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CONTACT PROBLEM FOR PERIODICALLY STRATIFIED HALF-SPACE AND RIGID FOUNDATION POSSESSING GEOMETRICAL SURFACE DEFECT

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A three-dimensional problem of frictionless contact interaction of a periodic two-layered elastic half-space and a rigid foundation with a local sloping surface recess is examined. The analysis is performed within the framework of a homogenized model with microlocal parameters. By constructing appropriate harmonic functions, the resulting boundary-value problem is reduced to some mixed problem of the potential theory. In dealing with its solution, the integro-differential singular equation of Newton's potential type for the function of gap height is obtained. To determine the unknown region of the gap the condition of smooth running of its faces is used. As an example, a certain form of the initial defect is considered and in this case the solution of the equation is found analytically by using an analogue of Dyson's theorem.

Key words: periodic two-layered half-space, local surface defect, singular integro-differential equation

1. Introduction

The wide use of layered composite materials in advanced engineering has evoked considerable attention of researchers. In recent decades there appeared a number of homogenized models of periodic layered material structures (see, for example, Bensoussan et al., 1978; Christensen, 1979; Bakhvalov and Panasenko, 1984). The common feature of the models is a procedure of averaging of the composite properties, in other words, a procedure of changing of a strictly heterogeneous structure by the homogeneous medium with mean properties. Such assumption simplifies mathematical treatment of specific problems and allows one to describe the general behaviour of a composite structure by the mean parameters (mean displacements, mean stresses). However, in the case of nonhomogeneous elastic bodies consisting of periodically repeated cells, the application of homogenized models seems to be more suitable. One of them is the homogenized model with microlocal parameters, developed by Woźniak (1987) and Matysiak and Woźniak (1988). This model makes it possible to evaluate not only mean but also local values of displacements and stresses in every material component of the stratified body. At the same time, the method is quite simple in the mathematical aspect: the determination of the mean as well as local parameters does not demand cumbersome mathematical calculations. Due to the simplicity of the mathematical apparatus and good physical description of the processes both at the macro- and microlevel, the homogenized model with the microlocal parameters has been successfully applied to numerous problems, whose solutions are of great practical importance for engineering, geomechanics, machine design industry, building industry and others (see a survey paper given by Matysiak, 1995).

Some crack problems for a periodically two-layered space, which are important from the standpoint of fracture mechanics, have been considered within the framework of the linear homogenized model with microlocal parameters in a series of papers given by Kaczyński and Matysiak (1988, 1994) and Kaczyński et al. (1994). The general approach to the investigation of three-dimensional static elastic and thermoelastic problems for a periodically stratified medium weakened by interface cracks was developed by Kaczyński (1993,1994). In those works, similarity between the governing equations of the homogenized model with microlocal parameters of a periodically stratified medium and the fundamental equations of a transversely isotropic solid was clearly shown. The method of solving is essentially based on this fact. The desired quantities (displacements, stresses, temperature) are represented by harmonic functions (co-called potentials) similar to those corresponding to transversely isotropic bodies, and the thus posed problems are reduced to classical boundary-value problems of harmonic potentials. In the present contribution the mentioned harmonic function method will be used to a new class of elasticity problems for periodically laminated bodies, which are closely related to crack problems from the point of view of mathematics.

The problem under study is referred to non-classical contact problems involving interactions of bodies with conformed surfaces (cf. Johnson, 1985).

Such a kind of the interaction has not been yet sufficiently investigated in the literature although it is quite typical for a lot of contacting joints. The surfaces of real bodies feature various geometrical defects such as recesses, protrusions, concavities, etc. Among reasons of their existence there are diverse processes of mechanical, physical and chemical nature, which take place during manufacturing of details and during operating of machines and mechanisms. These surface geometric defects, when the bodies are put into contact, lead to the appearance of zones, where the surfaces of bodies do not touch each other, so the intercontact gaps are created through these regions. The dimensions of the defects are small in comparison with the dimensions of the bodies, and therefore the zone of the intercontact gaps is small in comparison with the surface of the nominal contact. That is why in the literature the problems involving contact interactions of bodies with conformed surfaces are called as the problems of the local contact absence. In this field basic research has been performed by Martynyak (1985) in the plane case, Shvets et al. (1995, 1996) and by Kit and Monastyrskyy (1998, 1999) in the axisymmetrical case. In those papers integral equations were constructed and solved analytically or numerically. Recently, three-dimensional problems dealing with the local contact absence were considered by Martynyak (2000) and Kit et al. (2001). By using the method of harmonic potentials the considered problems were reduced to singular integral equations of Newton's potentials type.

The problem to be considered is the interaction of a periodic two-layered half-space with a rigid foundation in the absence of local contact caused by the presence of a surface geometric defect. In Section 2 we review governing equations of the homogenized model of the analysed composite. In Section 3 the formulation of the problem is performed. Section 4 is devoted to the solution of the resulting boundary value problem. As a result of the general investigation, a singular integro-differential equation is obtained. Its solution is presented by considering some special initial recess in Section 5.

2. Governing equations

Let us consider a microperiodically stratified half-space, in which every unit lamina of a small height δ consists of two homogeneous isotropic layers of heights δ_1 and δ_2, so $\delta = \delta_1 + \delta_2$ (as shown in Fig. 1a). Let λ_1, λ_2 and μ_1, μ_2 be Lamé's constants of the subsequent layers (denoted by 1 and 2). Referring to the Cartesian coordinate system (x_1, x_2, x_3) with its centre on the boundary of the half-space and the x_3-axis normal to the layering, denote at the point $\boldsymbol{x} = (x_1, x_2, x_3)$ the displacement vector by $\boldsymbol{u} = [u_1, u_2, u_3]$ and the stresses by σ_{11}, σ_{12}, σ_{22}, σ_{13}, σ_{23}, σ_{33}.

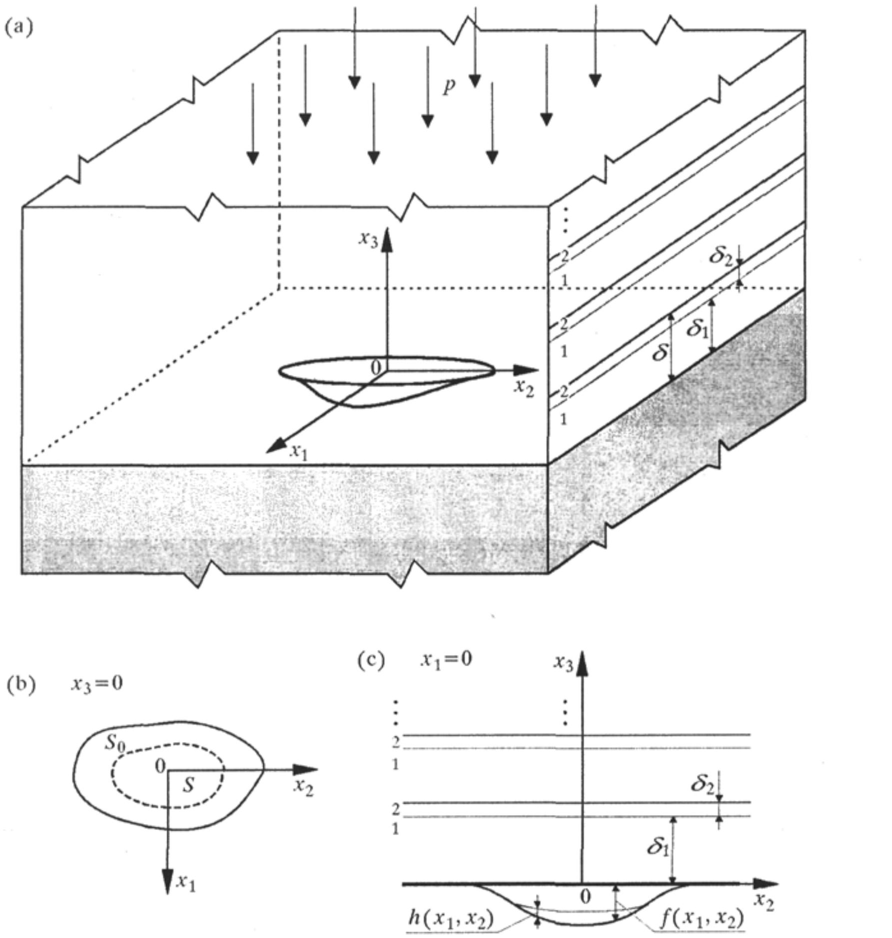

Fig. 1. Scheme of contact

We take into consideration the homogenized model of the linear elasticity with microlocal parameters, proposed by Woźniak (1987) and then developed by Matysiak and Woźniak (1988), characterised by the δ-periodic shape function, given by the formula

$$s(x_3) = \begin{cases} x_3 - \dfrac{\delta_1}{2} & x_3 \in \langle 0, \delta_1 \rangle \\ \dfrac{\delta_1 - \eta x_3}{1-\eta} - \dfrac{\delta_1}{2} & x_3 \in \langle \delta_1, \delta \rangle \end{cases} \qquad \eta = \frac{\delta_1}{\delta} \tag{2.1}$$

The model is based on the postulate of the microlocal approximation involving the following representation written in a symbolic form[1]

$$u_i(\boldsymbol{x}) = w_i(\boldsymbol{x}) + \underline{s(x_3)q_i(\boldsymbol{x})} \tag{2.2}$$

In the above, the unknown functions w_i and q_i are interpreted as macro-dispalcements and microlocal parameters, respectively. The underlined term in Eq. (2.2) represents the microdisplacements due to microperiodic material properties of the composite. Note, that for thin layers (δ is small) this term may be treated as small and can be neglected, but the derivative s' is a sectionally constant function that is not small even for small δ. Thus, the following approximations hold

$$u_i \cong w_i \qquad u_{i,\alpha} \cong w_{i,\alpha} \qquad u_{i,3} \cong w_{i,3} + s'(x_3)q_i \tag{2.3}$$

Following the procedure of the homogenization, described in cited papers, the system of differential equations for w_i and the system of algebraic equations for q_i (in the absence of body forces)are obtained

$$\begin{aligned}
&(\widetilde{\lambda} + \widetilde{\mu})w_{i,i\alpha} + \widetilde{\mu}w_{\alpha,ii} + [\lambda]q_{3,\alpha} + [\mu]q_{\alpha,3} = 0\\
&(\widetilde{\lambda} + \widetilde{\mu})w_{i,i3} + \widetilde{\mu}w_{3,ii} + ([\lambda] + 2[\mu])q_{3,3} + [\mu]q_{\gamma,\gamma} = 0\\
&\widetilde{\mu}q_\alpha + [\mu](w_{\alpha,3} + w_{3,\alpha}) = 0\\
&(\widehat{\lambda} + 2\widehat{\mu})q_3 + [\lambda]w_{i,i} + 2[\mu]w_{3,3} = 0
\end{aligned} \tag{2.4}$$

with the set of constants defined by the relations

$$\begin{aligned}
(\widetilde{\lambda}, \widetilde{\mu}) &= \eta(\lambda_1, \mu_1) + (1-\eta)(\lambda_2, \mu_2)\\
([\lambda], [\mu]) &= \eta(\lambda_1 - \lambda_2, \mu_1 - \mu_2)\\
(\widehat{\lambda}, \widehat{\mu}) &= \eta(\lambda_1, \mu_1) + \frac{\eta^2}{1-\eta}(\lambda_2, \mu_2)
\end{aligned}$$

Using Hooke's law and bearing Eqs. (2.2) and (2.3) in mind, the components of the stress tensor $\sigma_{ij}^{(l)}$ at the point $\boldsymbol{x}$ belonging to the layer of the lth kind are found to be

[1]Throughout this paper the Latin indices i, j run over 1,2,3 while the Greek indices α, β, γ run over 1,2 and summation over repeated subscripts is taken for granted; subscripts preceded by a comma indicate partial differentiation with respect to the corresponding coordinates. The index l, assumming values 1 or 2, is associated with layer 1 and 2, respectively.

$$
\begin{aligned}
\sigma^{(l)}_{\alpha\beta} &= \lambda_l \delta_{\alpha\beta}(w_{\gamma,\gamma} + w_{3,3} + s' q_3) + \mu_l (w_{\alpha,\beta} + w_{\beta,\alpha}) \\
\sigma^{(l)}_{\alpha 3} &= \mu_l (w_{\alpha,3} + w_{3,\alpha} + s' q_\alpha) \\
\sigma^{(l)}_{33} &= (\lambda_l + 2\mu_l)(w_{3,3} + s' q_3) + \lambda_l w_{\gamma,\gamma}
\end{aligned}
\tag{2.5}
$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, and $s' = 1$ if $\boldsymbol{x}$ belongs to 1st layer or $s' = -\eta/(1-\eta)$ if $\boldsymbol{x}$ belongs to 2nd layer.

It is possible to eliminate all microlocal parameters from Eqs. (2.4) and (2.5), and hence we arrive at the governing system of three linear partial differential equations of the second order with constant coefficients for the macrodisplacements w_i, and the stress-displacement relations as follows (see Kaczyński, 1993, 1994)

$$
\begin{aligned}
&\left\{
\begin{array}{l}
\dfrac{1}{2}(c_{11} + c_{12}) w_{\gamma,\gamma\alpha} + \dfrac{1}{2}(c_{11} - c_{12}) w_{\alpha,\gamma\gamma} + c_{44} w_{\alpha,33} + (c_{13} + c_{44}) w_{3,3\alpha} = 0 \\
(c_{13} + c_{44}) w_{\gamma,\gamma 3} + c_{44} w_{3,\gamma\gamma} + c_{33} w_{3,33} = 0
\end{array}
\right. \\
&\left\{
\begin{array}{l}
\sigma^{(l)}_{\alpha 3} = c_{44}(w_{\alpha,3} + w_{3,\alpha}) \\
\sigma^{(l)}_{33} = c_{13} w_{\gamma,\gamma} + c_{33} w_{3,3} \\
\sigma^{(l)}_{12} = \mu_l (w_{1,2} + w_{2,1}) \\
\sigma^{(l)}_{11} = d^{(l)}_{11} w_{1,1} + d^{(l)}_{12} w_{2,2} + d^{(l)}_{13} w_{3,3} \\
\sigma^{(l)}_{22} = d^{(l)}_{12} w_{1,1} + d^{(l)}_{11} w_{2,2} + d^{(l)}_{13} w_{3,3}
\end{array}
\right.
\end{aligned}
\tag{2.6}
$$

Positive coefficients appearing in the above equations, describing the material and geometrical characteristics of the subsequent layers, are given in Appendix A.

It is noteworthy to point out the close relation of Eqs. (2.6) to fundamental equations for a transversely isotropic solid. The difference manifests itself in the fact that the components of the stress tensor $\sigma^{(l)}_{\alpha\beta}$ are discontinuous at the interfaces. Let us observe that the condition of perfect mechanical bonding between the layers (the continuity of the stress vector at the interfaces) is satisfied, so hereafter we shall omit the index l in the components σ_{3j}. Finally, setting $\lambda_1 = \lambda_2 \equiv \lambda$, $\mu_1 = \mu_2 \equiv \mu$ entails $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$ and the well-known equations of elasticity for a homogeneous isotropic body with Lamé's constants λ, μ are recovered.

3. Formulation of the problem

The problem under study involves the investigation of frictionless contact for two periodically stratified bodies with geometrically perturbed surfaces. For the simplicity of reasoning we shall confine ourselves to the case when one of the bodies, called a substrate, is absolutely rigid and possesses a small surface recess.

Let the periodic two-layered half-space be put into contact with the rigid substrate due to a uniform compressive pressure p applied at infinity (see Fig. 1a)). We assume that this contact is frictionless and unilateral such that the normal contact stresses cannot be tensional, and mutual penetrating of bodies surfaces is forbidden.

The surface of the substrate contains a local small deviation from the plane in the form of the sloping recess. Let us denote by S_0 the zone of the occupation of this defect. The relief of the recess is described generally by a smooth function (see Fig. 1c))

$$f: \quad R^2 \ni (x_1, x_2) \;\mapsto\; f(x_1, x_2) \in R \tag{3.1}$$

with the assumptions

$$\begin{aligned}
&f(x_1, x_2) = 0 && \forall (x_1, x_2) \in R^2 - S_0 \\
&\max_{(x_1,x_2)\in S_0} \Big\{ \Big|\frac{\partial f}{\partial x_1}\Big|, \Big|\frac{\partial f}{\partial x_2}\Big| \Big\} \ll 1 \\
&\frac{\partial f}{\partial x_1}(x_1, x_2) = \frac{\partial f}{\partial x_2}(x_1, x_2) = 0 && \forall (x_1, x_2) \in \partial S_0
\end{aligned} \tag{3.2}$$

In the above, the third condition is in fact a consequence of the first one and of the smoothness of the function f. It means that the recess is of a sloping form, not having corner points.

Due to the surface unevenness, the intimate contact occurs not over the entire plane $x_3 = 0$ but in a region $S \subseteq S_0$ where the intercontact gap is created (see Fig. 1b)). The surfaces of the contacting bodies are assumed to be free of traction within the gap region because they do not touch each other. As a result of the smoothness of the initial defect, the dimensions of the zone of S, depending on the external load, are unknown a priori and must be determined from the additional condition given later.

Taking into account (3.1) and (3.2), we are able to pose the boundary conditions at the nominal contact surface, i.e. on the plane $x_3 = 0$.

Now, we are ready to write the following boundary conditions of the problem at hand

$$
\begin{aligned}
&\sigma_{33} = -p \qquad \sigma_{31} = \sigma_{32} = 0 && x_3 \to +\infty \\
&\sigma_{31}(x_1, x_2, 0) = \sigma_{32}(x_1, x_2, 0) = 0 && \forall(x_1, x_2) \in R^2 \\
&\sigma_{33}(x_1, x_2, 0) = 0 && \forall(x_1, x_2) \in S \\
&u_3(x_1, x_2, 0) = \begin{cases} f(x_1, x_2) & \forall(x_1, x_2) \in S_0 - S \\ 0 & \forall(x_1, x_2) \in R^2 - S_0 \end{cases}
\end{aligned}
\tag{3.3}
$$

In addition, one has the requirement of a unilateral constraint (the normal contact tractions must be compressive) and of the lack of mutual penetrating (the height h of the intercontact gap must not be negative)

$$
\begin{aligned}
&\sigma_{33}(x_1, x_2, 0) \leqslant 0 \\
&h(x_1, x_2) \equiv u_3(x_1, x_2, 0) - f(x_1, x_2) \geqslant 0 \qquad \forall(x_1, x_2) \in S
\end{aligned}
\tag{3.4}
$$

Finally, to determine the unknown zone S of the intercontact gap, the condition of smooth closure will be used in the form

$$
\frac{\partial h}{\partial x_1}(x_1, x_2) = \frac{\partial h}{\partial x_2}(x_1, x_2) = 0 \qquad \forall(x_1, x_2) \in \partial S \tag{3.5}
$$

4. Solution to the boundary value problem

Within the framework of the homogenized model presented in Section 2 and applying the principle of superposition to satisfy boundary conditions (3.3), the problem can be presented as a sum of two components, namely

$$
\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \widetilde{\boldsymbol{\sigma}} \qquad \boldsymbol{u} = \boldsymbol{u}^0 + \widetilde{\boldsymbol{u}} \tag{4.1}
$$

where the components with the superscript 0 describe the stresses and displacements in the layered half-space pressed against the substrate with the absolutely flat surface, and the components having tilde describe the perturbations caused by the existence of the surface defect. The problem of determination of

$\boldsymbol{\sigma}^0$ and $\boldsymbol{u}^0$ is trivial, so afterwards we concentrate attention on the perturbed problem with the boundary conditions

$$\begin{aligned}
&\widetilde{\sigma}_{33} = \widetilde{\sigma}_{31} = \widetilde{\sigma}_{32} = 0 && x_3 \to +\infty \\
&\widetilde{\sigma}_{31}(x_1,x_2,0) = \widetilde{\sigma}_{32}(x_1,x_2,0) = 0 && \forall(x_1,x_2) \in R^2 \\
&\widetilde{\sigma}_{33}(x_1,x_2,0) = p && \forall(x_1,x_2) \in S \\
&\widetilde{w}_3(x_1,x_2,0) = \begin{cases} f(x_1,x_2) \\ 0 \end{cases} && \begin{array}{l} \forall(x_1,x_2) \in S_0 - S \\ \forall(x_1,x_2) \in R^2 - S_0 \end{array}
\end{aligned} \tag{4.2}$$

Here we have used the fact that

$$\sigma^0_{33}(x_1,x_2,0) = -p \qquad\qquad u^0_3(x_1,x_2,0) \cong w^0_3(x_1,x_2,0) = 0$$

A method of convenient solving the above-mentioned problem was demonstrated by Kaczyński (1993). Taking into account $(4.2)_2$, the problem is reduced to determination of a single harmonic function (denoted by $\widehat{\varphi}(x_1,x_2,x_3)$ in the case $\mu_1 \neq \mu_2$ and by $\overline{\varphi}(x_1,x_2,x_3)$ in the case $\mu_1 = \mu_2 \equiv \mu$), which must satisfy certain conditions in the plane $x_3 = 0$ (see Kaczyński (1993) for more details). We restrict ourselves only to writing the values of special interest – normal displacements and normal stresses on the boundary of the half-space[2]:

— Case $\mu_1 \neq \mu_2$

$$\begin{aligned}
&\widetilde{w}_3(x_1,x_2,0) = [m_2(1+m_2)^{-1} - m_1(1+m_1)^{-1}]\widehat{\varphi}_{,3}(x_1,x_2,0) \\
&\widetilde{\sigma}_{33}(x_1,x_2,0) = c_{44}(t_2^{-1} - t_1^{-1})\widehat{\varphi}_{,33}(x_1,x_2,0)
\end{aligned} \tag{4.3}$$

— Case $\mu_1 = \mu_2 \equiv \mu$, $B = \frac{\lambda_1\lambda_2 + 2\mu[\eta\lambda_1 + (1-\eta)\lambda_2]}{(1-\eta)\lambda_1 + \eta\lambda_2 + 2\mu}$

$$\begin{aligned}
&\widetilde{w}_3(x_1,x_2,0) = -\frac{B+2\mu}{B+\mu}\overline{\varphi}_{,3}(x_1,x_2,0) \\
&\widetilde{\sigma}_{33}(x_1,x_2,0) = -2\mu\overline{\varphi}_{,33}(x_1,x_2,0)
\end{aligned} \tag{4.4}$$

The application of conditions $(4.2)_{3,4}$ yields a mixed problem in the potential theory of finding the harmonic function

$$\varphi = \begin{cases} \widehat{\varphi} & \text{for} \quad \mu_1 \neq \mu_2 \\ \overline{\varphi} & \text{for} \quad \mu_1 = \mu_2 \end{cases}$$

[2] All constants appearing below are defined in Appendix B.

in the half-space $x_3 \geqslant 0$, decaying at infinity (in view of $(4.2)_1$) and satisfying on the boundary the conditions

$$
\begin{aligned}
&M[\varphi_{,33}(x_1,x_2,x_3)]_{x_3=0} = p && \forall (x_1,x_2) \in S \\
&L[\varphi_{,3}(x_1,x_2,x_3)]_{x_3=0} = \begin{cases} f(x_1,x_2) & \forall (x_1,x_2) \in S_0 - S \\ 0 & \forall (x_1,x_2) \in R^2 - S_0 \end{cases}
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
M &= \begin{cases} c_{44}\Big(\dfrac{1}{t_2} - \dfrac{1}{t_1}\Big) & \text{for} \quad \mu_1 \neq \mu_2 \\ -2\mu & \text{for} \quad \mu_1 = \mu_2 = \mu \end{cases} \\
L &= \begin{cases} \dfrac{m_2}{1+m_2} - \dfrac{m_1}{1+m_1} & \text{for} \quad \mu_1 \neq \mu_2 \\ -\dfrac{B+2\mu}{B+\mu} & \text{for} \quad \mu_1 = \mu_2 = \mu \end{cases}
\end{aligned}
\tag{4.6}
$$

Following Martynyak (2000), we reduce the problem given by the above conditions to an integro-differential equation. To this end, the following representation for the unknown function φ through the potentials of single layers is used

$$
\begin{aligned}
\varphi(x_1,x_2,x_3) &= -\frac{1}{2\pi L} \iint\limits_S \frac{h(\xi_1,\xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}} + \\
&- \frac{1}{2\pi L} \iint\limits_{S_0} \frac{f(\xi_1,\xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2}}
\end{aligned}
\tag{4.7}
$$

It fulfils condition $(4.5)_2$, and after substituting to remaining condition $(4.5)_1$, one is led to a singular equation of Newton's potential type for the function h

$$
\varDelta \iint\limits_S \frac{h(\xi_1,\xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}} = \frac{2\pi L p}{M} - \varDelta \iint\limits_{S_0} \frac{f(\xi_1,\xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}}
\tag{4.8}
$$

where $\varDelta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ stands for two-dimensional Laplace operator.

The above equation is similar to that occurring in crack problems but in the present consideration the right-hand side possesses the term dependent on the initial relief of the surface, and the region of the integration S is not fixed.

Solving Eq. (4.8) for an arbitrary recess involves great difficulties. However, in certain cases (see an example below) the solution can be found analytically.

5. Example

Consider a circular recess of the radius b: $S_0 = \{(x_1, x_2) : x_1^2 + x_2^2 \leqslant b^2\}$. The relief of the defect is described by the formula

$$f(x_1, x_2) = -h_0 \sqrt{\Big(1 - \frac{x_1^2 + x_2^2}{b^2}\Big)^3} \qquad h_0 \ll b \tag{5.1}$$

It is easy to note that the contact problem may be considered as axially symmetric. Consequently, assume that the zone of the intercontact gap is in the form of a circle, namely $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leqslant a^2\}$ with the radius of the gap $a \leqslant b$ being unknown.

For the thus chosen form of the initial defect, the integral in the right-hand side of Eq. (4.8) can be calculated by the method devised by Khay (1993). Governing equation (4.8) takes the form

$$\Delta \iint\limits_S \frac{h(\xi_1, \xi_2)\, d\xi_1 d\xi_2}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}} = \frac{2\pi Lp}{M} - \frac{3\pi^2 h_0}{2b}\Big(1 - \frac{3x_1^2}{2b^2} - \frac{3x_2^2}{2b^2}\Big) \tag{5.2}$$

To solve it, we make use of an analogue of Dyson's theorem (Khay, 1993) that states:

if P_n is a polynomial of n-th degree (of two variables), then

$$\Delta \iint\limits_S \sqrt{1 - \frac{\xi_1^2}{a^2} - \frac{\xi_2^2}{a^2}} \frac{P_n(\xi_1, \xi_2)}{\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}}\, d\xi_1 d\xi_2$$

is a polynomial of n-th degree.

Bearing this statement in mind, we seek a solution to Eq. (5.2) in the following form

$$h(x_1, x_2) = \sqrt{1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{a^2}}(c_{00} + c_{10}x_1 + c_{01}x_2 + c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2) \tag{5.3}$$

where c_{ij} are unknown coefficients.

Substituting (5.3) into (5.2) and calculating the resulting integrals (see Khay, 1993), we arrive at the equality of two polynomials. Hence,a set of linear algebraic equations for c_{ij} can be easily obtained, and their solving gives

$$
\begin{aligned}
&c_{00} = a\Big(-\frac{2Lp}{\pi M} + \frac{3}{2}\frac{h_0}{b} - \frac{1}{2}\frac{h_0}{b}\frac{a^2}{b^2}\Big) \\
&c_{10} = c_{01} = c_{11} = 0 \qquad\qquad c_{20} = c_{02} = -\frac{ah_0}{b^3}
\end{aligned}
\tag{5.4}
$$

From the above it follows that the gap height is described by

$$
h(x_1,x_2) = \sqrt{a^2 - x_1^2 - x_2^2}\Big(-\frac{2Lp}{\pi M} + \frac{3h_0}{2b} - \frac{h_0 a^2}{2b^3} - \frac{h_0 x_1^2}{b^3} - \frac{h_0 x_2^2}{b^3}\Big) \tag{5.5}
$$

This expression is dependent on a – the radius of the gap, which is still unknown. For its determination we use the conditions of smooth closure given by (3.5). This leads us to a quadratic equation (note that two conditions of (3.5) yield the same equation)

$$
-\frac{2}{\pi}\frac{Lp}{M} + \frac{3h_0}{2b}\Big(1 - \frac{a^2}{b^2}\Big) = 0 \tag{5.6}
$$

from which we have

$$
a = b\sqrt{1 - \frac{4}{3\pi}\frac{b}{h_0}\frac{Lp}{M}} \tag{5.7}
$$

It is worth to note that by setting $\lambda_1 = \lambda_2$, $\mu_1 = \mu_2$ in Eqs. (5.5) and (5.7) we recover the results obtained by Kit and Monastyrskyy (1998) who used another method for determining the interaction of an isotropic homogeneous half-space and a rigid foundation.

The complete displacement-stress field can be found from the harmonic potential φ, given by Eq. (4.7), by virtue of (5.7) and (5.5).

A. Appendix

Denoting by $b_l = \lambda_l + 2\mu_l$ $(l = 1,2)$, $b = (1-\eta)b_1 + \eta b_2$, the positive coefficients in governing equations (2.6) are given by the following formulae

$$
\begin{aligned}
c_{11} &= \frac{b_1 b_2}{b} + \frac{4\eta(1-\eta)(\mu_1 - \mu_2)(\lambda_1 - \lambda_2 + \mu_1 - \mu_2)}{b} \\
c_{12} &= \frac{\lambda_1\lambda_2 + 2[\eta\mu_2 + (1-\eta)\mu_1][\eta\lambda_1 + (1-\eta)\lambda_2]}{b}
\end{aligned}
$$

$$c_{13} = \frac{(1-\eta)\lambda_2 b_1 + \eta\lambda_1 b_2}{b} \qquad c_{33} = \frac{b_1 b_2}{b}$$

$$c_{44} = \frac{\mu_1\mu_2}{(1-\eta)\mu_1 + \eta\mu_2} \qquad d_{11}^{(l)} = \frac{4\mu_l(\lambda_l + \mu_l) + \lambda_l c_{13}}{b_l}$$

$$d_{12}^{(l)} = \frac{2\mu_l\lambda_l + \lambda_l c_{13}}{b_l} \qquad d_{13}^{(l)} = \frac{\lambda_l c_{33}}{b_l}$$

B. Appendix

The constants appearing in Eqs. (4.3) and (4.4) are given as follows

$$t_1 = \frac{1}{2}(t_+ - t_-) \qquad t_2 = \tfrac{1}{2}(t_+ + t_-)$$
$$m_\alpha = \tfrac{c_{11}t_\alpha^{-2} - c_{44}}{c_{13}+c_{44}} \qquad \forall \alpha \in \{1,2\}$$

where

$$t_\pm = \sqrt{\frac{(A_\pm \pm 2c_{44})A_\mp}{c_{33}c_{44}}} \qquad A_\pm = \sqrt{c_{11}c_{33}} \pm c_{13}$$

Note that $t_1 t_2 = \sqrt{c_{11}/c_{33}}$, $m_1 m_2 = 1$.

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Zagadnienie kontaktowe dla periodycznie uwarstwionej półprzestrzeni i sztywnego podłoża z geometrycznym defektem powierzchniowym

Streszczenie

W pracy rozważono trójwymiarowe zagadnienie ściskania prowadzące do jednostronnego kontaktu bez tarcia periodycznie dwuwarstwowej półprzestrzeni sprężystej ze sztywnym podłożem zawierającym lokalny defekt (gładkie niewielkie wgłębienie). Zastosowano przybliżone podejście oparte na zhomogenizowanym modelu z parametrami lokalnymi. Wynikające zagadnienie brzegowe sformułowano w postaci mieszanego problemu teorii potencjału, który sprowadzono do osobliwego równania całkowo-różniczkowego na głębokość defektu z nieznaną powierzchnią kontaktu. Dla szczególnego kształtu wgłębienia uzyskano jego rozwiązanie z użyciem analogu twierdzenia Dysona.

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