose state is described by the equation

$$pV = \frac{m}{\mu^0} RT \tag{3.2}$$

where p, V, T, m, μ^0 are pressure, volume, absolute temperature, mass and molar mass of the gas, respectively. R is the absolute gas constant.

The contact is supposed to be frictionless. Along the gap normal stresses in the elastic half-plane are equal to the gas pressure p, which, for the time being, is unknown and should be found during solving the problem using equation (3.2) of the gas state.

Taking into account that the height of the depression is small in comparison with its length (3.1), in the framework of the linear theory of elasticity it is possible to write down boundary conditions on an undeformed boundary of the stratified half-plane that is on the axis Ox_1. They are as follows

$$\sigma_{12}(x_1, 0) = 0 \qquad x_1 \in D_0 \qquad\qquad \sigma_{22}(x_1, 0) = -p \qquad |x_1| \leqslant a \tag{3.3}$$

$$v(x_1, 0) = \begin{cases} r(x_1) & a < |x_1| < b \\ 0 & |x_1| > a \end{cases} \tag{3.4}$$

$$\sigma_{22}^\infty = -p^\infty \qquad\qquad \sigma_{12}^\infty = \sigma_{11}^\infty = 0 \qquad\qquad \sqrt{x_1^2 + x_2^2} \to \infty \tag{3.5}$$

4. Reduction of the problem to the Singular Integral Equation (SIE)

Kaczyński and Matysiak (1987) have shown that stresses and displacements in a stratified body can be presented through two arbitrary holomorphic functions φ and ψ of generalized complex variables z_m

$$\begin{aligned} &\sigma_{22} = 2\mathrm{Re}[\varPhi(z_1) + \varPsi(z_2)] && \sigma_{12} = 2\mathrm{Im}[s_1\varPhi(z_1) + s_2\varPsi(z_2)] \\ &\sigma_{11j} = 2\mathrm{Re}[c_{1j}\varPhi(z_1) + c_{2j}\varPsi(z_2)] && \\ &u = -2\mathrm{Re}[q_1\varphi(z_1) + q_2\psi(z_2)] && v = -2\mathrm{Re}[p_1\varphi(z_1) + p_2\psi(z_2)] \end{aligned} \tag{4.1}$$

where

$$z_m = x_1 + \mathrm{i}s_m x_2 \qquad s_m = \sqrt{\frac{A_1A_2 - 2BC - B^2 + (-1)^m\sqrt{D}}{2A_1C}}$$

$$D = (A_1A_2 - 2BC - B^2)^2 - 4A_1A_2C^2$$

and we assume that $D > 0$

$$\begin{aligned} &c_{mj} = \frac{(A_2 + s_m^2 B)D_j - (B + s_m^2 A_1)E_j}{A_1A_2 - B^2} \\ &D_j = \frac{\lambda_j}{\lambda_j + 2\mu_j}A_1 && E_j = \frac{4\mu_j(\lambda_j + \mu_j)}{\lambda_j + 2\mu_j} + \frac{\lambda_j}{\lambda_j + \mu_j}B \\ &p_m = \frac{s_m^2 A_1 + B}{A_1A_2 - B^2} && q_m = \mathrm{i}\frac{s_m^2 B + A_2}{s_m(A_1A_2 - B^2)} \\ &\varPhi(z_m) = \varphi'(z_m) && \varPsi(z_m) = \psi'(z_m) \end{aligned} \tag{4.2}$$

index $j = 1, 2, 3$ denotes the corresponding layer.

It is often convenient to use other presentations through only one complex potential defined not only at the area of body occupation, like the potentials φ and ψ in relations (4.1), but also in its complement to the whole plane. In the case of a isotropic half-plane, with the use of the technique of analytical continuation, it was done by Muskhelishvili (1953) and extended on the case of stratified bodies by Kryshtafovych and Matysiak (2001). Expanding the definition of the complex potential $\varPhi(z)$ to the whole plane by the formula (Kryshtafovych and Matysiak, 2001)

$$\varPhi(z_m) = -\frac{s_2 + s_1}{s_2 - s_1}\overline{\varPhi(\overline{z_m})} - \frac{2s_2}{s_2 - s_1}\overline{\varPsi(\overline{z_m})}$$

we can overwrite the relations for stresses and displacements (4.1) through one piece-wise holomorphic function $\varPhi(z)$

$$
\begin{aligned}
\sigma_{22} &= 2\mathrm{Re}[\varPhi(z_1) + t_1\varPhi(z_2) + t_2\overline{\varPhi(\overline{z_2})}] - (p^\infty - p)\\
\sigma_{12} &= 2\mathrm{Im}[s_1\varPhi(z_1) + s_2t_1\varPhi(z_2) + s_2t_2\overline{\varPhi(\overline{z_2})}]\\
\sigma_{11j} &= 2\mathrm{Re}[c_{1j}\varPhi(z_1) + c_{2j}t_1\varPhi(z_2) + c_{2j}t_2\overline{\varPhi(\overline{z_2})}]\\
u &= 2\mathrm{Re}[q_1\varPhi(z_1) + q_2t_1\varPhi(z_2) + q_2t_2\overline{\varPhi(\overline{z_2})}]\\
v &= 2\mathrm{Re}[p_1\varPhi(z_1) + p_2t_1\varPhi(z_2) + p_2t_2\overline{\varPhi(\overline{z_2})}]
\end{aligned}
\tag{4.3}
$$

where

$$
K = \frac{s_1s_2(s_1+s_2)A_1}{A_1A_2 - B^2} \qquad t_m = \frac{1}{2}\Big[(-1)^m\frac{s_1}{s_2} - 1\Big] \qquad m = 1,2 \tag{4.4}
$$

Having satisfied all boundary conditions (3.3) - (3.5) except for the second condition in (3.3) we received the following expressions for complex potentials via the height $h(x)$ of the intercontact gap

$$
\begin{aligned}
\varPhi(z_m) &= \frac{s_2K}{2\pi(s_2-s_1)}\int\limits_L \frac{h'(t)+r'(t)}{t-z_m}\,dt \qquad z_m \in D_2\\
\varPhi(z_m) &= -\frac{s_2K}{2\pi(s_2-s_1)}\int\limits_L \frac{h'(t)+r'(t)}{t-z_m}\,dt \qquad z_m \in D_1
\end{aligned}
\tag{4.5}
$$

Satisfying the last boundary condition, namely the second one in conditions (3.3), we obtained the SIE on the derivative of the gap height

$$
\int\limits_{-a}^{a} \frac{h'(t)}{t-x_1}\,dt = -\int\limits_{-b}^{b} \frac{r'(t)}{t-x_1}\,dt + \pi K(p^\infty - p) \equiv F(x_1) \qquad x_1 \in (-a,a) \tag{4.6}
$$

If one of the layers occupies the whole lamina and the thickness of two other layers equals zero ($\eta_1 = 1$, $\eta_2 = \eta_3 = 0$ or $\eta_2 = 1$, $\eta_1 = \eta_3 = 0$ or $\eta_3 = 1$, $\eta_1 = \eta_2 = 0$) then, instead of the stratified half-plane D_2, we will have an isotropic homogeneous one. From relations (4.2), it follows that $D = 0$, $s_1 = s_2 = 1$. In this case, presentations (4.1) are no more valid (Kaczyński and Matysiak, 1987), and in order to solve the problem correctly we should utilize well known presentations of stresses and displacements through the piece-wise holomorphic complex function for an isotropic half-plane (Muskhelishvili, 1953). With the use of such presentations, Martynyak (1985) have derived the singular integral equation that differs from equation (4.6) only by the coefficient K, which in the isotropic case equals

$$K = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)}$$

λ, μ – Lamé coefficients of the isotropic half-plane. In the case of the whole lamina occupied by only one layer we will receive the same expression for the coefficient K if we put $s_1 = s_2 = 1$ into relations (4.4). This means that in the case of the isotropic half-plane we can also use integral equation (4.6), though we can not use complex representations (4.3).

5. Solution to the problem

The elastic half-plane smoothly conjugates with the base at the points $x = \pm a$. Therefore, the function $h'(x)$ is equal to zero at these points

$$h'(\pm a) = 0 \tag{5.1}$$

The corresponding solution of SIE (4.6) is following

$$h'(x_1) = -\frac{\sqrt{a^2 - x_1^2}}{\pi} \int\limits_{-a}^{a} \frac{F(t)}{\sqrt{a^2 - t^2}(t - x_1)} \, dt \tag{5.2}$$

and exists under the imposed condition (Muskhelishvili, 1953) on the function $F(t)$

$$\int\limits_{-a}^{a} \frac{F(t)}{\sqrt{a^2 - t^2}} \, dt = 0 \tag{5.3}$$

A depression of the specific form

$$r(x_1) = -H\sqrt{\Big(1 - \frac{x_1^2}{b^2}\Big)^3} \qquad |x_1| \leqslant b \tag{5.4}$$

was considered.

After analytical calculations with the use of relations (4.6) and (5.4) formulas (5.2) and (5.3) take the form

$$h'(x_1) = -\frac{6H}{b^3} x\sqrt{a^2 - x_1^2} \qquad K(p^{\infty} - p) = \frac{3H}{2b}\Big(1 - \frac{a^2}{b^2}\Big) \tag{5.5}$$

The height of the gap at the points of contact must be equal to zero

$$h(a) = h(-a) = 0 \tag{5.6}$$

Integrating relation $(5.5)_1$ and using condition (5.6), we obtain the height of the gap

$$h(x_1) = \frac{H}{b^3}\sqrt{(a^2 - x_1^2)^3} \tag{5.7}$$

Having the height of the gap (5.7) we can calculate its volume per unit along the Ox_1 direction

$$V = \int\limits_{-a}^{a} h(t)\;dt = \frac{3}{8}H\pi\frac{a^4}{b^3} \tag{5.8}$$

To find the unknown parameters of our problem, namely the length of the gap $2a$, its volume V and the pressure of the intercontact gas p, we have to solve a system of nonlinear equations (3.2), $(5.5)_2$ and (5.8)

$$\begin{gathered} pV = \frac{m}{\mu^0}RT \qquad\qquad V = \frac{3}{8}H\pi\frac{a^4}{b^3} \\ K(p^\infty - p) = \frac{3H}{2b}\Big(1 - \frac{a^2}{b^2}\Big) \end{gathered} \tag{5.9}$$

With the aid of relations (4.3), (4.5), (5.4), $(5.5)_1$ we can obtain a formula for the contact pressure

$$\begin{aligned} p_c(x_1) &= p^\infty + \\ &+\frac{3H}{Kb^3}\Big[x_1 \operatorname{sgn} x_1\Big(S_-(|x_1| - a)\sqrt{x_1^2 - a^2} - S_-(|x_1| - b)\sqrt{x_1^2 - b^2}\Big) + \frac{a^2 - b^2}{2}\Big] \end{aligned}$$

where

$$S_-(y) = \begin{cases} 0 & \text{for} \quad y < 0 \\ 1 & \text{for} \quad y \geqslant 0 \end{cases}$$

6. Numerical analysis of the solution

All the calculations were carried out for the dimensionless variables

$$\begin{array}{lll} \eta_1 = \dfrac{\delta_1}{\delta} & \eta_2 = \dfrac{\delta_2}{\delta} & \eta_3 = 1 - \eta_1 - \eta_2 \\ \omega = \dfrac{\mu_1}{\mu_2} & \gamma = \frac{\mu_1}{\mu_3} & \\ \overline{a} = \dfrac{a}{b} & \overline{H} = \dfrac{H}{b} & \overline{h} = \dfrac{h}{b} \\ \overline{V} = \dfrac{V}{b^2} & \overline{p} = \dfrac{p}{\mu_1} & \overline{p}^{\infty} = \dfrac{p^{\infty}}{\mu_1} \end{array} \tag{6.1}$$

$$\overline{K} = \mu_1 K = \frac{s_1 s_2 (s_1 + s_2) \frac{A_1}{\mu_1}}{\frac{A_1}{\mu_1}\frac{A_2}{\mu_1} - \frac{B^2}{\mu_1^2}} \qquad \overline{m} = m\frac{RT}{\mu^0 \mu_1 b^2} \tag{6.2}$$

The ratio between the share modulus of the first and second layer was taken $\omega = 2$, and between the first and third layer – $\gamma = 4$. It means that the first layer is the stiffest one, while the third layer – the least stiff. Poisson's ratios of the first, second and third layers were taken $\nu_1 = 0.3$, $\nu_2 = 0.35$ and $\nu_3 = 0.4$, respectively. The maximum height of the depression was $\overline{H} = 0.01$. The solid line in Fig. 4 corresponds to the case when the mass of the gas within the gap equals $\overline{m} = 10^{-6}$ and the dashed line – to the case of $\overline{m} = 10^{-7}$.

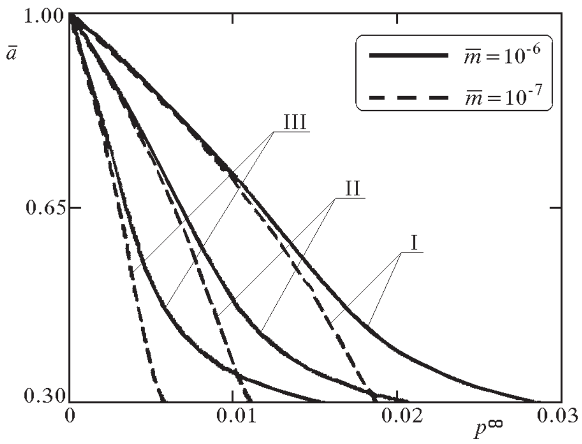

Fig. 4. Half-length of the gap versus the external pressure

We have considered three cases of relative layers thickness.

Case I – The first layer occupies the whole elastic half-plane. The thickness of the second and third layer equals 0 $(\eta_1 = 1, \eta_2 = 0)$.

Case II – The thickness of the first layer is 0.5, of the second layer is 0.3 and of the third one 0.2 $(\eta_1 = 0.5, \eta_2 = 0.3)$.

Case III – The third layer occupies the whole elastic half-plane. The thickness of the first and second layers equals 0 $(\eta_1 = 0, \eta_2 = 0)$.

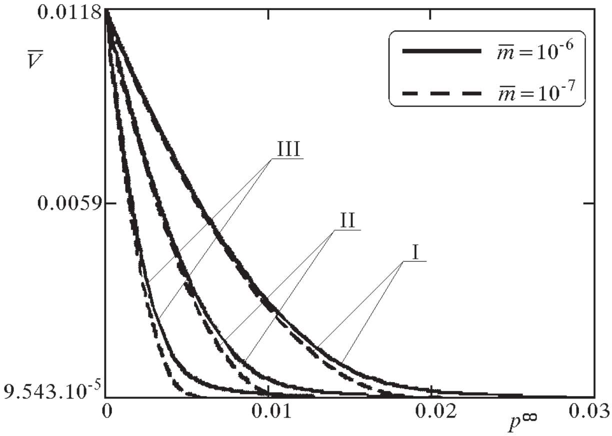

Fig. 5. Volume of the gap versus the external pressure

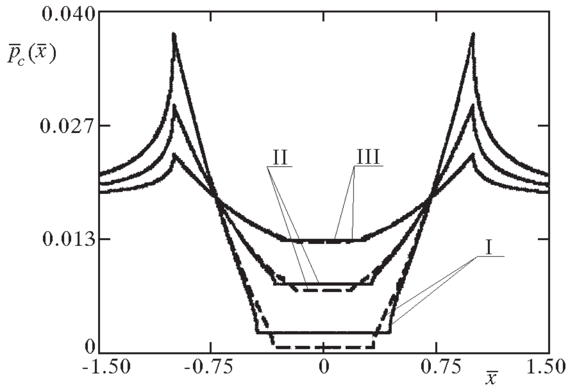

Fig. 6. Contact pressure distribution

In Fig. 4 the dependence of the gap half-length on the external pressure is presented. We can see that the length of the gap greatly depends on the relative thickness of the layers as well as on the mass of the gas. The stiffer combination of layers we have, the longer gap it is. Similarly, the more gas we put in the gap, the greater gap we get. The same dependence of the relative line location on the geometrical parameters of the layers and the mass of the gas can be seen in Fig. 5, where the volume of the gap versus the external pressure is presented. Nevertheless, the shape of the lines differs from the one shown in previous figure. The contact pressure distributions under the external load $\overline{p}^{\infty} = 0.018$ is shown in Fig. 6. At the ends of the initial depression we can see peaks of the contact pressure which are caused by the form of depression. The horizontal areas correspond to the pressure in the gap and their size, i.e. to the length of the gap. For the stiffest combination of the relative thickness of the layers the gas pressure is smallest and the peak values are biggest, while vice versa for the least stiff combination.

7. Conclusions

Numerical analysis of the solution to the problem revealed that the relative thickness of the layers of a stratified body and the amount of the gas within the intercontact gap have considerable influence on both contact pressure distribution and geometrical characteristics of the gap. It was established that the increasing of the gas amount and stiffness of the lamina enlarges the gap.

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Oddziaływanie sprężystej półprzestrzeni warstwowej na sztywną przeszkodę oddzieloną szczeliną wypełnioną gazem

Streszczenie

W pracy analizuje się oddziaływanie sprężystej półprzestrzeni warstwowo-niejednorodnej na ciało sztywne z walcową szczeliną wypełnioną gazem. Półprzestrzeń sprężystą poddaje się homogenizacji mikrolokalnej, a do rozwiązania otrzymanych równań modelowych stosuje się metodę potencjałów zespolonych. Analizowany problem sprowadza się do poszukiwania rozwiązania równania całkowego. W pracy uzyskuje się rozwiązania w postaci zamkniętej oraz przeprowadza analizę numeryczną zależności rozwartości szczeliny oraz jej objętości od obciążenia zewnętrznego.

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